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MATHEMATICS

# SLIPPAGE TESTS FOR A SET OF GAMMA-VARIATES

BY

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## 1. *Summary*

In this paper a generalization is given of the significance tests for the largest and the smallest respectively of a set of estimated normal variances as suggested by W. G. COCHRAN (1941) and one of the present authors (cf. R. DOORNBOS (1956)) respectively. These tests only deal with the case where the samples from which the variances are estimated all have the same size.

The present paper gives a treatment which is also valid for different sample sizes. Further we consider the power function of the tests with respect to the alternative hypothesis that one of the variances has slipped to the right or, in the case of the test for the smallest variance, to the left.

## 2. *Introduction and description of the tests*

Suppose we have a set of random variables

$$\mathbf{u}_1, \ldots, \mathbf{u}_k \text{ [2])} \tag{2.1}$$

distributed independently of one another according to gamma distributions with parameters $\alpha_1, \beta_1; \ldots; \alpha_k, \beta_k$ respectively; that is to say the density function of $\mathbf{u}_i$ is

$$f(u_i) = \frac{1}{\Gamma(\alpha_i)\,\beta_i^{\alpha_i}}\, u_i^{\alpha_i - 1}\, e^{-u_i/\beta_i}, \qquad 0 \leqq u_i \leqq \infty, \tag{2.2}$$

where $\alpha_i$ and $\beta_i$ are real positive numbers. As is well known the distribution of $\mathbf{t} = \boldsymbol{\chi}^2\sigma^2$, where $\boldsymbol{\chi}^2$ is distributed as a chi-square with $\nu$ degrees of freedom, is a special case of a gamma distribution, with parameters $\alpha = \nu/2$ and $\beta = 2\sigma^2$.

Now our problem is to find tests for the hypothesis

$$H_0\colon\ \beta_1 = \ldots = \beta_k = \beta, \text{ say,} \tag{2.3}$$

against the alternatives

$$\begin{cases} H_1\colon\ \beta_1 = \ldots = \beta_{i-1} = \beta_{i+1} = \ldots = \beta_k = \beta, \\ \qquad\qquad \beta_i = C_i\beta,\ C_i > 1, \end{cases} \tag{2.4}$$

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[1]) Report SP 49 of the Statistical Department of the Mathematical Centre.
[2]) Random variables are denoted by symbols printed in bold type.

for one unknown value of $i$ and

(2.5) $$\begin{cases} H_2\colon\ \beta_1 = \ldots = \beta_{i-1} = \beta_{i+1} = \ldots = \beta_k = \beta, \\ \beta_i = c_i \beta;\ 0 < c_i < 1, \end{cases}$$

for one unknown value of $i$.

For both tests we compute the ratios

(2.6) $$\mathbf{x}_j = \frac{\mathbf{u}_j}{\Sigma \mathbf{u}_i}, \quad (j = 1, \ldots, k).$$

Then, if we are testing $H_0$ against $H_1$, the following incomplete $B$-integrals are determined:

(2.7) $$\begin{cases} \mathbf{d}_j = \dfrac{1}{B(\alpha_j, A-\alpha_j)} \displaystyle\int_{\mathbf{x}_j}^{1} x^{\alpha_j - 1}(1-x)^{A-\alpha_j-1}\, dx \\ \quad = 1 - I_{\mathbf{x}_j}(\alpha_j, A-\alpha_j), \quad (j = 1, \ldots, k), \end{cases}$$

where $A = \sum \alpha_i$. Next we define the test statistic $\mathbf{d}$ by

(2.8) $$\mathbf{d} = \min \mathbf{d}_j.$$

If we reject $H_0$ when $\mathbf{d}$ takes a value $d \leqq \varepsilon/k$, the level of significance lies between $\varepsilon$ and $\varepsilon - \varepsilon^2/2$ as will be shown in section 4.

If all $\alpha_i$ have the same value, $\mathbf{d}$ corresponds to the smallest ratio $\mathbf{x}$ and our test reduces to COCHRAN's test.

