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MATHEMATICS

MAXIMUM LIKELIHOOD ESTIMATION OF ORDERED PROBABILITIES [1]

BY

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1. *Introduction*

The problem treated in this paper concerns the maximum likelihood estimation of partially or completely ordered probabilities.

Consider k independent series of independent trials, each trial resulting in a success or a failure. The i-th series consists of n_i trials with $\boldsymbol{a}_i$ [2]) successes and $\boldsymbol{b}_i = n_i - \boldsymbol{a}_i$ failures; π_i is the (unknown) probability of a success for each trial of the i-th series ($i=1, 2, \ldots, k$) and $\pi_1, \pi_2, \ldots, \pi_k$ satisfy the inequalities

$$\alpha_{i,j}(\pi_i - \pi_j) \leqq 0 \qquad (i, j = 1, 2, \ldots, k), \tag{1.1}$$

where

$$\left\{\begin{array}{ll} 1. & \alpha_{i,j} = -\alpha_{j,i}, \\ 2. & \alpha_{i,j} = 0 \text{ for } m_0 \text{ pairs of values } (i, j) \text{ with } i < j, \\ 3. & \alpha_{i,j} = 1 \text{ for } m_1 \text{ pairs of values } (i, j) \text{ with } i < j, \end{array}\right. \tag{1.2}$$

$$\left\{\begin{array}{l} m_0 + m_1 = \binom{k}{2}, \\ m_1 \geqq 1 \end{array}\right. \tag{1.3}$$

and, if $i<h<j$ then

$$\alpha_{i,j} = 1 \text{ if } \alpha_{i,h} = \alpha_{h,j} = 1 \text{ (transitivity).} \tag{1.4}$$

In section 2 and 3 methods will be described by means of which the maximum likelihood estimates of $\pi_1, \pi_2, \ldots, \pi_k$ may be found, i.e. the values of $x_1, x_2, \ldots, x_k$ which maximize

$$L = L(x_1, x_2, \ldots, x_k) \stackrel{def}{=} \sum_{i=1}^{k} \{a_i \lg x_i + b_i \lg (1 - x_i)\} \tag{1.5}$$

in the domain

$$D: \left\{\begin{array}{l} \alpha_{i,j}(x_i - x_j) \leqq 0, \\ 0 \leqq x_i \leqq 1 \end{array}\right. \qquad (i, j = 1, 2, \ldots, k). \tag{1.6}$$

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[1]) Report SP 50 of the Statistical Department of the Mathematical Centre.

[2]) Random variables will be distinguished from numbers (e.g. from the value they take in an experiment) by printing their symbols in bold type.

Unless explicitely stated otherwise L will only be considered in this domain D; the maximum likelihood estimates will throughout this paper be denoted by $p_1, p_2, \ldots, p_k$, the point in D where L assumes its maximum.

Let further

$$L_i = L_i(x_i) \overset{\text{def}}{=} a_i \lg x_i + b_i \lg(1-x_i) \quad (i = 1, 2, \ldots, k) \tag{1.7}$$

and

$$f_i = \frac{a_i}{n_i} \quad (i = 1, 2, \ldots, k). \tag{1.8}$$

Further the restrictions $\pi_i \leqq \pi_i$ (i.e. $\alpha_{i,j} = 1$) satisfying

$$\alpha_{i,h}\,\alpha_{h,j} = 0 \text{ for each } h \text{ between } i \text{ and } j \tag{1.9}$$

will be denoted by $R_1, R_2, \ldots, R_s$. Each R_λ thus corresponds with one pair (i, j), i.e. R_λ denotes the relation $\pi_i \leqq \pi_j$, satisfying (1.9); this pair (i, j) will be denoted by (i_λ, j_λ). Because of the transitivity relations (1.4) the system $R_1, R_2, \ldots, R_s$ is equivalent to (1.1).

In section 4 some examples will be given.

Remarks:

1. Every set of restrictions $R_1, R_2, \ldots, R_s$ represents a convex domain.
2. It will be clear that partially or completely ordered probabilities can always be numbered in such a way that they satisfy (1.1).
3. If $m_0 = 0$ then (1.1) is equivalent to

$$\pi_1 \leqq \pi_2 \leqq \ldots \leqq \pi_k. \tag{1.10}$$

This special case has been solved independently by Miriam Ayer, H. D. Brunk, G. M. Ewing, W. T. Reid, Edward Silverman [2] and the present author [1].

