Significance of the smallest of a set of estimated normal variances [1])

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S a m e n v a t t i n g

Een toets voor de kleinste van een aantal geschatte varianties van normale verdelingen.

Stel wij hebben k onafhankelijke steekproeven, ieder van dezelfde uitgebreidheid n, afkomstig uit normale verdelingen:

$$\begin{matrix} x_{11}, \ldots, x_{1n} \\ \vdots \quad\quad \vdots \\ x_{k1}, \ldots, x_{kn}. \end{matrix}$$

Wij toetsen de hypothese dat de varianties van de k normale verdelingen gelijk zijn tegen het alternatief, dat de steekproef met de kleinste variantie afkomstig is uit een verdeling die een kleinere variantie heeft dan de andere verdelingen.

De geschatte varianties zijn:

$$s_i^2 = \sum_{j=1}^{n} (x_{ij} - x_{i.})^2/(n - 1), \; (i = 1, \ldots, k),$$

waarin $x_{i.} = \sum_{j=1}^{n} x_{ij}/n$ *het gemiddelde van de* i^e *steekproef is. De kleinste onder de* s_i^2 *noemen wij* s_{min}^2*. De toetsingsgrootheid is nu*

$$a = s_{min}^2 / \sum_{i=1}^{k} s_i^2.$$

In dit artikel wordt de waarschijnlijkheidsverdeling van deze toetsingsgrootheid afgeleid, benevens een tabel van kritieke waarden behorende bij een onbetrouwbaarheidsdrempel liggende tussen 0,05 en 0,04875.

1. Introduction

In some cases a test for the equality of the variances of a set of normal distributions, which is powerful especially against the alternative that one of the variances is smaller than the others, the latter being equal or not, might be useful. When, for instance, one has to select out of a set of machines or

[1]) Report SP 45 of the Statistical Department of the Mathematical Centre, Amsterdam. Head of the department: Prof. Dr D. van Dantzig.

out of several processes, the one with the smallest variability of produced product, we need such a test, to decide whether the smallest variance is significantly smaller than the other ones.

The test given here, which meets this condition, is analogous to the test concerning the largest variance, developed by W. G. Cochran (1941) and also described in C. Eisenhart, M. W. Hastay and W. A. Wallis (1947), where more extensive tables of critical values are given. Like Cochran we only deal with the case where the samples all have the same size.

2. Description of the test

Suppose we have k groups of random variables [1])

$$(1) \qquad \begin{matrix} \underline{x}_{11}, & \dots, & \underline{x}_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \underline{x}_{k1}, & \dots, & \underline{x}_{kn}, \end{matrix}$$

completely independent of one another and normally distributed with common variance σ^2 and with means which may differ from group to group but not within the groups. The variance estimates

$$(2) \qquad \underline{s}^2 = \sum_{j=1}^{n} (\underline{x}_{ij} - \underline{x}_{i.})^2/(n-1), \; (i = 1, \dots, k)$$

where $\underline{x}_{i.} = \sum_{j=1}^{n} \underline{x}_{ij}/n$, are then distributed independently as $\underline{\chi}^2\sigma^2/\nu$, $\underline{\chi}^2$ having ν degrees of freedom ($\nu = n - 1$). We define

$$(3) \qquad \underline{a}_i = \underline{s}_i^2 / \sum_{j=1}^{k} \underline{s}_j^2.$$

The test-statistic $\underline{a}$ is the smallest of the ratios $\underline{a}_i$. The critical region consists of small values of $\underline{a}$.

In table 1 we give the lower 5% points for the distribution of $\underline{a}$, i.e. the critical values of $\underline{a}$ for the test. The significance level is not exactly equal to 0,05, but lies between 0,04875 and 0,05.

3. An optimum property of the test

D. R. Truax (1953) proved the following optimum property of Cochran's test. We assume that the variances of the k populations under con-

[1]) Random variables are denoted by underlined symbols.