Testing $H_0$ against $H_2$ requires computation of the integrals

(2.9) $$\begin{cases} \mathbf{e}_j = \dfrac{1}{B(\alpha_j, A-\alpha_j)} \displaystyle\int_{0}^{\mathbf{x}_j} x^{\alpha_j - 1}(1-x)^{A-\alpha_j-1}\, dx = 1 - \mathbf{d}_j \\ \quad = I_{\mathbf{x}_j}(\alpha_j, A-\alpha_j). \end{cases}$$

We reject $H_0$ if

(2.10) $$\mathbf{e} = \min \mathbf{e}_j \leqq \varepsilon/k.$$

The level of significance is again a number between $\varepsilon$ and $\varepsilon - \varepsilon^2/2$.

## 3. *An optimum property of the tests if* $\alpha_1 = \ldots = \alpha_k$

D. R. TRUAX (1953) proved an optimum property of COCHRAN's test. In exactly the same way one can prove that our tests are optimal in the following sense if $\alpha_1 = \ldots = \alpha_k$. Let $D_0$ be the decision that $H_0$ is true and let $D_{1j}$ be the decision that $H_0$ is false and that $\beta_j = \max(\beta_1, \ldots, \beta_k)$. Then, if $d = d_m$, i.e. if $d_m$ is the smallest of $d_1, \ldots, d_k$ the procedure

(3.1) $$\begin{cases} \text{if } d \leqq L_\varepsilon \text{ select } D_{1m}, \\ \text{if } d > L_\varepsilon \text{ select } D_0, \end{cases}$$

where $L_\varepsilon$ is a constant determined by the condition that $D_0$ should be selected with probability $1-\varepsilon$ if $H_0$ is true, maximizes the probability of making the correct decision if the hypothesis $H_1$ is true.

When the hypothesis $H_2$ is true the analogous optimum property holds for our second test. In both cases $\varepsilon/k$ is an approximation of the critical value of $\mathbf{d}$ and $\mathbf{e}$.

4. *Proofs of the results stated in* 2

To obtain the joint distribution of $\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}$ as given by (2.6) and of $\mathbf{U}=\mathbf{u}_1+ \ldots +\mathbf{u}_k$ we put

$$(4.1)\qquad \begin{cases} u_1 = x_1 U, \\ \vdots \\ u_{k-1} = x_{k-1} U, \\ u_{k-1} = U(1-x_1 \ldots - x_{k-1}). \end{cases}$$

The Jacobian of this transformation becomes $U^{k-1}$ and the joint distribution of $\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}$ and $\mathbf{U}$ is found to be given by the density function

$$(4.2)\qquad \begin{cases} g(x_1, \ldots, x_{k-1}, U) = \\ = \dfrac{\Gamma(A)}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_k)} x_1^{\alpha_1-1}\ldots x_{k-1}^{\alpha_{k-1}-1}(1-x_1\ldots-x_{k-1})^{\alpha_k-1}\dfrac{U^{A-1}e^{-U/\beta}}{\Gamma(A)\beta^A}, \end{cases}$$

where $A=\alpha_1+ \ldots +\alpha_k$.

Thus we see, as is well known that $\mathbf{U}$ has also a gamma distribution, with parameters $\alpha_1+ \ldots +\alpha_k=A$ and $\beta$ and moreover that the joint distribution of $\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}$ is given by

$$(4.3)\qquad h(x_1, \ldots, x_{k-1}) = \frac{\Gamma(A)}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_k)} x_1^{\alpha_1-1}\ldots x_{k-1}^{\alpha_{k-1}-1}(1-x_1\ldots-x_{k-1})^{\alpha_k-1}.$$