2. *The maximum likelihood estimates of $\pi_1, \pi_2, \ldots, \pi_k$*

In this section the following theorem will be proved.

Theorem I: *L possesses a unique maximum and if $p'_1, p'_2, \ldots, p'_k$ are the maximum likelihood estimates of $\pi_1, \pi_2, \ldots, \pi_k$ under the restrictions $R_1, \ldots, R_{\lambda-1}, R_{\lambda+1}, \ldots, R_s$ then the maximum likelihood estimates $p_1, p_2, \ldots, p_k$ under the restrictions $R_1, R_2, \ldots, R_s$ satisfy the relations*

$$\begin{cases} 1. \quad p_i = p'_i \quad (i = 1, 2, \ldots, k) \text{ if } p'_{i_\lambda} \leqq p'_{j_\lambda}, \\ 2. \quad p_{i_\lambda} = p_{j_\lambda} \text{ if } p'_{i_\lambda} > p'_{j_\lambda}. \end{cases} \tag{2.1}$$

Proof: The R_λ have not been arranged in a special order. Therefore we may, without any loss of generality, take $\lambda = s$.

The uniqueness of the maximum of L will be proved by induction. If $s = 0$ then it is well known that L possesses for any k a unique maximum and

$$p_i = f_i \quad (i = 1, 2, \ldots, k). \tag{2.2}$$

Further $s = 0$ if $k = 1$; therefore L possesses a unique maximum and (2.2) holds for $k=1$. Now suppose that it has been proved that L possesses a unique maximum for the following two cases:

$$(2.3)\qquad \begin{cases} 1. & k \text{ series of trials with } s-1 \text{ restrictions} \\ 2. & k-1 \text{ series of trials with } s-1 \text{ or less restrictions,} \end{cases}$$

where $k \geqq 2$, $s \geqq 1$ and consider a case with k series of trials and s restrictions. Then it follows from (2.3.1) that L possesses a unique maximum under the restrictions $R_1, R_2, \ldots, R_{s-1}$ i.e. there exists exactly one point $p_1', p_2', \ldots, p_k'$ satisfying these restrictions and maximizing L.

Now the following two cases may be distinguished.

1. $p_{i_s}' \leqq p_{j_s}'$; then $p_1', p_2', \ldots, p_k'$ satisfy all restrictions $R_1, R_2, \ldots, R_s$. Therefore in this case L possesses a unique maximum under the restrictions $R_1, R_2, \ldots, R_s$ and

$$(2.4)\qquad p_i = p_i' \quad (i = 1, 2, \ldots, k).$$

2. $p_{i_s}' > p_{j_s}'$; then (2.1.2) may be proved as follows. Consider a fixed point $x_1, x_2, \ldots, x_k$ in D with $x_{i_s} < x_{j_s}$. It will be clear that such points exist. Then if

$$(2.5)\qquad \begin{cases} X_i(\beta) \overset{def}{=} (1-\beta)\, x_i + \beta\, p_i' & (i = 1, 2, \ldots, k), \\ 0 \leqq \beta \leqq 1, \end{cases}$$

we have

$$(2.6)\qquad X_i(0) = x_i, \;\; X_i(1) = p_i' \quad (i = 1, 2, \ldots, k)$$

and for each β with $0 \leqq \beta \leqq 1$, $X_1(\beta), X_2(\beta), \ldots, X_k(\beta)$ is a point satisfying the restrictions $R_1, R_2, \ldots, R_{s-1}$, but not necessarily R_s (cf. remark 1 in the foregoing section). Therefore if

$$(2.7)\qquad \beta_0 \overset{def}{=} \frac{x_{j_s} - x_{i_s}}{x_{j_s} - x_{i_s} + p_{i_s}' - p_{j_s}'}$$

then

$$(2.8)\qquad \begin{cases} 1. & 0 < \beta_0 < 1, \\ 2. & X_{i_s}(\beta_0) = X_{j_s}(\beta_0), \end{cases}$$

i.e. $X_1(\beta_0), X_2(\beta_0), \ldots, X_k(\beta_0)$ is a point satisfying the restrictions

$$R_1, R_2, \ldots, R_s,$$

i.e. a point in D.

