0215 SP 53

KONINKL. NEDERL. AKADEMIE VAN WETENSCHAPPEN – AMSTERDAM
Reprinted from Proceedings, Series A, 60, No. 3 and Indag. Math., 19, No. 3, 1957

MATHEMATICS

PRIORITY IN WAITING LINE PROBLEMS [1]). I

BY

H. KESTEN AND J. TH. RUNNENBURG

(Communicated by Prof. D. VAN DANTZIG at the meeting of November 24, 1956)

1. *Introduction*

The object of this paper is to give a more detailed account of the situation, discussed in the first part of COBHAM's article [2]. We shall consider here the situation where customers of different priorities arrive at *one* counter to be served [2]).

2. *Description of the system*

We distinguish r priorities by the priority numbers $1, 2, \ldots, r$, where 1 stands for the highest and r for the lowest priority. Customers of priority number k will be called k-customers in the sequel. At time zero the counter is opened for servicing. At that moment, with probability $p_0(a_1, \ldots, a_r)$ a queue consisting of a_1 1-customers, ..., a_r r-customers is present (with $a_1 \geqslant 0, \ldots, a_r \geqslant 0$, $p_0(a_1, \ldots, a_r) \geqslant 0$, $\sum \lfloor a_1 \geqslant 0, \ldots, a_r \geqslant 0 \rfloor p_0(a_1, \ldots, a_r) = 1$) [3]). New k-customers arrive ($k \in \{1, \ldots, r\}$) according to the following law: the interval from time zero to the first arrival of a k-customer, and the intervals between arrivals of successive k-customers are mutually independent random variables with distributionfunction

$$G_k(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-\lambda_k x} & \text{for } x \geqslant 0, \end{cases} \tag{2.1}$$

where we assume $\lambda_k > 0$ for $k \in \{1, \ldots, r\}$. The servicetime is also stochastic and has the same distributionfunction $F_k(t)$ (continuous from the right) for all k-customers. All arrival intervals (including the intervals from time zero to the arrival of the first k-customer) and all servicetimes are mutually independent.

Servicing takes place for each priority in the order of arrival. If customers of different priorities are present when the counter becomes free to serve a new customer, that one with highest priority which came first to the counter, is the next to be served. If the counter becomes

[1]) Report SP 53 of the Statistical Department of the Mathematical Centre.

[2]) Questions, put to us by the N.V. Philips' Gloeilampenfabrieken, Eindhoven, Holland, gave rise to the present investigation.

[3]) The conditions under which a sum or a limit have to be taken are sometimes denoted by placing them between half square brackets $\lfloor \ \rfloor$. Summations are always over non-negative integers.

empty the next customer to be served is the first newly arriving customer. Servicing of a customer is never interrupted to make way for another customer.

Following D. G. KENDALL [10] we consider the moments at which customers leave the counter at the end of their servicetime. The customers are numbered (1, 2, ...) in the order in which they *leave* the counter, and

$$p_{k,n}(a_1, \ldots, a_r) \tag{2.2}$$

is defined as the probability that the n^{th} departing customer is a k-customer and leaves a queue consisting of a_1 1-customers, ..., a_r r-customers at the counter (for all $k \in \{1, \ldots, r\}$, $n \in \{1, 2, \ldots\}$ and $a_j \in \{0, 1, \ldots\}$ for $j \in \{1, \ldots, r\}$).

We introduce the generating functions

$$f_{k,n}(X_1, \ldots, X_r) \stackrel{\text{def}}{=} \sum \lfloor a_1 \geqslant 0, \ldots, a_r \geqslant 0 \rfloor\, p_{k,n}(a_1, \ldots, a_r)\, X_1^{a_1} \ldots X_r^{a_r} \tag{2.3}$$

for $|X_1| \leqslant 1, \ldots, |X_r| \leqslant 1$, the functions $\varphi_k(\alpha)$ and the moments of $F_k(t)$, defined by [1])

$$\varphi_k(\alpha) \stackrel{\text{def}}{=} \int_{0-}^{\infty} e^{-\alpha t}\, dF_k(t) \tag{2.4}$$

for $\operatorname{Re}\alpha \geqslant 0$ and

$$\mu_k^{(l)} \stackrel{\text{def}}{=} \int_{0-}^{\infty} t^l\, dF_k(t). \tag{2.5}$$

We exclude the case where $F_k(0) = 1$ for some k, i.e. we have $\mu_k^{(l)} > 0$ for all k and all real l and $\varphi_k(\alpha) < 1$ for all k and all $\alpha > 0$.

Finally let

$$H_{k,n}(t) \tag{2.6}$$

be the conditional distributionfunction of the waitingtime of the n^{th} departing customer, given that the n^{th} departing customer is a k-customer, and

$$\psi_{k,n}(\alpha) \stackrel{\text{def}}{=} \int_{0-}^{\infty} e^{-\alpha t}\, dH_{k,n}(t) \tag{2.7}$$

for all $k \in \{1, \ldots, r\}$ and $n \in \{1, 2, \ldots\}$.

We distinguish two cases:

$$\text{the case of nonsaturation, defined by } \sum_1^r {}_i\ \lambda_i \mu_i^{(1)} < 1$$

and

$$\text{the case of saturation, defined by } \sum_1^r {}_i\ \lambda_i \mu^{(1)} \geqslant 1.$$

For the case of nonsaturation we prove that the limits of $p_{k,n}(a_1, \ldots, a_r)$ and $f_{k,n}(X_1, \ldots, X_r)$ for $n \to \infty$ exist and that $H_{k,n}(t)$ tends to a distribution-

[1]) The integrals are Lebesgue-Stieltjesintegrals over the interval $0 \leqslant t < \infty$.

function $H_k(t)$ for $n \to \infty$. All these limits are independent of the initial situation, i.e. the probability distribution $\{p_0(a_1, \ldots, a_r)\}$. $H_k(t)$ is the distributionfunction of the waitingtime of an arbitrary k-customer in the stationary situation.

Using D. VAN DANTZIG's "method of collective marks" ([5], [6] and [7]), we derive recurrence relations (3.12) between the generating functions $f_{k,n}(X_1, \ldots, X_r)$ together with relations (3.16) connecting the $f_{k,n}(X_1, \ldots, X_r)$, $\psi_{k,n}(\alpha)$ and $\varphi_k(\alpha)$. From these relations we derive the relations (5.2) for the

$$f_k(X_1, \ldots, X_r) \overset{\text{def}}{=} \lim_{n \to \infty} f_{k,n}(X_1, \ldots, X_r),$$

which are then solved. From the relation (3.16) we derive (5.3), connecting $f_k(X_1, \ldots, X_r)$ and

$$\psi_k(\alpha) \overset{\text{def}}{=} \lim_{n \to \infty} \psi_{k,n}(\alpha).$$

Once the $f_k(X_1, \ldots, X_r)$ are solved, they are used, together with the last relation, to compute the first two moments of $H_k(t)$ and to derive an expression for $\psi_k(\alpha)$, for $k \in \{1, \ldots, r\}$. The first moment of $H_k(t)$ was given by COBHAM [2], but we did not understand his proof.

For the case of saturation we only state some results without proof.

We shall use some abbreviations to keep the formulae from becoming awkwardly long. With the understanding that on both sides of the equalitysign in (2.8) up to and including (2.14) indices may be added to the function symbols, we write [1])

$$f(X) \overset{\text{abb}}{=} f(X_1, \ldots, X_r), \tag{2.8}$$

$$g(X) \overset{\text{abb}}{=} \sum_1^r {}^i f_i(X), \tag{2.9}$$

$$f(u^k X v^l) \overset{\text{abb}}{=} f(u, \ldots, u, X_{k+1}, \ldots, X_{r-l}, v, \ldots, v), \tag{2.10}$$

i.e. the first k variables in (2.10) are equal to u, the last l variables are equal to v and the remaining variables (if any) are equal to the corresponding variables of $f(X)$ (we shall always have $k+l \leqslant r$). In the same way

$$f(U^{(k)} X) \overset{\text{abb}}{=} f(U_1, \ldots, U_{k-1}, X_k, \ldots, X_r), \tag{2.11}$$

$$f(U^{(k)} X v^l) \overset{\text{abb}}{=} f(U_1, \ldots, U_{k-1}, X_k, \ldots, X_{r-l}, v, \ldots, v), \tag{2.12}$$

$$f(y_{(k)} X) \overset{\text{abb}}{=} f(y_{k,1}, \ldots, y_{k,k-1}, X_k, \ldots, X_r), \tag{2.13}$$

$$f(y_{(k)} X v^l) \overset{\text{abb}}{=} f(y_{k,1}, \ldots, y_{k,k-1}, X_k, \ldots, X_{r-l}, v, \ldots, v). \tag{2.14}$$

We use

$$\lim_{X \to 1} f(X) \qquad (|X| < 1)$$

---

[1]) $\overset{\text{abb}}{=}$ is used, when on the *left* hand side of an equalitysign an abbreviation is introduced for an expression on the *right* hand side.

if we want to take

$$\lim_{X_1\to 1}\ \lim_{X_2\to 1} \dots \lim_{X_r\to 1} f(X)$$

where $X_1, \dots, X_r$ must remain *inside* the unit circle. The order in which the latter limits are taken is irrelevant unless otherwise stated.