But the same derivation gives us the general result

$$(4.4)\qquad \begin{cases} p(x_1, \ldots, x_i) = \\ = \dfrac{\Gamma(A)}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_i)\Gamma(\alpha_{i+1}+\ldots+\alpha_k)} x_1^{\alpha_1-1}\ldots x_i^{\alpha_i-1}(1-x_1\ldots-x_i)^{\alpha_{i+1}+\ldots+\alpha_k-1}, \\ \qquad\qquad (i=1, \ldots, k-1), \end{cases}$$

if we consider instead of $\mathbf{u}_1, \ldots, \mathbf{u}_k$ the $i+1$ variables $\mathbf{u}_1, \ldots, \mathbf{u}_i$ and $\mathbf{u}_{i+1}+ \ldots +\mathbf{u}_k$ which are also independent of one another and which have gamma distributions with parameters $\alpha_1, \ldots, \alpha_i, \alpha_{i+1}+ \ldots +\alpha_k$ and $\beta$.

We consider now a set of $k$ real numbers $g_1, \ldots, g_k$ $(0 \leqq g_i \leqq 1)$ and the probabilities defined by

$$(4.5)\qquad \begin{cases} p_i = P[\mathbf{x}_i \leqq g_i], \\ p_{i,j} = P[\mathbf{x}_i \leqq g_i \text{ and } \mathbf{x}_j \leqq g_j], \quad (i \neq j) \\ \vdots \\ q_i = P[\mathbf{x}_i > g_i] \\ q_{i,j} = P[\mathbf{x}_i > g_i \text{ and } \mathbf{x}_j > g_j], \quad (i \neq j) \\ \vdots \end{cases}$$

all computed under $H_0$. If we denote by $P$ the probability that at least one of the ratios $\mathbf{x}_i$ does not exceed the corresponding value $g_i$, we have

(4.6) $$P = \Sigma p_i - \Sigma p_{i,j} + \Sigma p_{i,j,l} \dots + (-1)^{k-1} p_{1,2,\dots,k},$$

where the $r$th summation is extended over all $p$'s with $r$ subscripts; hence the $r$th sum has $\binom{k}{r}$ terms (cf. M. FRÉCHET (1940), or W. FELLER (1950), chapter 4).

For $Q$, the probability that at least one of the $\mathbf{x}_i$ exceeds $g_i$, we have

(4.7) $$Q = \Sigma q_i - \Sigma q_{i,j} + \Sigma q_{i,j,l} \dots + (-1)^{k-1} q_{1,2,\dots,k}.$$

It follows from BONFERRONI's inequality (cf. M. FRÉCHET (1940), or W. FELLER (1950), chapter 4) that

(4.8) $$\Sigma p_i - \Sigma p_{i,j} \leqq P \leqq \Sigma p_i$$

and

(4.9) $$\Sigma q_i - \Sigma q_{i,j} \leqq Q \leqq \Sigma q_i.$$

In the latter part of this section we shall prove the inequalities

(4.10) $$p_{i,j} \leqq p_i \, p_j$$

and

(4.11) $$q_{i,j} \leqq q_i \, q_j.$$

If we now determine the numbers $g_{i,\varepsilon}$ so that all $p_i$ are equal to $\varepsilon/k$, or according to (4.4)

(4.12) $$P\,[\mathbf{x}_i \leqq g_{i,\varepsilon}] = I_{g_{i,\varepsilon}}(\alpha_i, A - \alpha_i) = \varepsilon/k,$$

then we get from (4.8) and (4.10)

$$\Sigma p_i - \Sigma p_i \, p_j \leqq \Sigma p_i - \Sigma p_{i,j} \leqq P_\varepsilon \leqq \Sigma p_i,$$

or

$$\varepsilon - (k-1)\varepsilon^2/2k \leqq P_\varepsilon \leqq \varepsilon,$$

or for $k \geqq 2$

(4.13) $$\varepsilon - \varepsilon^2/2 \leqq P_\varepsilon \leqq \varepsilon,$$

if $P_\varepsilon$ is defined by

(4.14) $$P_\varepsilon = P\,[\min(\mathbf{x}_i - g_{i,\varepsilon}) \leqq 0].$$

In the same way we get

(4.15) $$\varepsilon - \varepsilon^2/2 \leqq Q_\varepsilon \leqq \varepsilon,$$

if the numbers $G_{i,\varepsilon}$ are determined so as to make all $q_i$ equal to $\varepsilon/k$ and if $Q_\varepsilon$ is the probability that at least one of the ratios $\mathbf{x}_i$ exceeds the corresponding value $G_{i,\varepsilon}$.