TABLE I. *Approximated lower 5 percent points of $\underline{\alpha}$, number of variances k, degrees of freedom ν.*

k \ ν	1	2	3	4	5	6
2	0.00154	0.02500	0.06083	0.09430	0.12275	0.14663
3	0.0^3278 [1]	0.00837	0.02489	0.04262	0.05892	0.07331
4	0.0^4964	0.00418	0.01401	0.02546	0.03647	0.04647
5	0.0^4444	0.00251	0.00916	0.01736	0.02550	0.03306
6	0.0^4241	0.00167	0.00653	0.01280	0.01917	0.02518
7	0.0^4145	0.00119	0.00493	0.00992	0.01512	0.02008
8	0.0^5941	0.0^3895	0.00387	0.00799	0.01234	0.01654
9	0.0^5645	0.0^3696	0.00314	0.00661	0.01033	0.01395
10	0.0^5461	0.0^3557	0.00261	0.00558	0.00882	0.01200
12	0.0^5259	0.0^3380	0.00189	0.00418	0.00673	0.00926
15	0.0^5129	0.0^3238	0.00128	0.00294	0.00484	0.00676
20	0.0^6530	0.0^3132	0.0^3781	0.00188	0.00318	0.00453

[1]) 0.0^3278 stands for 0.000278, etc.

sideration are σ_i^2 ($i = 1, \ldots, k$). We say that the variance of the population has „slipped to the right'' if $\sigma_1^2 = \ldots = \sigma_{i-1}^2 = \sigma_{i+1}^2 = \ldots = \sigma_k^2 = \sigma^2$ and $\sigma_i^2 = \lambda^2\sigma^2$, where $\lambda > 1$. Let D_0 be the decision that all k variances are equal, and let D_j be the decision that D_0 is false and $\sigma_j^2 = \max(\sigma_1^2, \ldots, \sigma_k^2)$ for $j = 1, \ldots, k$. We want to find a statistical procedure which will select one of the $k + 1$ decisions $D_0, D_1, \ldots, D_k$ and which has the following properties:

a) When all variances are equal, D_0 should be selected with probability $1 - \alpha$, where α is a small positive number fixed prior to the experiment;

b) the procedure is symmetric, that is the probability of selecting D_i when σ_i^2 has slipped to the right is the same for all i when λ has the same value in all cases;

c) the procedure is invariant if all the observations are multiplied by the same constant; and

d) the procedure is invariant if some constant b_i is added to all the observations in the i^{th} population.

Then the procedure:

$$\text{if } \max_{i=1,\ldots,k} s_i^2 / \sum_{i=1}^{k} s_i^2 > L_\alpha \quad \text{select } D_m,$$

$$\text{if } \max_{i=1,\ldots,k} s_i^2 / \sum_{i=1}^{k} s_i^2 \leq L_\alpha \quad \text{select } D_0,$$

where L_α is a constant determined by property *a*) and $s_m^2 = \max s_i^2$, maximizes the probability of making the correct decision when one of the variances has slipped to the right, subject to the restrictions *a*), *b*), *c*) and *d*).

With some slight alterations the proof of T r u a x applies also to our test, which maximizes therefore the probability of making a correct decision when one of the variances has "slipped to the left".

4. The distribution of $\underline{a}$

If we denote by $P_r(a)$ the probability that r of the ratios $\underline{a}_i$ (specified in advance) do not exceed the value a, then the probability that $\underline{a}$ does not exceed a is given by

$$(4) \qquad p(a) = \binom{k}{1} P_1(a) - \binom{k}{2} P_2(a) + \binom{k}{3} P_3(a) \ldots + (-1)^k P_k(a).$$

This formula follows immediately from a wellknown theorem in probability theory, the proof of which can be found for instance in W. F e l l e r (1950) (Chapter 4, formula (1.5)).