Finally [1])

$$(2.15)\qquad pX \overset{\text{abb}}{=\!=} \sum_1^r{}^{i}\ p_i X_i,$$

and for all $k, l \in \{1, \dots, r\}$, with $k+l \leqslant r$,

$$(2.16)\qquad p(u^k, X) \overset{\text{abb}}{=\!=} \sum_1^k{}^{i}\ p_i u + \sum_{k+1}^r{}^{i}\ p_i X_i,$$

$$(2.16')\qquad p(u^k, X, \ ^l) \overset{\text{abb}}{=\!=} \sum_1^k{}^{i}\ p_i u + \sum_{k+1}^{r-l}{}^{i}\ p_i X_i + \sum_{r-l+1}^{r}{}^{i}\ p_i v,$$

$$(2.17)\qquad p(U^{(k)}, X) \overset{\text{abb}}{=\!=} \sum_1^{k-1}{}^{i}\ p_i U_i + \sum_k^r{}^{i}\ p_i X_i,$$

$$(2.17')\qquad p(U^{(k)}, X, v^l) \overset{\text{abb}}{=\!=} \sum_1^{k-1}{}^{i}\ p_i U_i + \sum_k^{r-l}{}^{i}\ p_i X_i + \sum_{r-l+1}^{r}{}^{i}\ p_i v,$$

$$(2.18)\qquad p(y_{(k)}, X) \overset{\text{abb}}{=\!=} \sum_1^{k-1}{}^{i}\ p_i y_{k,i} + \sum_k^r{}^{i}\ p_i X_i,$$

$$(2.18')\qquad p(y_{(k)}, X, v^l) \overset{\text{abb}}{=\!=} \sum_1^{k-1}{}^{i}\ p_i y_{k,i} + \sum_k^{r-l}{}^{i}\ p_i X_i + \sum_{r-l+1}^{r}{}^{i}\ p_i v.$$

3. *Recurrence relations for the system*

In order to apply the method of collective marks of D. VAN DANTZIG [5] and [6], we introduce an event E, which happens with probability $1-X_k$ whenever a k-customer arrives, thus

$$(3.1)\qquad 0 \leqslant X_k \leqslant 1 \text{ for each } k \in \{1, \dots, r\}.$$

The events E are independent for all customers. Any event E is called a "catastrophe" in D. VAN DANTZIG's papers. Its nature, however, is irrelevant. As only probabilities of other events, together with non-occurrence of any "catastrophe" are considered, it is irrelevant whether under occurrence of an event E the process continues or not.

We can now interprete $f_{k,n}(X)$ as a probability for

$$(3.2)\qquad p_{k,n}(a_1, \dots, a_r)\, X_1^{a_1} \dots X_r^{a_r}$$

is the probability, that at the n^{th} departure, $n \in \{1, 2, \dots\}$, one k-customer leaves the counter, a_1 1-customers, …, a_r r-customers remain at the counter

[1]) If $k = 1$ the first sum on the right hand side of (2.17) and (2.18) equals zero, if $k = r$ the last sum of (2.16); analogously for (2.16′), (2.17′) and (2.18′).

and with respect to none of the remaining customers the event E happened. Therefore

(3.3) $$f_{k,n}(X) = \sum \lfloor a_1 \geqslant 0, \ldots, a_r \geqslant 0 \rfloor \, p_{k,n}(a_1, \ldots, a_r) X_1^{a_1} \ldots X_r^{a_r}$$

is the probability, that at the n^{th} departure, $n \in \{1, 2, \ldots\}$, a k-customer leaves the counter and with respect to none of those remaining at the counter the event E happened. Further

(3.4) $$p_{i,n}(0, \ldots, 0, a_k, \ldots, a_r) X_k^{a_k} \ldots X_r^{a_r}$$

is the probability, that at the n^{th} departure, $n \in \{1, 2, \ldots\}$, an i-customer leaves the counter, a_k k-customers, ..., a_r r-customers remain at the counter and with respect to none of the customers remaining at the counter the event E happened. If $a_k > 0$ the next customer to be served is a k-customer, therefore for $k \in \{1, \ldots, r\}$ [1]) (using (2.10))

(3.5) $$\begin{cases} f_{i,n}(0^{k-1}X) - f_{i,n}(0^k X) = \\ = \sum \lfloor a_k \geqslant 1, a_{k+1} \geqslant 0, \ldots, a_r \geqslant 0 \rfloor \, p_{i,n}(0, \ldots, 0, a_k, \ldots, a_r) X_k^{a_k} \ldots X_r^{a_r} \end{cases}$$

is the probability, that at the n^{th} departure an i-customer leaves the counter, service on a k-customer starts and with respect to none of the customers left by the departing i-customer the event E happened.

Put

(3.6) $$\lambda \overset{\text{def}}{=} \lambda_1 + \ldots + \lambda_r.$$

Now

$$f_{i,n}(0^r) = p_{i,n}(0, \ldots, 0)$$

is the probability, that at the n^{th} departure an i-customer leaves and the counter becomes empty, while

(3.7) $$p_k \overset{\text{def}}{=} \frac{\lambda_k}{\lambda}$$

is the probability, that the first customer arriving after a given moment is a k-customer, therefore (using (2.9) and (2.10))

(3.8) $$p_k X_k g_n(0^r)$$

is the probability, that at the n^{th} departure, $n \in \{1, 2, \ldots\}$, the counter becomes empty and the next arriving customer is a k-customer, with respect to which the event E does not happen.

(3.9) $$\int_{0-}^{\infty} e^{-\lambda_1 t} \frac{(\lambda_1 t)^{a_1}}{a_1!} \ldots e^{-\lambda_r t} \frac{(\lambda_r t)^{a_r}}{a_r!} \, dF_k(t)$$

is the probability, that during the servicetime of a k-customer exactly

[1]) If $k = r$ then $f_{i,n}(0^k X)$ stands for $f_{i,n}(0^r)$.

a_1 1-customers, ..., a_r r-customers arrive, so (using (2.15))

$$(3.10)\quad \begin{cases} \varphi_k(\lambda(1-pX)) = \\ = \sum \lfloor a_1 \geqslant 0, \ldots, a_r \geqslant 0 \rfloor\, X_1^{a_1} \ldots X_r^{a_r} \int\limits_{0-}^{\infty} e^{-\lambda_1 t} \frac{(\lambda_1 t)^{a_1}}{a_1!} \ldots e^{-\lambda_r t} \frac{(\lambda_r t)^{a_r}}{a_r!}\, dF_k(t) \end{cases}$$

is the probability, that with respect to none of the customers, arriving during the servicetime of a k-customer, the event E happened.

Analogously

$$(3.11)\qquad \varphi_k(\lambda_k(1-X_k))$$

is the probability, that with respect to none of the customers with priority number k, arriving during the servicetime of a k-customer, the event E happened.

Now the probability that at the $(n+1)^{st}$ departure a k-customer leaves and that neither to him nor to those remaining at the counter the event E happened is equal to the probability that at the n^{th} departure *either* an i-customer leaves the counter (for i equal to 1, 2, ... or r), service on a k-customer starts and to those remaining at the counter (the k-customer under service included) the event E did not happen *or* the counter becomes empty and the first customer arriving is a k-customer, with respect to whom the event E did not happen *and* (in any case) during the servicetime of that k-customer no customers, with respect to whom the event E happened, arrive. This equality can be written in the following way, using (3.3), (3.5), (3.8) and (3.10) with their interpretations

$$(3.12)\quad X_k f_{k,n+1}(X) = \{g_n(0^{k-1}X) - g_n(0^k X) + p_k X_k g_n(0^r)\}\, \varphi_k(\lambda(1-pX)).$$

This relation is valid for $k \in \{1, \ldots, r\}$, $n \in \{1, 2, \ldots\}$ and all real X_k satisfying $0 \leqslant X_k \leqslant 1$, because of the arbitrariness of the event E. If at the moment the counter is opened for service, with probability $p_0(a_1, \ldots, a_r)$ a queue consisting of a_1 1-customers, ..., a_r r-customers is present and

$$(3.13)\qquad g_0(X) \overset{\text{def}}{=} \sum \lfloor a_1 \geqslant 0, \ldots, a_r \geqslant 0 \rfloor\, p_0(a_1, \ldots, a_r)\, X_1^{a_1} \ldots X_r^{a_r},$$

then (3.12) is true for $n=0$ as well.

For $0 \leqslant X_i \leqslant 1$, $i \neq k$ and $0 < X_k \leqslant 1$ we can solve (3.12) for $f_{k,n+1}(X)$ once $g_n(X)$ is known for those values of X. But then we can find $f_{k,n+1}(X)$ (and $g_n(X)$) for all X satisfying $|X_1| \leqslant 1, \ldots, |X_r| \leqslant 1$ by analytic continuation for each $k \in \{1, \ldots, r\}$. Therefore (3.12) holds generally for each $k \in \{1, \ldots, r\}$, $n \in \{0, 1, 2, \ldots\}$ and $|X_1| \leqslant 1, \ldots, |X_r| \leqslant 1$.