As the procedure described in section 2 to test $H_0$ against the sets of alternatives $H_2$ and $H_1$ respectively gives us the probabilities $P_\varepsilon$ and

$Q_\varepsilon$ respectively of rejecting $H_0$ when $H_0$ is true, these probabilities lie between the bounds stated there.

We now proceed to prove the inequalities (4.10) and (4.11). First it is shown that (4.10) and (4.11) are equivalent. We have

$$p_i = 1 - q_i \text{ and } p_j = 1 - q_j$$

and consequently

$$p_i(1-p_j) = q_j(1-q_i). \tag{4.16}$$

Further

$$p_i - p_{i,j} = q_j - q_{i,j} \; (= P[\mathbf{x}_i \leqq g_j \text{ and } \mathbf{x}_j > g_j]). \tag{4.17}$$

From (4.16) and (4.17) we obtain

$$p_i p_j - p_{i,j} = q_i q_j - q_{i,j}, \tag{4.18}$$

which proves the equivalence of (4.10) and (4.11). Thus it is sufficient to prove (4.10) [1]) and we need only consider values $g_i$ and $g_j$ such that $g_i + g_j \leqq 1$, for when $g_i + g_j > 1$, $q_{i,j} = 0$ and so (4.11) and (4.10) are obviously true.

It is easily seen that (4.10) is equivalent with

$$\frac{p_{i,j}}{p_j} \leqq \frac{p_i - p_{i,j}}{q_j}, \tag{4.19}$$

or

$$P\,[\mathbf{x}_i \leqq g_i \,|\, \mathbf{x}_j \leqq g_j] \leqq P\,[\mathbf{x}_i \leqq g_i \,|\, \mathbf{x}_j > g_j]. \tag{4.20}$$

From (4.4) it follows that the left hand member $L(g_i, g_j)$ of (4.20) equals

$$C \frac{\int\limits_0^{g_j}\int\limits_0^{g_i} x_j^{\alpha_j - 1} x_i^{\alpha_i - 1} (1 - x_i - x_j)^{A - \alpha_i - \alpha_j - 1}\, dx_i\, dx_j}{\int\limits_0^{g_j} x_j^{\alpha_j - 1} (1 - x_j)^{A - \alpha_j - 1}\, dx_j},$$

where

$$C = \frac{\Gamma(A - \alpha_j)}{\Gamma(\alpha_i)\,\Gamma(A - \alpha_i - \alpha_j)}.$$

Putting $x_i = v(1 - x_j)$ we get

$$\left\{\begin{aligned} L(g_i, g_j) &= C \frac{\int\limits_0^{g_j}\int\limits_0^{g_i/(1-x_j)} v^{\alpha_i - 1}(1-v)^{A-\alpha_i-\alpha_j-1} x_j^{\alpha_j - 1}(1-x_j)^{A-\alpha_j-1}\, dv\, dx_j}{\int\limits_0^{g_j} x_j^{\alpha_j - 1}(1-x_j)^{A-\alpha_j-1}\, dx_j} \\ &\leqq \int\limits_0^{g_i/(1-g_j)} v^{\alpha_i - 1}(1-v)^{A-\alpha_i-\alpha_j-1}\, dv. \end{aligned}\right. \tag{4.21}$$