Further $L\{X_1(\beta), X_2(\beta), \ldots, X_k(\beta)\}$ is for fixed values of $x_1, x_2, \ldots, x_k$ a function of β, say $g(\beta)$, and from (1.5) and (2.5) it follows that

$$(2.9)\qquad \frac{d^2 g(\beta)}{d\beta^2} = \sum_{i=1}^{k} (p_i' - x_i)^2 \, \frac{-n_i\{X_i(\beta)\}^2 + 2a_i X_i(\beta) - a_i}{\{X_i(\beta)\}^2\{1 - X_i(\beta)\}^2}.$$

Further we have

$$(2.10)\qquad -n_i\{X_i(\beta)\}^2 + 2a_i X_i(\beta) - a_i \begin{cases} < 0 \text{ for all } \beta \text{ if } 0 < a_i < n_i, \\ = -n_i\{X_i(\beta)\}^2 \text{ if } a_i = 0, \\ = -n_i\{1 - X_i(\beta)\}^2 \text{ if } a_i = n_i \end{cases}$$

and from (2.9) and (2.10) it follows, $p_i' - x_i$ being $\neq 0$ for at least one value of i (viz. for $i = i_s$ or/and $i = j_s$), that

$$\frac{d^2 g(\beta)}{d\beta^2} < 0. \tag{2.11}$$

Further we have

$$L\{p_1', p_2', \ldots, p_k'\} > L\{X_1(\beta), X_2(\beta), \ldots, X_k(\beta)\} \tag{2.12}$$

for each β with $0 \leqq \beta < 1$ and from (2.6), (2.11) and (2.12) it follows that $g(\beta)$ is an increasing function of β in the interval $0 \leqq \beta \leqq 1$.

Thus for each point $x_1, x_2, \ldots, x_k$ in D with $x_{i_s} < x_{j_s}$, D contains a point $X_1, X_2, \ldots, X_k$ with

$$\begin{cases} 1. & X_{i_s} = X_{j_s}, \\ 2. & L(X_1, X_2, \ldots, X_k) > L(x_1, x_2, \ldots, x_k), \end{cases} \tag{2.13}$$

i.e. $L(x_1, x_2, \ldots, x_k)$ attains its maximum under the restrictions $R_1, R_2, \ldots, R_s$ for $x_{i_s} = x_{j_s}$. Substituting this into (1.5) the two terms with $i = i_s$ and $i = j_s$ are reduced to one term of the form

$$(a_{i_s} + a_{j_s}) \lg x_{i_s} + (b_{i_s} + b_{j_s}) \lg (1 - x_{i_s}).$$

This means that the two series of trials in question are to be pooled. The uniqueness of the maximum of L under these restrictions then follows from (2.3.2).

By applying theorem I repeatedly and using the well known solution of the problem for the case that $s = 0$ a solution may be obtained. This may, however, lead to a rather complicated procedure, which can often be simplified by applying the special theorems mentioned in the following section.

3. *Some special theorems*

Theorem II: *If $\alpha_{i,j}(f_i - f_j) \leqq 0$ for each pair of values (i, j) then*

$$p_i = f_i \quad (i = 1, 2, \ldots, k). \tag{3.1}$$

Proof: This follows immediately from the fact that in this case the maximum of L in D coincides with the maximum of L in the domain: $0 \leqq x_i \leqq 1$ $(i = 1, 2, \ldots, k)$. The theorem also follows from theorem I.

The following theorem will be immediately clear.

Theorem III: *If $i_1, i_2, \ldots, i_\nu$ is a set of values satisfying*

$$\alpha_{i,i_1} = \alpha_{i,i_2} = \ldots = \alpha_{i,i_\nu} = 0 \quad \text{for each } i \neq i_1, i_2, \ldots, i_\nu \tag{3.2}$$

then the maximum likelihood estimates of $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_\nu}$ are those values of $x_{i_1}, x_{i_2}, \ldots, x_{i_\nu}$ which maximize $L_{i_1} + L_{i_2} + \ldots + L_{i_\nu}$ in the domain

$$D': \begin{cases} \alpha_{i_h, i_{h'}} (x_{i_h} - x_{i_{h'}}) \leqq 0, \\ 0 \leqq x_{i_h} \leqq 1 \end{cases} \qquad (h, h' = 1, 2, \ldots, \nu). \tag{3.3}$$

For the proof of the theorems IV and V we need the following lemma.