We might try to express $f_{k,n+1}(X)$ as a function of $g_0(X)$ only, by repeated application of (3.12) and so eliminating $g_l(X)$ with $l \geqslant 1$. This however is not practicable, the more so as $f_{k,n+1}(X)$ for $X_k=0$ can be found from (3.12) only by dividing both sides by X_k for $X_k \neq 0$ and taking the limits for $X_k \to 0$, which leads to partial differential quotients in the expression for $f_{k,n+1}(X)$ for $X_k=0$.

Analogous to (3.11) and its interpretation we have:

$$\psi_{k,n}(\lambda_k(1-X_k)) \tag{3.14}$$

is the probability, that if at the n^{th} departure a k-customer leaves the counter, with respect to none of the customers with priority number k arriving during his waitingtime, the event E happened.

Finally

$$f_{k,n}(1^{k-1}X\,1^{r-k}) \tag{3.15}$$

is the probability, that at the n^{th} departure a k-customer leaves the counter and with respect to none of the customers with priority number k which remain at the counter the event E happened. Now this is equal to the probability that at the n^{th} departure a k-customer leaves and that with respect to none of the customers with priority number k arriving either during his waitingtime or during his servicetime the event E happened.

Therefore we have

$$f_{k,n}(1^{k-1}X\,1^{r-k}) = f_{k,n}(1^r)\,\psi_{k,n}(\lambda_k(1-X_k))\,\varphi_k(\lambda_k(1-X_k)), \tag{3.16}$$

for $k \in \{1, \ldots, r\}$, $n \in \{1, 2, \ldots\}$ and for all X_k satisfying $0 \leqslant X_k \leqslant 1$. This may again be generalized by analytic continuation. Therefore (3.16) holds for all X_k satisfying $|X_k| \leqslant 1$.

We can now summarize our results. From (3.16) we have, that $\psi_{k,n}(\alpha)$ is a function of $f_{k,n}(X)$ and $\varphi_k(\alpha)$. The functions $f_{k,n}(X)$ are known to satisfy (3.12), but cannot be solved explicitly from those relations in terms of $g_0(X)$. However, as we are interested in the behaviour of the system in the long run, we will use (3.12) and (3.16) to find $\lim_{n\to\infty} \psi_{k,n}(\alpha)$.

The relations (3.12) and (3.16) can also be derived in a more formal way than it has been done here.

4. *Convergence to a stationary distribution*

Before making use of the relations (3.12) and (3.16) we shall prove some results connected with the convergence of the $p_{k,n}(a_1, \ldots, a_r)$ for $n \to \infty$, which justify the method of the next section.

Let us say that the system is in the state $(k; a_1, \ldots, a_r)$ at the departure of the n^{th} customer if the n^{th} departing customer is a k-customer and if he leaves for every $i \in \{1, \ldots, r\}$ a_i i-customers at the counter. Then all transition probabilities from a state at the n^{th} departure to any state at the $(n+1)^{\text{st}}$ departure are independent of n and can easily be calculated. By considering only the moments, at which a customer leaves the system, we thus obtain a Markof chain, with a denumerable number of states. Let us denote this Markof chain by M. For every state there is a positive probability to reach in a finite number of steps a state where a departing customer leaves an empty counter, and from this situation any state can again be reached in any number of steps. We conclude that M is an irreducible and aperiodic Markof chain (cf. FELLER [8] for the terminology and classification of states in Markof chains). From Corollary 1 in FELLER [8] (p. 328) it follows immediately, that $\lim_{n\to\infty} p_{k,n}(a_1, \ldots, a_r)$ exists and is independent of the initial distribution.

In the case of nonsaturation $(\sum_1^r{}^i \lambda_i \mu_i^{(1)} < 1)$ all states are ergodic. To prove this, we need a theorem of FOSTER [9], which was given by MOUSTAFA [12] in the following slightly generalized form:

Theorem 4.1. An irreducible, aperiodic Markof chain represented by the Markof matrix $\|p_{i,j}\|$ $(i, j = 1, 2, \ldots)$ is ergodic if for some $\varepsilon > 0$ and some integer i_0, there exists a non-negative solution $\{y_i\}$ of the inequalities

$$\sum_1^\infty{}^j p_{i,j}\, y_j \leqslant y_i - \varepsilon \quad \text{for} \quad i > i_0, \tag{4.1}$$

$$\sum_1^\infty{}^j p_{i,j}\, y_j < \infty \qquad \text{for} \quad i \leqslant i_0. \tag{4.2}$$

We note that $\sum_1^\infty{}^j p_{i,j} y_j$ can be regarded as the expectation after one step, if we start in the i^{th} state, of a random variable [1] $\mathbf{y}$, taking values y_j with probabilities $p_{i,j}$.

Theorem 4.2. If $\sum_1^r{}^i \lambda_i \mu_i^{(1)} < 1$, all states in the Markof chain M are ergodic.

Proof: This theorem is an application of Th. 4.1. The states of M can be characterized by $(k; a_1, \ldots, a_r)$, i.e. the priority number of the leaving customer and the number of customers of each priority left by him. With each state we associate a number y. By definition $y = \sum_1^r{}^i a_i \mu_i^{(1)}$ for the state $(k; a_1, \ldots, a_r)$, i.e. y is the expectation of the time needed to serve the remaining customers and as such non-negative. If we start in the situation $(k; 0, \ldots, 0, a_l, \ldots, a_r)$ with $a_l > 0$ for an $l \leqslant r$, the next customer to be served is an l-customer and the expectation of $\mathbf{y}$ after one step is then

$$\begin{aligned} &\sum_1^{l-1}{}^i \lambda_i \mu_i^{(1)} \mu_l^{(1)} + (a_l + \lambda_l \mu_l^{(1)} - 1)\, \mu_l^{(1)} + \sum_{l+1}^r{}^i (a_i + \lambda_i \mu_l^{(1)})\, \mu_i^{(1)} = \\ &= \sum_l^r{}^i a_i \mu_i^{(1)} + \mu_l^{(1)} \{ \sum_1{}^i \lambda_i \mu_i^{(1)} - 1 \} \leqslant \\ &\leqslant \sum_l^r{}^i a_i \mu_i^{(1)} - \varepsilon \end{aligned}$$

[1]) Random variables are distinguished from numbers by printing their symbols in bold type.

where

$$\varepsilon \stackrel{\text{def}}{=} \min_{1 \leq l \leq r} \mu_l^{(1)} \{1 - \sum_1^r{}^i \lambda_i \mu_i^{(1)}\}.$$

In fact the expected number of i-customers arriving during the service-time of an l-customer is $\lambda_i \mu_l^{(1)}$, and one l-customer leaves the system at the end of this step. Therefore (4.1) is satisfied in this case. If we start in the state $(k; 0, \ldots, 0)$, the expectation of $\mathbf{y}$ after one step is finite, so (4.2) is satisfied for the r states with $a_1 = a_2 = \ldots = a_r = 0$.

Thus Th. 4.2 follows.

Corollary. If we define $p_k(a_1, \ldots, a_r) = \lim_{n \to \infty} p_{k,n}(a_1, \ldots, a_r)$ we have:

$$p_k(a_1, \ldots, a_r) > 0 \text{ for all } k \in \{1, \ldots, r\},\ a_i \geqslant 0$$

$$\sum_1^r{}^k \sum \lfloor a_1 \geqslant 0, \ldots, a_r \geqslant 0 \rfloor\, p_k(a_1, \ldots, a_r) = 1$$

and the $p_k(a_1, \ldots, a_r)$ form a stationary distribution for the Markof chain M. This is an immediate consequence of Th. 4.2 and Th. 2, p. 325, in Feller [8].

To prove also the convergence of $\sum \lfloor S \rfloor p_{k,n}(a_1, \ldots, a_r)$ where the summation is over an arbitrary set S of states, and the convergence of moments of the queue length, we need the following theorem.

Theorem 4.3. Let an irreducible, aperiodic ergodic Markof chain be represented by the Markof matrix $\|p_{i,j}\|$ $(i, j = 1, 2, \ldots)$. If

$$\pi_j \stackrel{\text{def}}{=} \lim_{n \to \infty} p_{i,j}^{(n)},$$

where $p_{i,j}^{(n)}$ are the n step transitionprobabilities (these limits exist, are positive and independent of i; cf. Feller [8], p. 325) then we have for any non-negative state function F_j

$$\lim_{n \to \infty} \sum_1^\infty{}^j p_{i,j}^{(n)} F_j = \sum_1^\infty{}^j \pi_j F_j \quad \text{for every } i. \tag{4.3}$$

Proof: As $\lim_{n \to \infty} p_{i,j}^{(n)} = \pi_j$ and $F_j \geqslant 0$ we have for all positive integers s

$$\liminf_{n \to \infty} \sum_1^\infty{}^j p_{s,j}^{(n)} F_j \geqslant \sum_1^\infty{}^j \pi_j F_j, \tag{4.4}$$

because if $\varepsilon > 0$ and N is such that [1])

$$\sum_1^N{}^j \pi_j F_j \geqslant \sum_1^\infty{}^i \pi_j F_j - \varepsilon,$$

---

[1]) If $\sum_1^\infty{}^j \pi_j F_j = \infty$, only some obvious changes are necessary.

we have

$$\liminf_{n\to\infty} \sum_1^\infty {}^j p_{s,j}^{(n)} F_j \geqslant \liminf_{n\to\infty} \sum_1^N {}^j p_{s,j}^{(n)} F_j = \sum_1^N {}^j \pi_j F_j \geqslant \sum_1^\infty {}^j \pi_j F_j - \varepsilon$$

for every $\varepsilon > 0$, whence (4.4) holds.