[1]) The following proof, which is substantially simpler than another one which was developed by the authors, has been found by H. Kesten, assistant of the Statistical Department, as a special case of the proof of the more general inequality

$$p_{i,j,\ldots,l} \leqq p_i p_j \cdots p_l.$$

Similarly the right hand member $R(g_i, g_j)$ of (4.20) is found to be equal to

$$(4.22)\quad \begin{cases} C\dfrac{\displaystyle\int_{g_j}^{1}\int_{0}^{\min(g_i,1-x_j)} x_i^{\alpha_i-1}x_j^{\alpha_j-1}(1-x_i-x_j)^{A-\alpha_i-\alpha_j-1}\,dx_i\,dx_j}{\displaystyle\int_{g_j}^{1} x_j^{\alpha_j-1}(1-x_j)^{A-\alpha_j-1}\,dx_j} \\ \qquad\qquad \geqq C\displaystyle\int_{0}^{g_i/(1-g_j)} v^{\alpha_i-1}(1-v)^{A-\alpha_i-\alpha_j-1}\,dv. \end{cases}$$

So it follows from (4.21) and (4.22) that (4.20) holds.

## 5. *The power of the tests*

In this section we shall derive upper and lower bounds for the probabilities of making a correct decision, following the procedure described in section 3, under the hypotheses $H_1$ and $H_2$.

In the first case, i.e. when $H_1$ is true, we assume that $\beta_i$ is the parameter which has slipped to the right, i.e. $\beta_i = C_i\beta$, $C_i > 1$. Then we prove that $Q_i$, the probability of making the correct decision lies between the limits

$$(5.1)\qquad \{1-I_{B_i}(\alpha_i, A-\alpha_i)\}(1-\varepsilon) \leqq Q_i \leqq \{1-I_{B_i}(\alpha_i, A-\alpha_i)\},$$

where

$$(5.2)\qquad B_i = \frac{G_{i,\varepsilon}}{C_i-(C_i-1)G_{i,\varepsilon}},$$

where $G_{i,\varepsilon}$ is determined so as to make

$$(5.3)\qquad I_{G_{i,\varepsilon}}(\alpha_i, A-\alpha_i) = 1-\varepsilon/k.$$

When $C_i$ becomes large $Q_i$ converges to the upper bound given by the right hand member of (5.1).

When $H_2$ is true and $\beta_j$ has slipped to the left, i.e. $\beta_j - c_j\beta$, $0 \leqq c_j < 1$ the following limits can be derived for $P_j$, the probability of making the correct decision in this case.

$$(5.4)\qquad \{I_{b_j}(\alpha_j, A-\alpha_j)\}(1-\varepsilon) \leqq P_j \leqq I_{b_j}(\alpha_j, A-\alpha_j),$$

where

$$(5.5)\qquad b_j = \frac{g_{j,\varepsilon}}{c_j+(1-c_j)g_{j,\varepsilon}}$$

and $g_{j,\varepsilon}$ is determined from

$$(5.6)\qquad I_{g_{j,\varepsilon}}(\alpha_j, A-\alpha_j) = \varepsilon/k.$$

Again for small values of $c_j$

$$P_j \approx I_{b_j}(\alpha_j, A-\alpha_j).$$

In order to prove (5.1) we may assume without loss of generality that $i=1$ and then we put $\mathbf{u}_1/C_1 = \mathbf{v}_1$, thus $\mathbf{v}_1$ has a gamma distribution with parameters $\alpha_1$ and $\beta$. The probability $Q_1$ of making the correct decision is

$$Q_1 = P[\mathbf{d}_1 = \min \mathbf{d}_j \text{ and } \mathbf{d}_1 < \varepsilon/k]$$
$$\geqq P[\mathbf{d}_1 < \varepsilon/k \text{ and } \mathbf{d}_2 > \varepsilon/k \ldots \text{ and } \mathbf{d}_k > \varepsilon/k]$$
$$= P[\mathbf{d}_1 < \varepsilon/k] - P[(\mathbf{d}_1 < \varepsilon/k \text{ and } \mathbf{d}_2 < \varepsilon/k); \ldots; \text{ or } (\mathbf{d}_1 < \varepsilon/k \text{ and } \mathbf{d}_k < \varepsilon/k)].$$