Equation (4) contains all the probabilities $P_1, \ldots, P_k$ and is therefore rather unsuitable for the computation of significance levels of $\underline{a}$. Therefore we use only the first term of (4) and determine the approximate critical value a_α from

$$(5) \qquad P_1(a_\alpha) = \alpha/k.$$

To investigate the accuracy of this approximation we need the following inequalities:

$$(6) \qquad P_2(a) \leqq \{P_1(a)\}^2 \quad (0 \leqq a \leqq 1),$$

$$(7) \qquad \binom{k}{1} P_1(a) - \binom{k}{2} P_2(a) \leqq p(a) \leqq \binom{k}{1} P_1(a).$$

Formula (6) will be derived in the appendix. The second inequality is known as one of the inequalities of B o n f e r r o n i (cf. F e l l e r (1950), (Chapter 4, formula (6.7), for the case $m = 1$). Like (4), formula (7) is rather simple to prove. We may illustrate this for the case $k = 3$. In the general case we consider three possible events A_1, A_2 and A_3, which, as the outcome of an experiment, may occur separately or together. In figure 1 the possible outcomes of the experiment are sketched as regions in the corresponding sample space.

From this we read directly that

$$(8) \qquad P[A_1] + P[A_2] + P[A_3] - P[A_1 \text{ and } A_2] - P[A_1 \text{ and } A_3] - \\ - P[A_2 \text{ and } A_3] \leqq P[A_1 \text{ or } A_2 \text{ or } A_3] \leqq P[A_1] + P[A_2] + P[A_3],$$

because the domains corresponding to the three members of (8) respectively are

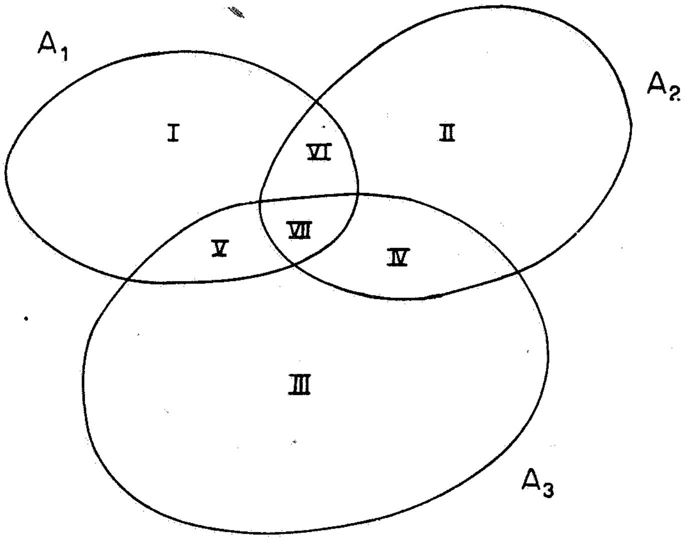

Figure 1. Unions and intersections of three events.

I + II + III + IV + V + VI,
I + II + III + IV + V + VI + VII and
I + II + III + 2.IV + 2.V + 2.VI + 3.VII,

respectively. Applying (8) to our case, where A_1 stands for the event $\underline{a}_1 \leqq a$ and so on, we get immediately (7).

Combining now (6) and (7) we get

$$(9) \qquad k\, P_1(a) - \tfrac{1}{2}\, k\, (k-1)\, \{P_1(a)\}^2 \leqq p(a) \leqq k\, P_1(a).$$

Substituting (5) into (9) gives us

$$k \cdot \frac{\alpha}{k} - \tfrac{1}{2} k\, (k-1) \frac{\alpha^2}{k^2} \leqq p\,(a_\alpha) \leqq k\, \frac{\alpha}{k},$$

or

$$(10) \qquad \alpha - \tfrac{1}{2}\, \frac{k-1}{k}\, \alpha^2 \leqq p\,(a_\alpha) \leqq a_\alpha,$$

or for $$k \geqq 2$$

$$(11) \qquad \alpha - \tfrac{1}{2}\, \alpha^2 \leqq p\,(a_\alpha) \leqq \alpha,$$

that is, lower 5 percent levels of $\underline{a}$ determined from (5) will correspond to $0.4875 \leqq p \leqq 0.05$. These approximate levels are given in table 1.