The proof of (4.3) is completed, if $\sum_1^\infty {}^j \pi_j F_j = \infty$. If $\sum_1^\infty {}^j \pi_j F_j < \infty$ we proceed as follows. We know, that π_j is always positive, $\sum_1^\infty {}^j \pi_j = 1$, and $\pi_j = \sum_1^\infty {}^i \pi_i p_{i,j}^{(n)}$ for all positive integers n (cf. FELLER [8], p. 325). Therefore we have for a fixed $N \geqslant s$ and every n

$$\sum_1^\infty {}^j \pi_j F_j = \sum_1^\infty {}^i \pi_i \sum_1^\infty {}^j p_{i,j}^{(n)} F_j \geqslant \sum \llcorner 1 \leqslant i \leqslant N, i \neq s \lrcorner\, \pi_i \sum_1^\infty {}^j p_{i,j}^{(n)} F_j + \pi_s \sum_1^\infty {}^j p_{s,j}^{(n)} F_j$$

so

$$\sum_1^\infty {}^j \pi_j F_j \geqslant \limsup_{n\to\infty} \{ \sum \llcorner 1 \leqslant i \leqslant N, i \neq s \lrcorner\, \pi_i \sum_1^\infty {}^j p_{i,j}^{(n)} F_j + \pi_s \sum_1^\infty {}^j p_{s,j}^{(n)} F_j \} \geqslant$$

$$\geqslant \sum \llcorner 1 \leqslant i \leqslant N, i \neq s \lrcorner\, \pi_i \liminf_{n\to\infty} \sum_1^\infty {}^j p_{i,j}^{(n)} F_j + \pi_s \limsup_{n\to\infty} \sum_1^\infty {}^j p_{s,j}^{(n)} F_j \geqslant$$

$$\geqslant \sum \llcorner 1 \leqslant i \leqslant N, i \neq s \lrcorner\, \pi_i \sum_1^\infty {}^j \pi_j F_j + \pi_s \limsup_{n\to\infty} \sum_1^\infty {}^j p_{s,j}^{(n)} F_j .$$

Now take $N \to \infty$

$$\sum_1^\infty {}^j \pi_j F_j \geqslant (1 - \pi_s) \sum_1^\infty {}^j \pi_j F_j + \pi_s \limsup_{n\to\infty} \sum_1^\infty {}^j p_{s,j}^{(n)} F_j .$$

As $\pi_s > 0$ this leads to

$$\limsup_{n\to\infty} \sum_1^\infty {}^j p_{s,j}^{(n)} F_j \leqslant \sum_1^\infty {}^j \pi_j F_j \tag{4.5}$$

for all s.

From (4.5) together with (4.4) we have (4.3).

Remark 1. The theorem remains true for arbitrary state functions F_j with

$$\sum_1^\infty {}^j \pi_j |F_j| < \infty$$

as can be seen by writing

$$F_j = F_j^+ - F_j^- ,$$

where

$$F_j^+ \overset{\text{def}}{=} \frac{|F_j| + F_j}{2},$$

$$F_j^- \overset{\text{def}}{=} \frac{|F_j| - F_j}{2}.$$

Remark 2. If the Markof chain we consider has a probability $p_i^{(0)}$ of being in the state i in the initial situation ($p_i^{(0)} \geqslant 0$ and $\sum_1^\infty{}^i\, p_i^{(0)} = 1$), then by Th. 4.3

$$\lim_{n\to\infty} \sum_1^\infty{}^i \sum_1^\infty{}^j\, p_i^{(0)}\, p_{i,j}^{(n)}\, F_j = \sum_1^\infty{}^j\, \pi_j\, F_j,$$

provided F_j is bounded.

From the convergence of $p_{k,n}(a_1, \ldots, a_r)$ follows only the existence of $\lim_{n\to\infty} f_{k,n}(X)$, if $|X_i| < 1$ for all $i \in \{1, \ldots, r\}$. We may now conclude, that if $|X_i| \leqslant 1$ for all $i \in \{1, \ldots, r\}$

$$\lim_{n\to\infty} f_{n,k}(X) = \sum \lfloor a_1 \geqslant 0, \ldots, a_r \geqslant 0 \rfloor\, p_k(a_1, \ldots, a_r)\, X_1^{a_1} \ldots X_r^{a_r}.$$

This follows if we take the state function

$$F(i; a_1, \ldots, a_r) \stackrel{\text{def}}{=} \begin{cases} X_1^{a_1} \ldots X_r^{a_r} & \text{if} \quad i = k, \\ 0 & \text{if} \quad i \neq k. \end{cases}$$

Thus

$$f_k(X) \stackrel{\text{def}}{=} \lim_{n\to\infty} f_{k,n}(X) = \sum \lfloor a_1 \geqslant 0, \ldots, a_r \geqslant 0 \rfloor\, p_k(a_1, \ldots, a_r)\, X_1^{a_1} \ldots X_r^{a_r}$$

is a power series with positive coefficients, which converges for

$$|X_i| < 1, i \in \{1, \ldots, r\}, \text{ and as } \sum_1^r{}^k \sum \lfloor a_1 \geqslant 0, \ldots, a_r \geqslant 0 \rfloor\, p_k(a_1, \ldots, a_r) = 1,$$

we conclude that

$$\lim_{X\to 1} f_k(1^{k-1}\, X\, 1^{r-k}) = f_k(1^r) \qquad (|X| < 1). \tag{4.6}$$

Remark 3. From Th. 4.3 we also conclude that

$$\lim_{n\to\infty} \sum_1^r{}^k \sum \lfloor a_1 \geqslant 0, \ldots, a_r \geqslant 0 \rfloor\, a_j\, p_{k,n}(a_1, \ldots, a_r) =$$

$$= \sum_1^r{}^k \sum \lfloor a_1 \geqslant 0, \ldots, a_r \geqslant 0 \rfloor\, a_j\, p_k(a_1, \ldots, a_r),$$

i.e. the expected length of the queue of j-customers at the n^{th} departure tends to the expected length of the queue of j-customers derived from the stationary distribution, and analogously for the higher moments of the queue length, *provided* the initial state is fixed, i.e. $p_0(b_1, \ldots, b_r) = 1$ for a given initial state $(b_1, \ldots, b_r)$.

Theorem 4.4. If $\sum_1^r{}^i\, \lambda_i \mu_i^{(1)} < 1$, the conditional distributionfunctions of the waitingtimes $H_{k,n}(t)$ $(k \in \{1, \ldots, r\})$ converge to a non-degenerate distributionfunction $H_k(t)$ with

$$\psi_k(\alpha) \stackrel{\text{def}}{=} \int_{0-}^\infty e^{-\alpha t}\, dH_k(t)$$

satisfying

$$(4.7)\quad f_k\left(1,\ldots,1,1-\frac{\alpha}{\lambda_k},1,\ldots,1\right)=f_k(1^r)\,\psi_k(\alpha)\,\varphi_k(\alpha)\quad\text{for}\quad\left|1-\frac{\alpha}{\lambda_k}\right|\leqslant 1.$$

Proof. A distributionfunction of a non-negative random variable is uniquely determined if its Laplace transform is given on an interval which lies in the right half plane, because the Laplace transform of such a distributionfunction is analytic for all arguments with positive real part, and can thus be determined uniquely by analytic continuation, so that the uniqueness theorem for the inverse of a Laplace transform may be applied (cf. D. V. WIDDER [14] Th. 5a, p. 57 and Th. 6.3, p. 63).

From (3.16) follows the convergence of $\psi_{k,n}(\alpha)$ for $\left|1-\frac{\alpha}{\lambda_k}\right|\leqslant 1$ as $\lim\limits_{n\to\infty} f_{k,n}(1^r)>0$ and $\varphi_k(\alpha)>0$.

We can now follow a standard method (compare e.g. LÉVY [11], p. 49, proof of Th. 17 [2]) to prove that $H_{k,n}(t)$ converges to a function $H_k(t)$ with $\psi_k(\alpha)=\lim\limits_{n\to\infty}\psi_{k,n}(\alpha)$ satisfying (4.7). $H_k(t)$ is a monotonic non-decreasing function continuous from the right and satisfies $H_k(t)=0$ for $t<0$ and $\lim\limits_{t\to\infty} H_k(t)=1$, as from (4.7) $\lim\limits_{\alpha\to 0}\psi_k(\alpha)=1$. This proves Th. 4.4.

All the foregoing theorems concerning the queuing problem are valid only if $\sum_1^r{}^i\,\lambda_i\mu_i^{(1)}<1$. In the case of saturation $(\sum_1^r{}^i\,\lambda_i\mu_i^{(1)}\geqslant 1)$ analogous theorems can be proved, although we did not succeed so far in finding simple proofs. In fact one can prove:

If $\sum_1^s{}^i\,\lambda_i\mu_i^{(1)}<1$ and $\sum_1^{s+1}{}^i\,\lambda_i\mu_i^{(1)}\geqslant 1$ we have

$$\lim_{n\to\infty}\sum\lfloor a_{s+2}\geqslant 0,\ldots,a_r\geqslant 0\rfloor\,p_{k,n}(a_1,\ldots,a_r)=0$$

and

$$\lim_{n\to\infty}\sum\lfloor a_{s+1}\geqslant 0,\ldots,a_r\geqslant 0\rfloor\,p_{k,n}(a_1,\ldots,a_r)$$

exists and is positive.