Lemma I: *If* $x_1, x_2, \ldots, x_k$ *is any point in* D *such that for some pair of values* (i, j)

$$(3.4)\quad \begin{cases} 1. & x_i < x_j, \\ 2. & f_i \geqq f_j \\ 3. & \alpha_{h,i} \geqq \alpha_{h,j} \quad \text{for each } h < j, \\ 4. & \alpha_{i,h} \leqq \alpha_{j,h} \quad \text{for each } h > i, \end{cases}$$

then a number x *exists satisfying*

$$(3.5)\quad \begin{cases} 1. & x_1, x_2, \ldots, x_k \text{ is also a point in } D \text{ if } x \text{ is substituted for } x_i \text{ and } x_j, \\ 2. & L_i(x) + L_j(x) > L_i(x_i) + L_j(x_j). \end{cases}$$

Proof:

The following cases may be distinguished

$$(3.6)\quad x_i < x_j \leqq f_i; \text{ then take } x = x_j,$$

$$(3.7)\quad f_i \leqq x_i < x_j; \text{ then take } x = x_i,$$

$$(3.8)\quad x_i < f_i < x_j; \text{ then take } x = f_i.$$

It may easily be proved that this number x satisfies (3.5.2). For (3.6) e.g. we have

$$(3.9)\quad L_j(x) = L_j(x_j)$$

and

$$(3.10)\quad x_i < x \leqq f_i.$$

From (3.10) follows

$$(3.11)\quad L_i(x) > L_i(x_i)$$

and (3.5.2) follows from (3.9) and (3.11).

For the cases (3.7) and (3.8) it may be proved in a similar way by means of (3.4.2) that L satisfies (3.5.2).

In order to prove that this number x satisfies (3.5.1) it is sufficient to prove that

$$(3.12)\quad \begin{cases} 1. & \alpha_{h,i}(x_h - x) \leqq 0 \quad \text{for each } h < i, \\ 2. & \alpha_{i,h}(x - x_h) \leqq 0 \quad \text{for each } h > i, \\ 3. & \alpha_{h,j}(x_h - x) \leqq 0 \quad \text{for each } h < j, \\ 4. & \alpha_{j,h}(x - x_h) \leqq 0 \quad \text{for each } h > j. \end{cases}$$

From the fact that x satisfies

$$(3.13)\quad x_i \leqq x \leqq x_j$$

and the fact that $x_1, x_2, \ldots, x_k$ is a point in D it follows that

$$(3.14)\quad \begin{cases} 1. & \alpha_{h,i}(x_h - x) \leqq \alpha_{h,i}(x_h - x_i) \leqq 0 \text{ for each } h < i, \\ 2. & \alpha_{j,h}(x - x_h) \leqq \alpha_{j,h}(x_j - x_h) \leqq 0 \text{ for each } h > j. \end{cases}$$

Further it follows from (3.4.3) and (3.4.4) that

$$(3.15)\quad \begin{cases} 1. & \alpha_{i,h}(x-x_h) = \alpha_{j,h}(x-x_h) \text{ for each } h > i \text{ with } \alpha_{i,h} = 1, \\ 2. & \alpha_{h,j}(x_h-x) = \alpha_{h,i}(x_h-x) \text{ for each } h < j \text{ with } \alpha_{h,j} = 1 \end{cases}$$

and (3.12) follows from (3.14) and (3.15).

Theorem IV: *If for some pair of values* (i, j) *with* $i<j$

$$(3.16)\qquad \alpha_{i,j}(f_i-f_j) > 0$$

and

$$(3.17)\qquad \begin{cases} 1. & \alpha_{i,h} = \alpha_{h,j} = 0 \text{ for each } h \text{ between } i \text{ and } j, \\ 2. & \alpha_{h,i} = \alpha_{h,j} \quad \text{for each } h < i, \\ 3. & \alpha_{i,h} = \alpha_{j,h} \quad \text{for each } h > j, \end{cases}$$

then

$$(3.18)\qquad p_i = p_j.$$

Proof: From (3.16) and (3.17) it follows that

$$(3.19)\qquad \begin{cases} 1. & f_i > f_j, \\ 2. & \alpha_{h,i} = \alpha_{h,j} \text{ for each } h < j, \\ 3. & \alpha_{i,h} = \alpha_{j,h} \text{ for each } h > i. \end{cases}$$

Now suppose that $x_1, x_2, \ldots, x_k$ is a point in D with

$$(3.20)\qquad x_i < x_j.$$

From lemma I, (3.19) and (3.20) it follows then that a number x exists such that $x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_k$ is a point in D and

$$(3.21)\qquad L_i(x)+L_j(x) > L_i(x_i)+L_j(x_j).$$

Thus for each point $x_1, x_2, \ldots, x_k$ in D with $x_i < x_j$ a point $x'_1, x'_2, \ldots, x'_k$ in D exists with

$$(3.22)\qquad \begin{cases} 1. & x'_i = x'_j, \\ 2. & L(x'_1, x'_2, \ldots, x'_k) > L(x_1, x_2, \ldots, x_k), \end{cases}$$

i.e. L attains its maximum for $x_i = x_j$ and (3.18) then follows from the uniqueness of this maximum.