Thus the following inequality holds

$$(5.7) \quad P[\mathbf{d}_1 < \varepsilon/k] - \sum_{j=2}^{k} P[\mathbf{d}_1 < \varepsilon/k \text{ and } \mathbf{d}_j < \varepsilon/k] \leqq Q_1 \leqq P[\mathbf{d}_1 < \varepsilon/k].$$

We have

$$P[\mathbf{d}_1 < \varepsilon/k] = P[\mathbf{x}_1 > G_{1,\varepsilon}]$$
$$= P\left[\frac{C_1 \mathbf{v}_1}{\mathbf{v}_1 + \mathbf{u}_2 + \ldots + \mathbf{u}_k + (C_1 - 1)\mathbf{v}_1} > G_{1,\varepsilon}\right]$$
$$= P\left[\frac{\mathbf{v}_1}{\mathbf{v}_1 + \mathbf{u}_2 + \ldots + \mathbf{u}_k} > \frac{G_{1,\varepsilon}}{C_1 - (C_1 - 1)G_{1,\varepsilon}}\right]$$
$$= P\left[\frac{\mathbf{v}_1}{\mathbf{v}_1 + \mathbf{u}_2 + \ldots + \mathbf{u}_k} > B_1\right] \quad \text{(cf. (5.2))}.$$

The distribution of $\mathbf{v}_1/(\mathbf{v}_1 + \mathbf{u}_2 + \ldots + \mathbf{u}_k)$ is the distribution of $\mathbf{x}_1$ under $H_0$ and therefore known. In fact

$$(5.8) \quad P[\mathbf{d}_1 < \varepsilon/k] = 1 - I_{B_1}(\alpha_1, A - \alpha_1).$$

Further we have

$$(5.9) \quad \begin{cases} P[\mathbf{d}_1 < \varepsilon/k \text{ and } \mathbf{d}_j < \varepsilon/k] \\ = P\left[\frac{\mathbf{v}_1}{\mathbf{v}_1 + \mathbf{u}_2 + \ldots + \mathbf{u}_k} > B_1 \text{ and } \frac{\mathbf{u}_j}{\mathbf{v}_1 + \ldots + \mathbf{u}_k + (C_1 - 1)\mathbf{v}_1} > G_{1,\varepsilon}\right] \\ \leqq P\left[\frac{\mathbf{v}_1}{\mathbf{v}_1 + \mathbf{u}_2 + \ldots + \mathbf{u}_k} > B_1 \text{ and } \frac{\mathbf{u}_j}{\mathbf{v}_1 + \ldots + \mathbf{u}_k} > G_{1,\varepsilon}\right] \leqq \\ \text{(according to (4.11))} \\ \leqq P\left[\frac{\mathbf{v}_1}{\mathbf{v}_1 + \mathbf{u}_2 + \ldots + \mathbf{u}_k} > B_1\right] \cdot P\left[\frac{\mathbf{u}_j}{\mathbf{v}_1 + \ldots + \mathbf{u}_k} > G_{j,\varepsilon}\right] \\ = [1 - I_{B_1}(\alpha_1, A - \alpha_1)]\,\varepsilon/k. \end{cases}$$

Substituting (5.8) and (5.9) into (5.7) we get

$$(5.10) \quad \begin{cases} [1 - I_{B_1}(\alpha_1, A - \alpha_1)]\,(1 - \varepsilon) \leqq [1 - I_{B_1}(\alpha_1, A - \alpha_1)]\,(1 - (k-1)\,\varepsilon/k) \\ \leqq Q_1 \leqq [1 - I_{B_1}(\alpha_1, A - \alpha_1)], \end{cases}$$

which proves (5.1). When $C_1$ is large $P[\mathbf{d}_j < \varepsilon/k]$ will for $j \neq 1$ be much smaller than $\varepsilon/k$ and therefore in that case $Q_1$ converges to its upper bound.

The inequalities (5.4) can be derived in the same way.