If we define

$$\overline{p_k}(a_1,\ldots,a_s)\overset{\text{def}}{=}\lim_{n\to\infty}\sum\lfloor a_{s+1}\geqslant 0,\ldots,a_r\geqslant 0\rfloor\,p_{k,n}(a_1,\ldots,a_r)$$

we have for $k\in\{1,\ldots,r\}$

$$\lim_{n\to\infty} f_{k,n}(X\,1^{r-s})=\sum\lfloor a_1\geqslant 0,\ldots,a_s\geqslant 0\rfloor\,\overline{p_k}(a_1,\ldots,a_s)\,X_1^{a_1}\ldots X_s^{a_s}$$

whereas

$$\lim_{X\uparrow 1}\sum_1^{s+1}{}^k f_{k,n}(X\,1^{r-s})=\sum_1^{s+1}{}^k\sum\lfloor a_1\geqslant 0,\ldots,a_s\geqslant 0\rfloor\,\overline{p_k}(a_1,\ldots,a_s)=1.$$

$H_{k,n}(t)$ converges to a non-degenerate distributionfunction if $k \leqslant s$ and $\lim_{n\to\infty} H_{k,n}(t)=0$ for every finite t if $k \geqslant s+1$. If $k \leqslant s$ the moments of $H_{k,n}(t)$ do not necessarily converge to those of $H_k(t)$, i.e. we cannot conclude

$$\lim_{n\to\infty} \int_{0-}^{\infty} t^j \, dH_{k,n}(t) = \int_{0-}^{\infty} t^j \, dH_k(t) \quad \text{for} \quad k \leqslant s. \tag{4.8}$$

An example will show, that in some cases (4.8) does not hold. Take $s+2 \leqslant r$ and $\mu_{s+2}^{(1)}=\infty$. If we start from an initial situation with $a_1= \dots =a_{s+1}=0$, $a_{s+2}>0$ it is clear that $\int_0^\infty t\,dH_{k,n}(t)=\infty$ $(n \in \{1, 2, \dots\})$, whereas $\int_0^\infty t\,dH_k(t)$ is not necessarily infinite for $k \leqslant s$.

MATHEMATICS

PRIORITY IN WAITING LINE PROBLEMS [1]). II

BY

H. KESTEN and J. TH. RUNNENBURG

(Communicated by Prof. D. v. DANTZIG at the meeting of December 29, 1956)

5. *The case of nonsaturation*

In section 4 we proved that in the case of nonsaturation, i.e. if

$$\sum_1^r {}_i\, \lambda_i \mu_i^{(1)} < 1, \tag{5.1}$$

for $k \in \{1, \ldots, r\}$ and all X with $|X_1| \leqslant 1, \ldots, |X_r| \leqslant 1$

$$f_k(X) \stackrel{\text{def}}{=} \lim_{n\to\infty} f_{k,n}(X)$$

exists.

According to Theorem 4.4 in this case the limits

$$H_k(t) \stackrel{\text{def}}{=} \lim_{n\to\infty} H_{k,n}(t)$$

for all real t and

$$\psi_k(\alpha) \stackrel{\text{def}}{=} \lim_{n\to\infty} \psi_{k,n}(\alpha)$$

for $\operatorname{Re} \alpha \geqslant 0$ also exist and $\psi_k(\alpha)$ satisfies

$$\psi_k(\alpha) = \int_{0-}^{\infty} e^{-\alpha t}\, dH_k(t).$$

For $k \in \{1, \ldots, r\}$ and $|X_1| \leqslant 1, \ldots, |X_r| \leqslant 1$ we have from (3.12)

$$X_k f_k(X) = \{g(0^{k-1} X) - g(0^k X) + p_k X_k g(0^r)\}\, \varphi_k(\lambda(1-pX)) \tag{5.2}$$

while (3.16) leads to

$$f_k(1^{k-1} X\, 1^{r-k}) = f_k(1^r)\, \psi_k(\lambda_k - \lambda_k X_k)\, \varphi_k(\lambda_k - \lambda_k X_k). \tag{5.3}$$

From (5.2) we conclude (for $|X_1| \leqslant 1, \ldots, |X_r| \leqslant 1$ and arbitrary U_j satisfying $|U_1| \leqslant 1, \ldots, |U_{k-1}| \leqslant 1$)

$$\frac{f_k(X)}{\varphi_k(\lambda(1-pX))} = \frac{f_k(U^{(k)} X)}{\varphi_k(\lambda(1-p(U^{(k)}, X)))}, \tag{5.4}$$

for $X_k \neq 0$ (and by analytic continuation for $X_k = 0$ as well) and also

$$\sum_1^r {}_i\, \frac{X_i f_i(X)}{\varphi_i(\lambda(1-pX))} = \sum_1^r {}_i\, \{g(0^{i-1} X) - g(0^i X) + pXg(0^r)\}. \tag{5.5}$$

---

[1]) Report S211 (VP 11) of the Statistical Department of the Mathematical Centre.

Formula (5.5) simplifies to

$$(5.6)\qquad \sum_1^r{}^i f_i(X)\,\frac{X_i-\varphi_i(\lambda(1-pX))}{\varphi_i(\lambda(1-pX))}=g\,(0^r)\,(pX-1).$$

To determine $f_k(X)$ we introduce $y_{k,1}, \ldots, y_{k,k-1}$, defined (for $k \in \{2, \ldots, r\}$) by

$$(5.7)\qquad y_{k,i}-\varphi_i\,(\lambda\,(1-\sum_1^{k-1}{}^j p_j\,y_{k,j}-\sum_k^r{}^j p_j\,X_j))=0$$

for $i \in \{1, \ldots, k-1\}$. The $y_{k,i}$ are thus functions of $X_k, \ldots, X_r$. We shall prove (always assuming (5.1)):

Lemma 5.1. Equations (5.7) have for every set of complex numbers $X_k, \ldots, X_r$, satisfying $\sum_k^r{}^i \lambda_i \operatorname{Re} X_i<\sum_k^r{}^i \lambda_i$ exactly one solution for $y_{k,1},\ldots,y_{k,k-1}$, with $|y_{k,1}|<1, \ldots, |y_{k,k-1}|<1$.

Proof: Consider the equation

$$(5.8)\qquad z-\sum_1^{k-1}{}^i \lambda_i\,\varphi_i\,(\lambda-z-\sum_k^r{}^j \lambda_j\,X_j)=0.$$

By Rouché's Theorem (cf. TITCHMARSH [13], p. 116): "If $p(z)$ and $q(z)$ are analytic inside and on a closed contour C, and $|q(z)|<|p(z)|$ on C, then $p(z)$ and $p(z)+q(z)$ have the same number of zeros inside C", taking $p(z)\overset{\text{def}}{=}z$, $q(z)\overset{\text{def}}{=}-\sum_1^{k-1}{}^i \lambda_i\varphi_i\,(\lambda-z-\sum_k^r{}^j \lambda_jX_j)$ and for C the circle $|z|=\sum_1^{k-1}{}^i \lambda_i$ we have that

$$z-\sum_1^{k-1}{}^i \lambda_i\,\varphi_i\,(\lambda-z-\sum_k^r{}^j \lambda_j\,X_j)$$

has exactly one zero $z_k=z_k\,(X_k, \ldots, X_r)$ with $|z_k|<\sum_1^{k-1}{}^i \lambda_i$ for a fixed set of complex numbers $X_k, \ldots, X_r$, satisfying

$$\sum_k^r{}^i \lambda_i \operatorname{Re} X_i < \sum_k^r{}^i \lambda_i.$$

If we now take

$$y_{k,i}=\varphi_i\,(\lambda-z_k-\sum_k^r{}^j \lambda\;X_j)$$

equations (5.7) are solved and

$$|y_{k,i}|<1$$

because $\operatorname{Re}\,(\lambda-z_k-\sum_k^r{}^j \lambda_jX_j)>0$ and $|\varphi_i(\alpha)|<1$ for $\operatorname{Re}\alpha>0$. A second solution $y^*_{k,i}$ leads to $z^*_k\overset{\text{def}}{=}\sum_1^{k-1}{}^i \lambda_iy^*_{k,i}$ where z^*_k satisfies (5.8) and $|z^*_k|<\sum_1^{k-1}{}^i \lambda_i$. But then $z^*_k=z_k$ and therefore $y^*_{k,i}=y_{k,i}$.

Lemma 5.2. The solution z_k of (5.8) is an analytic function of the variables $X_k, \ldots, X_r$ for all $X_k, \ldots, X_r$ satisfying

$$\sum_k^r{}^i \lambda_i \operatorname{Re} X_i < \sum_k^r{}^i \lambda_i.$$

Remark (cf. Bochner and Martin [1], p. 30). A function $f(z_1, \ldots, z_l)$ is an analytic function of the l complex variables $z_1, \ldots, z_l$ in a certain region, if in some neighbourhood of every point $(z_1^0, \ldots, z_l^0)$ of that region it is the sum of an absolutely convergent powerseries in $z_1 - z_1^0, \ldots, z_l - z_l^0$.