5. Computation of the tabulated values

Cochran (1941) has shown that the probability $Q_1(a)$ that a specified one of the ratios $\underline{a}_i$ exceeds a is equal to

$$(12) \qquad Q_1(a) = \int_a^1 x^{\nu/2-1} (1-x)^{\nu(k-1)/2-1} / B(\nu/2, \nu(k-1)/2).$$

So $P_1(a) = 1 - Q_1(a)$ is given by

$$(13) \qquad P_1(a) = \int_0^a x^{\nu/2-1} (1-x)^{\nu(k-1)/2-1} / B(\nu/2, \nu(k-1)/2) = $$
$$= I_a(\nu/2, \nu(k-1)/2).$$

For this incomplete beta-function the following expansion follows easily

$$(14) \qquad P_1(a) = a^{\nu/2} \binom{\nu k/2 - 1}{\nu/2} \sum_{j=0}^{\infty} \binom{\nu(k-1)/2 - 1}{j} \frac{\nu}{\nu+2j} (-a)^j.$$ [1]

Now a_α must be determined from

$$a_\alpha^{-\nu/2} = \frac{k}{\alpha} \binom{\nu k/2 - 1}{\nu/2} \sum_{j=0}^{\infty} \binom{\nu(k-1)/2 - 1}{j} \frac{\nu}{\nu+2j} (-a)^j.$$

We approximated a_α successively by $a_1, a_2, \ldots$ in the following way

$$a_1^{-\nu/2} = \frac{k}{\alpha} \binom{\nu k/2 - 1}{\nu/2},$$

$$a_2^{-\nu/2} = \frac{k}{\alpha} \binom{\nu k/2 - 1}{\gamma/2} \left\{ 1 - \binom{\nu(k-1)/2 - 1}{1} \frac{\nu}{\nu+2} a_1 \right\},$$

$$a_3^{-\nu/2} = \frac{k}{\alpha} \binom{\nu k/2 - 1}{\nu/2} \left\{ 1 - \binom{\nu(k-1)/2 - 1}{1} \frac{\nu}{\nu+2} a_2 + \right.$$
$$\left. + \binom{\nu(k-1)/2 - 1}{2} \frac{\nu}{\nu+4} a_2^2 \right\},$$

etc.

Within the limits of the table we constructed the values of a_α are small and therefore the convergence is rather fast. In fact we found that the fourth approximation gave sufficient accuracy.

The 2.5, 1 and 0,5 percentage points of the incomplete beta-function are tabulated by C. M. Thompson (1941), so for $\alpha = 0.05$ and $k = 2{,}5$ and 10 we could check our calculations by reading the values directly form these

[1] When $\nu(k-1)/2$ is an integer this expansion breaks off at $j = \nu(k-1)/2 - 1$.

tables. For $\nu = 1$ we used the reversed expansion given by C. M. Thompson (1941), p. 159.

6. Appendix

It is shown first that the inequality (6) is equivalent with

$$Q_2(a) \leqq \{Q_1(a)\}^2, \tag{15}$$

where $Q_2(a)$ and $Q_1(a)$ respectively stand for the probabilities that a pair and one of the ratios $\underline{a}_i$ (specified in advance) respectively exceed a. We have

$$Q_1(a) = 1 - P_1(a)$$

and consequently

$$P_1(a)\{1 - P_1(a)\} = Q_1(a)\{1 - Q_1(a)\}. \tag{16}$$

Further the equality

$$P_1(a) - P_2(a) = Q_1(a) - Q_2(a) \tag{17}$$

holds, both sides being equal to the probability that one of a pair of specified ratios a_i exceeds and the other does not exceed a. From (16) and (17) we obtain

$$P_2(a) - \{P_1(a)\}^2 = Q_2(a) - \{Q_1(a)\}^2, \tag{18}$$

which proves the equivalence of the inequalities (6) and (15).