Remarks:

4. This theorem is also related to theorem I. If R_λ represents the restriction $\pi_i \leqq \pi_j$ it follows from (3.22) that L attains its maximum under the restrictions $R_1, \ldots, R_{\lambda-1}, R_{\lambda+1}, \ldots, R_s$ for $x_i \geqq x_j$, giving $p'_i \geqq p'_j$; from (2.1) then follows: $p_i = p_j$.

5. If $m_0 = 0$, i.e. if the probabilities $\pi_1, \pi_2, \ldots, \pi_k$ satisfy the inequalities

$$(3.23)\qquad \pi_1 \leqq \pi_2 \leqq \ldots \leqq \pi_k,$$

then each pair of values (i, j) with $j = i+1$ satisfies (3.17). Therefore we have in this case

$$(3.24)\qquad p_i = p_{i+1} \text{ for each } i \text{ with } f_i > f_{i+1}.$$

From theorem IV it follows that if there is a pair of values (i, j) satisfying (3.16) and (3.17) then the problem may be reduced to the case of $k-1$ series of trials with $s-1$ (or less) restrictions by substituting $x_i = x_j$ into $L(x_1, x_2, \ldots, x_k)$, i.e. by pooling the i-th and j-th series of trials.

Theorem V: *If (i, j) is a pair of values satisfying*

$$f_i \leqq f_j \tag{3.25}$$

and

$$\begin{cases} 1. & \alpha_{i,j} = 0, \\ 2. & \alpha_{h,i} \leqq \alpha_{h,j} \text{ for each } h < i, \\ 3. & \alpha_{i,h} \geqq \alpha_{j,h} \text{ for each } h > j, \end{cases} \tag{3.26}$$

then

$$p_i \leqq p_j. \tag{3.27}$$

Proof: Suppose $x_1, x_2, \ldots, x_k$ is a point in D with

$$x_i > x_j. \tag{3.28}$$

From lemma I, (3.25), (3.26) and (3.28) it follows then in the same way as in theorem IV that for each point $x_1, x_2, \ldots, x_k$ in D with $x_i > x_j$ a point $x_1', x_2', \ldots, x_k'$ in D exists with

$$\begin{cases} 1. & x_i' = x_j', \\ 2. & L(x_1', x_2', \ldots, x_k') > L(x_1, x_2, \ldots, x_k). \end{cases} \tag{3.29}$$

Now D also contains points with $x_i < x_j$ and therefore L attains its maximum for $x_i \leqq x_j$; (3.27) then follows from the uniqueness of the maximum.

By means of theorem V a new restriction may be introduced. This is sometimes useful as may be seen from example 2 of section 4.

Theorem VI: *If (i, j) is a pair of values with*

$$\alpha_{i,j} = 0, \tag{3.30}$$

if D' is the sub-domain of D where $x_i \leqq x_j$ and if $p_1', p_2', \ldots, p_k'$ is the point where L assumes its maximum in D' then

$$\begin{cases} 1. & p_1 = p_1', p_2 = p_2', \ldots, p_k = p_k' \text{ if } p_i' < p_j', \\ 2. & p_i \geqq p_j \text{ if } p_i' = p_j'. \end{cases} \tag{3.31}$$

Proof:

First consider the case that $p_i' < p_j'$. For this case it may be proved that $p_i < p_j$, by showing that $p_i \geqq p_j$ leads to a contradiction. Because if $p_i \geqq p_j$ then it may be proved in the same way as in theorem I that L attains its maximum in D' for $x_i = x_j$, i.e. then we have $p_i' = p_j'$. Thus $p_i < p_j$ if $p_i' < p_j'$ and from the uniqueness of the maximum of L then follows

$$p_1 = p_1', p_2 = p_2', \ldots, p_k = p_k' \text{ if } p_i' < p_j'. \tag{3.32}$$

Now consider the case that $p_i' = p_j'$, i.e. the case that L attains its maximum in D' for $x_i = x_j$; then L attains its maximum in D for $x_i \geqq x_j$, i.e. $p_i \geqq p_j$.