Proof: Consider the point $X_k = c_k, \ldots, X_r = c_r$, with $\sum_k^r{}^i \lambda_i \operatorname{Re} c_i < \sum_k^r{}^i \lambda_i$. By using theorem 9 from Bochner and Martin [1], p. 39 (for the special case $k=1$ in their notation), we prove that z_k is analytic in the point $(c_k, \ldots, c_r)$. The theorem reads in our notation:

If the function $F(z, X_k, \ldots, X_r)$ is an analytic function of $r-k+1$ (complex) variables in the neighbourhood of the point $(a, c_k, \ldots, c_r)$, if $F(a, c_k, \ldots, c_r) = 0$ and if $\frac{\partial F}{\partial z} \neq 0$ for $z = a$, $X_k = c_k, \ldots, X_r = c_r$ then the equation

$$F(z, X_k, \ldots, X_r) = 0$$

has a unique solution

$$z_k = z_k(X_k, \ldots, X_r)$$

equal to a for $X_k = c_k, \ldots, X_r = c_r$ and analytic in the neighbourhood of the point $(c_k. \ldots, c_r)$.

We take

$$F(z, X_k, \ldots, X_r) \stackrel{\text{def}}{=} z - \sum_1^{k-1}{}^i \lambda_i \varphi_i (\lambda - z - \sum_k^r{}^j \lambda_j X_j)$$

with $z, X_k, \ldots, X_r$ as (functionally) independent complex variables. This function is analytic in the neighbourhood of $(a, c_k, \ldots, c_r)$, if

$$\operatorname{Re} a + \sum_k^r{}^j \lambda_j \operatorname{Re} c_j < \lambda.$$

This holds in particular for $a = z_k(c_k, \ldots, c_r)$, where z_k is the only zero of $F(z, X_k, \ldots, X_r)$ with $|z_k| < \sum_1^{k-1}{}^i \lambda_i$ (cf. proof of lemma 5.1), because $\operatorname{Re} z_k + \sum_k^r{}^j \lambda_j \operatorname{Re} c_j < \sum_1^{k-1}{}^i \lambda_i + \sum_k^r{}^j \lambda_j = \lambda$. Furthermore $F(z_k(c_k, \ldots, c_r), c_k, \ldots, c_r) = 0$ as z_k satisfies (5.11) and

$$\left|\frac{\partial F}{\partial z}\right| = \left| 1 - \sum_1^{k-1}{}^i \lambda_i \frac{\partial \varphi_i}{\partial z} \right| \geqslant 1 - \sum_1^{k-1}{}^i \lambda_i \mu_i^{(1)} > 0,$$

for $z=a$, $X_k=c_k, \ldots, X_r=c_r$, because

$$\left|\frac{\partial\varphi_i}{\partial z}\right| = \left|\int_0^\infty t \exp\{-t(\lambda - z - \sum_k^r{}^j \lambda_j X_j)\}\, dF_i(t)\right| \leqq \int_0^\infty t\, dF_i(t) = \mu_i^{(1)}$$

and (5.1) holds. Therefore the equation (5.8) has a unique solution $z_k = z_k(X_k, \ldots, X_r)$ equal to $z_k(c_k, \ldots, c_r)$ for $X_k=c_k, \ldots, X_r=c_r$, which is *analytic* (only this is new) in the neighbourhood of $(c_k, \ldots, c_r)$.

Lemma 5.3. If we keep

$$\sum_k^r{}^i \lambda_i \operatorname{Re} X_i < \sum_k^r{}^i \lambda\ ,$$

we have

$$\lim z_k(X_k, \ldots, X_r) = \sum_1^{k-1}{}^i \lambda_i \text{ for } \sum_k^r{}^i \lambda_i (X_i - 1) \to 0. \tag{5.9}$$

Proof: As for $\operatorname{Re} x \geqslant 0$

$$|1 - e^{-x}| = \left|\int_0^1 x e^{-xt}\, dt\right| \leqslant |x|$$

and

$$\operatorname{Re}(\lambda - z_k - \sum_k^r{}^j \lambda_j X_j) > 0,$$

we have

$$\begin{aligned} &\left|z_k - \sum_1^{k-1}{}^i \lambda_i\right| \leqslant \\ &\leqslant \sum_1^{k-1}{}^i \lambda_i \int_{0-}^\infty \left|\exp\{-(\lambda - z_k - \sum_k^r{}^j \lambda_j X_j)t\} - 1\right| dF_i(t) \leqslant \\ &\leqslant \sum_1^{k-1}{}^i \lambda_i \mu_i \left|\lambda - z_k - \sum_k^r{}^j \lambda_j X_j\right| \leqslant \\ &\leqslant \sum_1^{k-1}{}^i \lambda_i \mu_i \left|\sum_1^{k-1}{}^j \lambda_j - z_k\right| + \sum_1^{k-1}{}^i \lambda_i \mu_i \left|\sum_k^r{}^j \lambda_j (1 - X_j)\right| \end{aligned}$$

and therefore if we take $\sum_k^r{}^i \lambda_i (X_i - 1) \to 0$, keeping $\sum_k^r{}^i \lambda_i (\operatorname{Re} X_i - 1) < 0$, we must have $z_k \to \sum_1^{k-1}{}^i \lambda_i$.

Lemma 5.4.

$$\lim \left(\frac{\partial}{\partial X_k}\right)^l z_k(X_k, \ldots, X_r) \text{ for } \sum_k^r{}^i \lambda_i (X_i - 1) \to 0$$

exists, if we keep $\sum_k^r{}^i \lambda_i \operatorname{Re} X_i < \sum_k^r{}^i \lambda_i$, for every $l \in \{0, 1, \ldots, m\}$, if $\mu_i^{(m)} < \infty$ for every $i, k \in \{1, \ldots, r\}$ and $m \geqslant 1$.

Proof: $\left(\frac{\partial}{\partial X_k}\right) z_k(X_k, \ldots, X_r)$ can be obtained by partial differentiation of

$$z_k = \sum_1^{k-1}{}^i \lambda_i \varphi_i (\lambda - z_k - \sum_k^r{}^j \lambda_j X_j)$$

with respect to X_k, for $\sum_k^r{}^i \lambda_i \operatorname{Re} X_i < \sum_k^r{}^i \lambda_i$ and solving for $\frac{\partial z_k}{\partial X_k}$. We obtain a fraction, from which we find the higher partial derivatives by ordinary partial differentiation, applying the chain rule and substituting for those derivatives already obtained. Remembering that $\mu_i^{(m)} < \infty$ implies that $\varphi_i(\alpha)$ is an m times continuously differentiable function for $\operatorname{Re} \alpha \geqslant 0$, which may be differentiated under the integralsign, that (5.1) holds and lemma 5.3, we can easily verify the statement of the lemma.

If

$$A_{kl} \stackrel{\text{abb}}{=} \lim \left(\frac{\partial}{\partial X_k}\right)^l z_k (X_k, \ldots, X_r) \text{ for } \sum_k^r{}^i \lambda_i (X_i - 1) \to 0,$$

$$\text{keeping } \sum_k^r{}^i \lambda_i (X_i - 1) < 0,$$

we find

$$A_{k1} = \frac{\lambda_k \sum_1^{k-1}{}^i \lambda_i \mu_i^{(1)}}{1 - \sum_1^{k-1}{}^i \lambda_i \mu_i^{(1)}}, \tag{5.10}$$

$$A_{k2} = \frac{\lambda_k^2 \sum_1^{k-1}{}^i \lambda_i \mu_i^{(2)}}{(1 - \sum_1^{k-1}{}^i \lambda_i \mu_i^{(1)})^3}, \tag{5.11}$$

$$A_{k3} = \frac{\lambda_k^3 \sum_1^{k-1}{}^i \lambda_i \mu_i^{(3)}}{(1 - \sum_1^{k-1}{}^i \lambda_i \mu_i^{(1)})^4} + \frac{3 \lambda_k^3 (\sum_1^{k-1}{}^i \lambda_i \mu_i^{(2)})^2}{(1 - \sum_1^{k-1}{}^i \lambda_i \mu_i^{(1)})^5}. \tag{5.12}$$

From (5.6) we obtain, substituting $X_i = y_{k,i}$ for $i \in \{1, \ldots, k-1\}$

$$\begin{cases} \dfrac{X_k - \varphi_k(\lambda(1 - p(y_{(k)}, X)))}{\varphi_k(\lambda(1 - p(y_{(k)}, X)))} f_k(y_{(k)} X) = \\ = -\sum_{k+1}^r{}^i \dfrac{X_i - \varphi_i(\lambda(1 - p(y_{(k)}, X)))}{\varphi_i(\lambda(1 - p(y_{(k)}, X)))} f_i(y_{(k)} X) + (p(y_{(k)}, X) - 1)\, g(0^r) \end{cases} \tag{5.13}$$

and by using (5.4) we have

$$\begin{cases} \dfrac{X_k - \varphi_k(\lambda(1 - p(y_{(k)}, X)))}{\varphi_k(\lambda(1 - pX))} f_k(X) = \\ = -\sum_{k+1}^r{}^i \dfrac{X_i - \varphi_i(\lambda(1 - p(y_{(k)}, X)))}{\varphi_i(\lambda(1 - pX))} f_i(X) + (p(y_{(k)}, X) - 1)\, g(0^r) \end{cases} \tag{5.14}$$

for all X_j satisfying $|X_j| \leqslant 1$ for $j \neq k$ and $|X_k| < 1$.