The sum of all $\underline{a}_i$ being equal to 1 (cf. (3)), $Q_2(a) = 0$ if $a \geqq \frac{1}{2}$. Thus for $a \geqq \frac{1}{2}$ equation (15) is obviously true and therefore (6) must also hold. To point out the validity of (15) for all values of a between 0 and 1, Cochran (1941) says that when it is given that one of the $\underline{a}_i$ is large this diminishes the probability of the other $a_j (j \neq i)$ being large.

The following counterexample[1]) shows however that this is not true in all cases, although it is true, as will be shown later, for the distributions under consideration.

Suppose the variables $\underline{x}_1$, $\underline{x}_2$ and $\underline{x}_3$ can take the values 1 and 2 respectively with probabilities p and $1 - p$ respectively. Defining

$$\underline{y}_i = \underline{x}_i / (\underline{x}_1 + \underline{x}_2 + x_3) \quad (i = 1,2,3),$$

and C being some real number, $\frac{1}{4} \leqq C < \frac{1}{3}$ we obtain

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[1]) This counterexample was constructed by H. Kesten, assistant of the Statistical Dept. of the Mathematical Centre, Amsterdam.

$$P[\underline{y}_1 \leqq C] = p\,(1 - p^2)$$
$$P[\,\underline{y}_1 \leqq C \text{ and } \underline{y}_2 \leqq C] = p^2\,(1 - p).$$

Consequently

$$P[\underline{y}_1 \leqq C \text{ and } \underline{y}_2 \leqq C] > \{P[\underline{y}_1 \leqq C]\}^2,$$

if $\frac{1}{4} \leqq C < \frac{1}{3}$ and $1 > p > \frac{1}{2}(\sqrt{5} - 1)$.

Therefore it is necessary to derive (6) explicitely for the case $a < \frac{1}{2}$. The following proof was found in co-operation with the Department of Pure Mathematics of the Mathematical Centre.[1])

Cochran showed that the simultaneous distribution of a given pair $\underline{a}_1$ and $\underline{a}_2$ of the ratios $\underline{a}_i$ is given by

$$f\,(a_1,a_2) = C_1 \,.\, (a_1 a_2)^{\nu/2-1}\,(1 - a_1 - a_2)^{\nu(k-2)/2-1},$$

where

$$C_1^{-1} = B(\nu/2,\, \nu(k-1)/2)\; B(\nu/2,\, \nu(k-2)/2).$$

So, for $a \leqq \frac{1}{2}$ we have

(19) $$P_2(a) = C_1 \int_0^a \int_0^a (x_1\, x_2)^{\nu/2-1}\,(1 - x_1 - x_2)^{\nu(k-2)/2-1}\, dx_1\, dx_2.$$

We put

$$\nu/2 = s,\; \nu(k-2)/2 = t,\; B(s,t)/B(s,\, s+t) = C_2,$$

and

$$C_1^{-1}\,\{[P_1(a)]^2 - P_2(a)\} = (\text{cf. } (13)) =$$
$$= C_2\,\{\int_0^a x^{s-1}\,(1-x)^{s+t-1}\,dx\}^2 - \int_0^a\int_0^a (x_1\,x_2)^{s-1}\,(1 - x_1 - x_2)^{t-1}\,dx_1\,dx_2 = \varphi(a).$$

We shall prove that there exists a point a_0 with $0 < a < \frac{1}{2}$, such that $\varphi(a)$ is steadily increasing for $0 < a < a_0$ and steadily decreasing for $a_0 < a < \frac{1}{2}$. The relation (6) will then be proved completely.