Thus if the maximum likelihood estimates of $\pi_1, \pi_2, \ldots, \pi_k$ in D' are known then the problem is solved by means of theorem VI if $p_i' < p_j'$ (cf. (3.31.1)) or the new restriction $\pi_j \leqq \pi_i$ may be introduced if $p_i' = p_j'$ (cf. (3.31.2)).

Remark:

6. Theorem V is related to theorem VI. Taking in theorem V for D' the subdomain of D where $x_j \leqq x_i$, it follows from (3.29) that $p_j' = p_i'$. From (3.31.2) then follows: $p_i \leqq p_j$.

In general it is much more simple to apply the theorems II–V than I or VI. In some situations however II–V are not applicable and then I or VI have to be used. This will be illustrated in the examples of the following section.

4. *Examples*

Example 1 (complete ordering; theorems II and IV):

Suppose $k=4$, $m_0=0$ $(\pi_1 \leqq \pi_2 \leqq \pi_3 \leqq \pi_4)$ and

$$(4.1)\qquad \left\{\begin{array}{lllll} i & 1 & 2 & 3 & 4 \\ a_i & 4 & 3 & 10 & 8 \\ n_i & 10 & 5 & 30 & 15 \\ f_i & 0{,}4 & 0{,}6 & 0{,}33 & 0{,}53. \end{array}\right.$$

From (4.1) and (3.24) it follows that

$$(4.2)\qquad p_2 = p_3$$

and the problem is reduced to the case of $k-1=3$ series of trials by pooling the second and third series of trials:

$$(4.3)\qquad \left\{\begin{array}{llll} i & 1 & 2(+3) & 4 \\ a_i' & 4 & 13 & 8 \\ n_i' & 10 & 35 & 15 \\ f_i' & 0{,}4 & 0{,}37 & 0{,}53. \end{array}\right.$$

From (4.3) and (3.24) it then follows that

$$(4.4)\qquad p_1 = p_2$$

and the problem is reduced to the case of $k-2=2$ series of trials with

$$(4.5)\qquad \left\{\begin{array}{lll} i & 1(+2+3) & 4 \\ a_i'' & 17 & 8 \\ n_i'' & 45 & 15 \\ f_i'' & 0{,}38 & 0{,}53. \end{array}\right.$$

From (4.5), (4.2), (4.4) and theorem II then follows

(4.6) $$p_1 = p_2 = p_3 = 0{,}38, \quad p_4 = 0{,}53.$$

Example 2 (incomplete ordering; theorems II, IV and V):
Suppose $k=5$, $m_1=6$, $m_0=4$

(4.7) $$\begin{cases} i & 1 & 2 & 3 & 4 & 5 \\ a_i & 7 & 13 & 15 & 2 & 12 \\ n_i & 10 & 20 & 30 & 5 & 15 \\ f_i & 0{,}7 & 0{,}65 & 0{,}5 & 0{,}4 & 0{,}8 \end{cases}$$

and

(4.8) $$\alpha_{1,2} = \alpha_{1,3} = \alpha_{2,4} = \alpha_{3,5} = 1.$$

Then the pair of values $i=2$, $j=4$ satisfies (3.16) and (3.17). Therefore we have

(4.9) $$p_2 = p_4$$

and the problem is reduced to the case of $k-1=4$ series of trials with $m_1'=4$, $m_0'=2$,

(4.10) $$\begin{cases} i & 1 & 2\,(+4) & 3 & 5 \\ a_i' & 7 & 15 & 15 & 12 \\ n_i' & 10 & 25 & 30 & 15 \\ f_i' & 0{,}7 & 0{,}6 & 0.5 & 0{,}8 \end{cases}$$

and

(4.11) $$\alpha'_{1,2} = \alpha'_{1,3} = \alpha'_{3,5} = 1.$$

For these 4 series of trials the pair $i=3$, $j=2$ and the pair $i=2$, $j=5$ satisfy (3.25) and (3.26). From theorem V then follows that L attains its maximum for

(4.12) $$x_1 \leqq x_3 \leqq x_2 \leqq x_5$$

and from (4.9), (4.10) and (4.12) follows

(4.13) $$p_1 = p_3 = 0{,}55, \; p_2 = p_4 = 0{,}6, \; p_5 = 0{,}8.$$

Example 3 (incomplete ordering; theorem I or VI):
Suppose $k=4$, $m_0=m_1=3$,

(4.14) $$\begin{cases} i & 1 & 2 & 3 & 4 \\ a_i & 7 & 18 & 13 & 10 \\ n_i & 10 & 30 & 20 & 25 \\ f_i & 0{,}7 & 0{,}6 & 0{,}65 & 0{,}4 \end{cases}$$

and

(4.15) $$\alpha_{1,2} = \alpha_{1,4} = \alpha_{3,4} = 1.$$

For this case the theorems II–V cannot be applied and we use therefore theorem I.