We have

$$X_k = \varphi_k(\lambda(1-p(y_{(k)}, X)))$$

only for

$$X_k = y_{k+1,k}(X_{k+1}, \ldots, X_r)$$

and therefore the $f_k(X)$ can be obtained successively for all $X_1, \ldots, X_r$ satisfying $|X_j| \leqslant 1$ for $j \neq k$ and $|X_k| < 1$ (either directly or else by analytic continuation) from (5.14), starting with $f_r(X)$ if $g(0^r)$ is known. We shall not try to obtain the $f_k(X)$ explicitly, but use (5.14) in the sequel.

The constant $g(0^r)$ is determined by the condition

$$g(1^r) = 1. \tag{5.15}$$

If we take $X_i = 1$ in (5.14) for $i \neq k$ and, keeping $|X_k| < 1$, take $X_k \to 1$, we have from lemma 5.3, that both sides of (5.14) tend to zero. It can be seen, that $X_k \neq \varphi_k(\lambda(1-p(y_{(k)}, X, 1^{r-k})))$ for $X_k \neq 1$, therefore, always keeping $|X_k| < 1$ and using l'Hopitals' rule

$$\left\{\begin{aligned} f_k(1^r) &= \lim_{X_k \to 1} f_k(1^{k-1}\, X\, 1^{r-k}) = \\ &= -\sum_{k+1}^{r}{}^i f_i(1^r) \lim_{X_k \to 1} \frac{1-\varphi_i(\lambda(1-p(y_{(k)}, X, 1^{r-k})))}{X_k - \varphi_k(\lambda(1-p(y_{(k)}, X, 1^{r-k})))} + \\ &+ g(0^r) \lim_{X_k \to 1} \frac{p(y_{(k)}, X, 1^{r-k}) - 1}{X_k - \varphi_k(\lambda(1-p(y_{(k)}, X, 1^{r-k})))} = \\ &= \sum_{k+1}^{r}{}^i f_i(1^r) \frac{\mu_i^{(1)}(\lambda_k + A_{k,1})}{1-\mu_k^{(1)}(\lambda_k + A_{k,1})} + \frac{g(0^r)}{\lambda} \frac{\lambda_k + A_{k,1}}{1-\mu_k^{(1)}(\lambda_k + A_{k,1})} \end{aligned}\right. \tag{5.16}$$

or with (5.10)

$$f_k(1^r) = \frac{\sum_{k+1}^{r}{}^i f_i(1^r)\, \mu_i^{(1)} \lambda_k + g(0^r)\, p_k}{1 - \sum_{1}^{k}{}^i \lambda_i \mu_i^{(1)}}. \tag{5.17}$$

Solving (5.17) for $f_k(1^r)$ leads to

$$f_k(1^r) = \frac{p_k\, g(0^r)}{1 - \sum_{1}^{r}{}^i \lambda_i \mu_i^{(1)}} \qquad \text{for } k \in \{1, \ldots, r\}. \tag{5.18}$$

Because

$$g(1^r) = 1, \tag{5.19}$$

we finally have

$$f_k(1^r) = p_k \tag{5.20}$$

and

$$g(0^r) = 1 - \sum_{1}^{r}{}^i \lambda_i \mu_i^{(1)}. \tag{5.21}$$

We thus proved

Theorem 5.1. The functions $f_k(X)$ satisfy the equations

$$(5.22)\quad \begin{cases} f_k(X)=\dfrac{\varphi_k(\lambda(1-pX))}{X_k-\varphi_k(\lambda(1-p(y_{(k)},X)))}\cdot \\ \cdot\left\{-\sum\limits_{k+1}^{r}{}^{i} f_i(1^{i-1}X)\,\dfrac{X_i-\varphi_i(\lambda(1-p(y_{(k)},X)))}{\varphi_i(\lambda(1-p(1^{i-1},X)))}+(1-\sum\limits_{1}^{r}{}^{i}\lambda_i\mu_i^{(1)})\,(p(y_{(k)},X)-1)\right\} \end{cases}$$

for $|X_i|\leqslant 1 (i\neq k)$, $|X_k|<1$, $X_k\neq y_{k+1,k}(X_{k+1},\dots,X_r)$ and all $k\in\{1,\dots,r\}$. They can be obtained successively from these equations starting from $k=r$.

The derivation of (5.20) and (5.21) here given is unnecessarily long and complicated, but the same method leads us to the moments of the waitingtime distribution as we shall now show.

In section 4 we proved that in the nonsaturated case the $f_k(X)$ are powerseries with non-negative coefficients, absolutely convergent for $|X_1|\leqslant 1,\dots,|X_r|\leqslant 1$ and $k\in\{1,\dots,r\}$. If we differentiate a function of this kind n times ($n\in\{0,1,\dots\}$) with respect to one of its arguments and take the limits (in any order) $X_1\to 1,\dots,X_r\to 1$, keeping $|X_i|<1$ for all $i\in\{1,\dots,r\}$, then either the resulting expression is finite and the powerseries for this derivative converges for $|X_1|\leqslant 1,\dots,|X_r|\leqslant 1$ or the limit is $+\infty$. Moreover in all cases we have

$$(5.23)\qquad \left\{\left(\frac{\partial}{\partial X_k}\right)^n f_k(X)\right\}_{X_1=\dots=X_r=1}=\lim_{X\to 1}\left(\frac{\partial}{\partial X_k}\right)^n f_k(X)\qquad (|X|<1).$$

From (5.22) we see, that

$$(5.24)\qquad \lim_{X\to 1}\left(\frac{\partial}{\partial X_k}\right)^n f_k(X)\qquad (|X|<1)$$

exists if $\varphi_j(\lambda(1-pX))$ is $(n+1)$-times differentiable with respect to X_k for $j\in\{1,\dots,r\}$ and $k\in\{1,\dots,r\}$. This is certainly the case if the $(n+1)^{\mathrm{st}}$ moments of all $F_l(x)$ ($l\in\{1,\dots,r\}$) exist. If (as the only alternative) at least one of these moments is $+\infty$, then we find from (5.22)

$$(5.25)\qquad \lim_{X\to 1}\left(\frac{\partial}{\partial X_k}\right)^n f_k(X)=+\infty\qquad (|X|<1).$$

If we take $X_i=1$ for $i\neq k$ in (5.22), differentiate with respect to X_k, then let $X_k\to 1$ and use (5.10), (5.11) and (5.20) the result is

$$(5.26)\qquad \left(\frac{\partial f_k}{\partial X_k}\right)_{X_1=\dots=X_r=1}=\frac{\lambda_k^2\mu_k^{(1)}}{\lambda}+\frac{\lambda_k^2\sum\limits_{1}^{r}{}^{i}\lambda_i\mu_i^{(2)}}{2\lambda(1-\sum\limits_{1}^{k-1}{}^{i}\lambda_i\mu_i^{(1)})(1-\sum\limits_{1}^{k}{}^{i}\lambda_i\mu_i^{(1)})},$$

whilst we find in the same way from the second partial derivative of (5.22) with respect to X_k

$$(5.27)\begin{cases}\left(\dfrac{\partial^2 f_k}{\partial X_k^2}\right)_{X_1=\ldots=X_r=1} = \dfrac{\lambda_k^3\mu_k^{(2)}}{\lambda} + \dfrac{\lambda_k^3 u_k^{(1)}\sum_1^r{}^i\, \lambda_i\mu_i^{(2)}}{\lambda(1-\sum_1^{k-1}{}^i\,\lambda_i\mu_i^{(1)})(1-\sum_1^{k}{}^i\,\lambda_i\mu_i^{(1)})} + \\ + \dfrac{\lambda_k^3\sum_1^r{}^i\,\lambda_i\mu_i^{(3)}}{3\lambda(1-\sum_1^{k-1}{}^i\,\lambda_i\mu_i^{(1)})^2(1-\sum_1^{k}{}^i\,\lambda_i\mu_i^{(1)})} + \dfrac{\lambda_k^3\sum_1^r{}^i\,\lambda_i\mu_i^{(2)}\sum_1^k{}^j\,\lambda_j\mu_j^{(2)}}{2\lambda(1-\sum_1^{k-1}{}^i\,\lambda_i\mu_i^{(1)})^2(1-\sum_1^{k}{}^i\,\lambda_i\mu_i^{(1)})^2} + \\ + \dfrac{\lambda_k^3\sum_1^r{}^i\,\lambda_i\mu_i^{(2)}\sum_1^{k-1}{}^j\,\lambda_j\mu_j^{(2)}}{2\lambda(1-\sum_1^{k-1}{}^i\,\lambda_i\mu_i^{(1)})^3(1-\sum_1^{k}{}^i\,\lambda_i\mu_i^{(1)})}.\end{cases}$$

From (5.3) we have by differentiating with respect to X_k

$$(5.28)\begin{cases}\left(\dfrac{\partial f_k}{\partial X_k}\right)_{X_1=\ldots=X_r=1} = f_k(1^r)\left(\dfrac{d}{dX_k}\{\varphi_k(\lambda_k(1-X_k))\,\psi_k(\lambda_k(1-X_k'))\}\right)_{X_k=1} = \\ = f_k(1^r)\,\{\lambda_k\mu_k^{(1)} + \lambda_k\,\mathscr{E}\mathbf{w}_k\},\end{cases}$$

$$(5.29)\qquad \left(\frac{\partial^2 f_k}{\partial X_k^2}\right)_{X_1=\ldots=X_r=1} = f_k(1^r)\,\{\lambda_k^2\mu_k^{(2)} + 2\lambda_k^2\mu_k^{(1)}\,\mathscr{E}\mathbf{w}_k + \lambda_k^2\,\mathscr{E}\mathbf{w}_k^2\},$$

if $\mathscr{E}\mathbf{w}_k$ and $\mathscr{E}\mathbf{w}_k^2$ are the first and second moment of the stationary waitingtime distribution $H_k(t)$ respectively.