## 6. *Tables and nomograms*

To obtain the values $d_j$ and $e_j$ as defined by (2.7) and (2.9) and to evaluate the power functions (5.4) and (5.5) we need suitable tables or nomograms of the incomplete $B$ function.

When all $\alpha_i$ are equal, the smallest $d_j'$ corresponds to the largest ratio $x_j$ and the smallest $e_i$ corresponds to the smallest ratio $x_i$. Further the critical values $G_{i,\varepsilon}$ of $x_i$ when testing the largest ratio and $g_{i,\varepsilon}$ for testing the smallest ratio are then all equal:

$$G_{i,\varepsilon} = G_\varepsilon \quad (i = 1, \ldots, k),$$
$$g_{i,\varepsilon} = g_\varepsilon \quad (i = 1, \ldots, k).$$

Therefore in this case it suffices to have tables with these critical values with entries $k$ and the common parameter value $\alpha$. These tables may be found in C. Eisenhart, M. W. Hastay and W. A. Wallis (1947) ($\varepsilon = 0.05$ and 0.01) for the first test and in R. Doornbos (1956) ($\varepsilon = 0.05$) for the second one.

When unequal values among the $\alpha_i$ occur the minimum $d$ value may be found in most cases by means of Pearson's tables of the incomplete function (K. Pearson (1934)).

The smallest $e$ value, however, will, when it lies in the neighbourhood of $\varepsilon/k$ and $k$ is not very small, correspond to such a small ratio $x$ that Pearson's tables are not suitable for our purpose. In this case and also if the parametervalues $A - \alpha_j$ are larger than 50, the nomograms of H. O. Hartley and E. R. Fitch (1951) may be used to obtain an approximation to the $e$-values.

To demonstrate the use of these charts we consider the following, fictitious, example. Suppose we have a group of ten machines turning out the same product, and we measure some property $t$ on each specimen. Suppose we are interested in finding out whether one of the machines produces the product more regularly than the other ones do.

From the $i$th machine we have $n_i$ observations $t_{ij}$ $(j = 1, \ldots, n_i)$. For each machine the sum $u_i$ of squared deviations from the mean value is computed:

$$u_i = \sum_{j=1}^{n_i} (t_{ij} - t_{i.})^2,$$

where

$$t_{i.} = \frac{1}{n_i} \sum_{j=1}^{n_i} t_{ij}.$$

Assuming that the $t_{ij}$ are independent observations from a normal distribution $N(\mu_i, \sigma_i)$, $u_i$ is an observation from a gamma distribution with parameters $\alpha_i = (n_i - 1)/2$ and $\beta_i = 2\sigma_i^2$.

Fictitions $u$-values are given in table 6.1.

Clearly machine no 5 will give the smallest $e$ value. So we must evaluate

$$e = I_{0.028}\,(7{,}63) = 1 - I_{0.972}\,(63{,}7).$$

From the chart we find that $I_{0.972}$ (63,7) $\approx$ 0.996. Thus $e$ is smaller than 0.05/10 and therefore we can reject the hypothesis that all $\sigma_i$ are equal

TABLE 6.1

| Machine | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | total |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Number of observations | 10 | 15 | 21 | 23 | 15 | 11 | 31 | 15 | 3 | 6 | 150 |
| $\alpha_i$ | 4.5 | 7 | 10 | 11 | 7 | 5 | 15 | 7 | 1 | 2.5 | 70 |
| $u_i$ | 45.9 | 109.6 | 112.8 | 142.0 | 25.7 | 123.0 | 182.0 | 106.4 | 12.8 | 46.5 | 906.7 |
| $x = \frac{u_j}{\Sigma u_j}$ | 0.051 | 0.121 | 0.124 | 0.157 | 0.028 | 0.136 | 0.201 | 0.117 | 0.014 | 0.051 | 1.000 |

at the level of significance 0.05 and it may be concluded that the fifth machine is the most accurate one.

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[1] This chart has also been included in Biometrika Tables for Statisticians, Vol. I (Cambridge 1954).