Take $i_\lambda=1$ and $j_\lambda=1$ (i.e. omit the restriction $\pi_1\leqq\pi_4$), then p_1', p_2', p_3', p_4' are those values of x_1, x_2, x_3, x_4 which maximize L in the domain

$$\left\{\begin{array}{l} x_1 \leqq x_2,\ x_3 \leqq x_4, \\ 0 \leqq x_i \leqq 1 \quad (i = 1, 2, 3, 4). \end{array}\right. \tag{4.16}$$

From theorem III and IV then follows

$$p_1' = p_2' = 0{,}63,\ p_3' = p_4' = 0{,}51 \tag{4.17}$$

and from theorem I and (4.17) (cf. (2.1.2))

$$p_1=p_4. \tag{4.18}$$

In this way the problem is reduced to the case of $k-1=3$ series of trials with

$$\left\{\begin{array}{llll} i & 3 & 1\,(+4) & 2 \\ a_i' & 13 & 17 & 18 \\ n_i' & 20 & 35 & 30 \\ f_i' & 0{,}65 & 0{,}49 & 0{,}6 \end{array}\right. \tag{4.19}$$

and

$$\alpha_{3,1}' = \alpha_{1,2}' = 1. \tag{4.20}$$

From (4.18), (4.19) and (4.20) follows

$$p_1=p_3=p_4=0{,}55,\ p_2=0{,}6. \tag{4.21}$$

This problem may also be solved by means of theorem VI as follows. If we take $i=2$, $j=3$ then D' is the domain

$$\left\{\begin{array}{l} x_1 \leqq x_2 \leqq x_3 \leqq x_4, \\ 0 \leqq x_i \leqq 1 \quad (i = 1, 2, 3, 4). \end{array}\right. \tag{4.22}$$

The estimates p_1', p_2', p_3', p_4' then follow from (3.24):

$$p_1' = p_2' = p_3' = p_4' = 0{,}56. \tag{4.23}$$

From theorem VI and (4.23) then follows (cf. (3.31.2))

$$p_2\geqq p_3. \tag{4.24}$$

Introducing the restriction $\pi_3\leqq\pi_2$ the problem is reduced to the case of 4 series of trials with (4.14) and

$$\alpha_{1,2}' = \alpha_{1,4}' = \alpha_{3,2}' = \alpha_{3,4}' = 1. \tag{4.25}$$

Then the pair $i=3$, $j=1$ and the pair $i=4$, $j=2$ satisfy (3.25) and (3.26). From theorem V then follows that L attains its maximum for

$$x_3\leqq x_1\leqq x_4\leqq x_2 \tag{4.26}$$

and from (4.14), (4.26) and (3.24) then follows

$$p_1=p_3=p_4=0{,}55,\ p_2=0{,}6. \tag{4.27}$$

Example 4 (incomplete ordering; theorem VI):

Suppose $k=8$, $m_0=13$, $m_1=15$,

$$(4.22)\qquad \begin{cases} i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ a_i & 8 & 22 & 13 & 25 & 20 & 21 & 32 & 2 \\ n_i & 10 & 40 & 20 & 50 & 30 & 50 & 50 & 5 \\ f_i & 0,8 & 0,55 & 0,65 & 0,5 & 0,67 & 0,42 & 0,64 & 0,4 \end{cases}$$

and

$$(4,23)\qquad \alpha_{1,2} = \alpha_{1,4} = \alpha_{3,4} = \alpha_{4,5} = \alpha_{5,6} = \alpha_{5,8} = \alpha_{7,8} = 1.$$

For this case the theorems II—V cannot be applied and we use therefore theorem VI.