We note that on account of the logarithmic convexity of the Γ-function we have

$$C_2 = B\,(s,\,t)/B\,(s,\, s+t) = \Gamma\,(t)\,\Gamma\,(2s+t)/\Gamma\,(s+t)\,\Gamma\,(s+t) > 1.$$

Differentiating $\varphi(a)$ with respect to a we get

$$\varphi'(a) = 2C_2\, a^{s-1}\,(1-a)^{s+t-1} \int_0^a x^{s-1}(1-x)^{s+t-1}\,dx$$
$$-\,2\int_0^a (ax)^{s-1}\,(1 - a - x)^{t-1}\,dx.$$

[1]) It was published before by the Dept. of Pure Mathematics of the Mathematical Centre as „Report Z.W. 1954-013, An inequality involving Beta-functions, by R. Doornbos, H. J. A. Duparc, C. G. Lekkerkerker and W. Peremans."

By applying the substitution $x = u\,(1 - a)$ in the last integral we get

$$\varphi'(a) = 2a^{s-1}\,(1 - a)^{s+t-1}\{C_2 \int_0^a x^{s-1}\,(1 - x)^{s+t-1}$$

$$- \int_0^{a/(1-a)} u^{s-1}\,(1 - u)^{t-1}\,du\} = 2a^{s-1}\,(1 - a)^{s+t-1}\,\varphi_1(a), \text{ say.}$$

Next differentiating $\varphi_1(a)$ we find

$$\varphi_1{}'(a) = C_2\,a^{s-1}\,(1 - a)^{s+t-1} - \frac{1}{(1 - a)^2}\left(\frac{a}{1 - a}\right)^{s-1}\left(\frac{1 - 2a}{1 - a}\right)^{t-1} =$$

$$= a^{s-1}\,(1 - a)^{-(s+t)}\,\{C_2\,(1 - a)^{2(s+t)-1} - (1 - 2a)^{t-1}\} =$$

$$= a^{s-1}\,(1 - a)^{-(s+t)}\,.\,\varphi_2(a), \text{ say.}$$

Clearly

$$(20)\qquad \begin{cases} \varphi_1(0) = 0 \\ \varphi_1(\tfrac{1}{2}) < C_2 \int_0^1 x^{s-1}\,(1 - x)^{s+t-1}\,dx - \int_0^1 x^{s-1}\,(1 - x)^{t-1}\,dx = \\ \qquad = C_2\,B\,(s, s + t) - B\,(s, t) = 0 \\ \varphi_2(0) = C_2 - 1 > 0,\ \varphi_2\,(\tfrac{1}{2}) > 0. \end{cases}$$

Further we have $\varphi_2(a) = 0$ if and only if

$$1 - 2a = C_2{}^{1/(t-1)}\,(1 - a)^{\{2(s+t)-1\}/(t-1)} \qquad \text{in the case } t \neq 1,$$

$$1 = C_2\,(1 - a)^{2s+1} \qquad \text{in the case } t = 1.$$

Since for fixed real $\alpha \neq 1$ the function $f(x) = x^\alpha$ is either a convex or a concave function for $x > 0$, it follows that $\varphi_2\,(a) = 0$ for at most two values of a.

Since $\varphi_2(a)$ is the derivative of $\varphi_1(a)$, apart from a positive factor for $a \neq 0$, it follows from the above result and the relations (20) that $\varphi_1\,(a)$ is equal to zero for $a = 0$, positive for small values of a, negative for $a = \frac{1}{2}$ and that $\varphi_1(a)$ has at most two extrema in the interval $(0, \frac{1}{2})$. Hence $\varphi_1(a)$ has exactly two extrema and is exactly once equal to zero (at a_0, say) in the interval $0 < a < \frac{1}{2}$. Moreover $\varphi_1(a)$ is positive for $0 < a < a_0$ and negative for $a_0 < a < \frac{1}{2}$. The function $\varphi_1(a)$ being the derivative of $\varphi(a)$, apart from a positive factor (for $a \neq 0$), the proof is completed.

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