Taking for (i, j) the pair $i=6$, $j=7$ the estimates $p_1', p_2', \ldots, p_8'$ are the values of $x_1, x_2, \ldots, x_8$ which maximize L in the domain

$$(4.24)\qquad D': \begin{cases} x_1 \leqq x_2,\ x_1 \leqq x_4,\ x_3 \leqq x_4 \leqq x_5 \leqq x_6 \leqq x_7 \leqq x_8, \\ 0 \leqq x_i \leqq 1. \quad (i=1, 2, \ldots, 8). \end{cases}$$

These estimates may be found as follows. From theorem IV follows

$$(4.25)\qquad p_5' = p_6',\ p_7' = p_8'.$$

Thus the problem of finding $p_1', p_2', \ldots, p_8'$ is reduced to the case of $k-2=6$ series of trials with $m_0'=5$, $m_1'=10$,

$$(4.26)\qquad \begin{cases} i & 1 & 2 & 3 & 4 & 5\,(+6) & 7\,(+8) \\ a_i' & 8 & 22 & 13 & 25 & 41 & 34 \\ n_i' & 10 & 40 & 20 & 50 & 80 & 55 \\ f_i' & 0,8 & 0,55 & 0,65 & 0,5 & 0,51 & 0,62 \end{cases}$$

and

$$(4.27)\qquad \alpha_{1,2}' = \alpha_{1,4}' = \alpha_{3,4}' = \alpha_{4,5}' = \alpha_{5,7}' = 1.$$

To these 6 series of trials we apply theorem I, taking for R_λ the restriction $\pi_1 \leqq \pi_4$. Then we find by means of the theorems II, III and IV in a similar way as in example 3

$$(4.28)\qquad p_1' = p_4'.$$

In this way the problem of finding $p_1', p_2', \ldots, p_8'$ is reduced to the case of $k-3=5$ series with $m_0''=2$, $m_1''=8$,

$$(4.29)\qquad \begin{cases} i & 3 & 1\,(+4) & 2 & 5\,(+6) & 7\,(+8) \\ a_i'' & 13 & 33 & 22 & 41 & 34 \\ n_i'' & 20 & 60 & 40 & 80 & 55 \\ f_i'' & 0,65 & 0,55 & 0,55 & 0,51 & 0,62 \end{cases}$$

and

$$(4.30)\qquad \alpha_{3,1}'' = \alpha_{1,2}'' = \alpha_{1,5}'' = \alpha_{5,7}'' = 1.$$

From theorem IV then follows

$$p_1' = p_3' \tag{4.31}$$

and the problem is reduced to the case of $k-4=4$ series of trials with $m_0''''=2$, $m_1''''=4$

(4.32)

i	1 (+3+4)	2	5 (+6)	7 (+8)
a_i''''	46	22	41	34
n_i''''	80	40	80	55
f_i''''	0,58	0,55	0,51	0,62

and

$$\alpha_{1,2}'''' = \alpha_{1,5}'''' = \alpha_{1,7}'''' = 1. \tag{4.33}$$

From theorem V then follows

$$p_2' \leqq p_7' \text{ and } p_5' \leqq p_2' \tag{4.34}$$

and from (3.24), (4.25), (4.28) and (4.31)

$$p_1' = p_3' = p_4' = p_5' = p_6' = 0{,}54,\ p_2' = 0{,}55,\ p_7' = p_8' = 0{,}62. \tag{4.35}$$

From theorem VI and (4.35) it follows, p_6' being smaller than p_7' (cf. (3.31.1))

$$p_i = p_i' \quad (i = 1,2,\ldots,8). \tag{4.36}$$

The method used in this example for finding the estimates $p_1, p_2, \ldots, p_8$ is not the only possible way. The problem may also be solved by exclusively applying theorem I, taking any of the restrictions $R_1, R_2, \ldots, R_8$ for R_λ or by exclusively applying theorem VI, taking for i and j any of the pairs of values (i, j) with $\alpha_{i,j}=0$.

Remark:

7. The indicated procedure may be generalized to several cases of ordered parameters of other probability distributions, e.g. parameters of Poisson distributions and means of normally distributed variables with known variances. This generalization and the properties of the estimates are being investigated.

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REFERENCES

1. Ayer, Miriam, H. D. Brunk, G. M. Ewing, W. T. Reid, Edward Silverman, An empirical distribution function for sampling with incomplete information, Ann. Math. Stat. 26, 641–647 (1955).
2. Eeden, Constance van, Maximum likelihood estimation of ordered probabilities, Report S 188 (VP 5) of the Statistical Department of the Mathematical Centre (Amsterdam, 1956).