On combining (5.26), (5.27), (5.28) and (5.29) we obtain:

Theorem 5.2. The first and second moment of the stationary waitingtime distribution $H_k(t)$, for $k\in\{1,\ldots,r\}$, are respectively

$$(5.30)\qquad \mathscr{E}\mathbf{w}_k = \frac{\sum_1^r{}^i\,\lambda_i\mu_i^{(2)}}{2(1-\sum_1^{k-1}{}^i\,\lambda_i\mu_i^{(1)})(1-\sum_1^{k}{}^i\,\lambda_i\mu_i^{(1)})}$$

and

$$(5.31)\begin{cases}\mathscr{E}\mathbf{w}_k^2 = \dfrac{\sum_1^r{}^i\,\lambda_i\mu_i^{(3)}}{3(1-\sum_1^{k-1}{}^i\,\lambda_i\mu_i^{(1)})^2(1-\sum_1^{k}{}^i\,\lambda_i\mu_i^{(1)})} + \\ + \dfrac{\sum_1^r{}^i\,\lambda_i\mu_i^{(2)}\sum_1^k{}^j\,\lambda_j\mu_j^{(2)}}{2(1-\sum_1^{k-1}{}^i\,\lambda_i\mu_i^{(1)})^2(1-\sum_1^{k}{}^i\,\lambda_i\mu_i^{(1)})^2} + \dfrac{\sum_1^r{}^i\,\lambda_i\mu_i^{(2)}\sum_1^{k-1}{}^j\,\lambda_j\mu_j^{(2)}}{2(1-\sum_1^{k-1}{}^i\,\lambda_i\mu_i^{(1)})^3(1-\sum_1^{k}{}^i\,\lambda_i\mu_i^{(1)})}.\end{cases}$$

Our (5.30) is COBHAM's formula (3) (see [2]).

The function $\psi_k(\alpha)$ can be found from (5.3), at least for $|1-\frac{\alpha}{\lambda_k}|<1$,

$$(5.32)\qquad \psi_k(\alpha) = \frac{\lambda f_k\left(1,\ldots,1,1-\frac{\alpha}{\lambda_k},1,\ldots,1\right)}{\lambda_k\varphi_k(\alpha)},$$

which, if combined with (5.22), leads to

$$(5.33)\quad \psi_k(\alpha) = \frac{(1-\sum_1^r{}^i\,\lambda_i\mu_i^{(1)})\,(-\sum_1^{k-1}{}^j + z_k^* - \alpha) - \sum_{k+1}^r{}^i\,\lambda_i(1-\varphi_i(\sum_1^{k-1}{}^j\,\lambda_j - z_k^* + \alpha))}{\lambda_k - \alpha - \lambda_k\varphi_k(\sum_1^{k-1}{}^j\,\lambda_j - z_k^* + \alpha)},$$

where $z_1^* = 0$ and $z_k^* = z_k^*(\alpha)$ satisfies (5.8) for $X_k = 1 - \frac{\alpha}{\lambda}$, $X_i = 1$ ($i \neq k$) and $k \geqslant 2$) i.e.

$$z_k^* - \sum_1^{k-1} {}^i \lambda_i \varphi_i \left(\sum_1^{k-1} {}^j \lambda_j - z_k^* + \alpha \right) = 0. \tag{5.34}$$

Therefore $\psi_1(\alpha)$ *is explicitly given by*

$$\psi_1(\alpha) = -\frac{\left(1 - \sum_1^r {}^i \lambda_i \mu_i^{(1)}\right)\alpha + \sum_2^r {}^i \lambda_i (1 - \varphi_i(\alpha))}{\lambda_1 - \alpha - \lambda_1 \varphi_1(\alpha)}, \tag{5.35}$$

while $\psi_k(\alpha)$ for $k \in \{2, \ldots, r\}$ contains the z_k^*.

As an illustration we give the following example:

Take $r = 2$, $F_1(x) = F_2(x) = 1 - \exp\left(-\frac{x}{\mu}\right)$, then

$$\varphi_1(\alpha) = \varphi_2(\alpha) = \frac{1}{\alpha\mu + 1}, \tag{5.36}$$

$$\psi_1(\alpha) = \frac{1 - \lambda_1\mu + \alpha\mu + \alpha\lambda\mu^2}{1 - \lambda_1\mu + \alpha\mu}, \tag{5.37}$$

$$\psi_2(\alpha) = \frac{(1 - \lambda\mu)(-\lambda_1 + z_2^* - \alpha)\{(\lambda_1 - z_2^* + \alpha)\mu + 1\}}{(\lambda_2 - \alpha)\{(\lambda_1 - z_2^* + \alpha)\mu + 1\} - \lambda_2}, \tag{5.38}$$

which leads to the following waitingtime distributions ($t \geqslant 0$)

$$H_1(t) = 1 - \lambda\mu \exp\left\{-\frac{(1 - \lambda_1\mu)t}{\mu}\right\}, \tag{5.39}$$

$$\left\{\begin{aligned} &H_2(t) = 1 - \lambda\mu + \frac{\lambda^2\mu}{\lambda_2}\left(1 - \exp\left\{-\frac{\lambda_2(1 - \lambda\mu)t}{\lambda\mu}\right\}\right) + \\ &- 2\lambda_1(1 - \lambda\mu)\int_0^t ds \int_0^s \frac{I_1(2u\sqrt{\lambda_1/\mu})}{2u\sqrt{\lambda_1\mu}} \exp\left\{-\frac{\lambda_1 + \lambda^2\mu}{\lambda\mu}u\right\} du, \end{aligned}\right. \tag{5.40}$$

where $I_1(x)$ is the modified Besselfunction of the first order and of the first kind, i.e.

$$I_1(x) = \sum_0^\infty {}^n \frac{x^{2n+1}}{2^{2n+1} n! (n+1)!}. \tag{5.41}$$

The result (5.40) contradicts equation (27) as given by R. E. Cox [4].

6. *The case of saturation*

If (5.1) is not satisfied, we can find a positive integer s, $0 \leqq s < r$, such that

$$\sum_1^s {}^i \lambda_i \mu_i^{(1)} < 1, \quad \sum_1^{s+1} {}^i \lambda_i \mu_i^{(1)} \geqslant 1. \tag{6.1}$$

In section 4 we stated already without proof, that

$$f_k(X) \stackrel{\text{def}}{=} \lim_{n\to\infty} f_{k,n}(X) \tag{6.2}$$

exists for $k \in \{1, \ldots, r\}$ and that

$$f_k(X) = 0 \tag{6.3}$$

if at least one X_j satisfies $|X_j| < 1$ for $j \in \{s+1, \ldots, r\}$.

As a consequence of (6.3), it cannot be true that

$$f_k(1^r) = \lim_{X \to 1} f_k(X), \tag{6.4}$$

as the right hand side in (6.4) equals 0 and

$$g(1^r) = 1. \tag{6.5}$$

The functions $f_k(X)$ thus cannot be powerseries with positive coefficients and the method of section 4 cannot be applied.

But if instead of $f_k(X)$ only $f_k(X 1^{r-s})$ is considered, we can repeat the argument of section 5 with some alterations.

From (5.2) and (6.3) we have at once

$$f_k(X) = 0 \tag{6.6}$$

for $k \in \{s+2, \ldots, r\}$ and $|X_i| \leqslant 1$, for all $i \in \{1, \ldots, r\}$.

If for $k \in \{1, \ldots, r\}$

$$\bar{f}_k(X) \stackrel{\text{def}}{=} f_k(X\, 1^{r-s}) \tag{6.7}$$

one can prove that for $k \in \{1, \ldots, s+1\}$, $\bar{f}_k(X)$ again is a powerseries with non-negative coefficients, absolutely convergent for $|X_1| \leqslant 1, \ldots, |X_s| \leqslant 1$ and satisfying

$$\sum_1^{s+1}{}^k \bar{f}_k(1^r) = 1.$$

From (5.2) we have for $k \in \{1, \ldots, s\}$ and $X_{s+1} = \ldots = X_r = 1$

$$X_k \bar{f}_k(X) = \sum_1^{s+1}{}^i \{\bar{f}_i(0^{k-1} X\, 1^{r-s}) - \bar{f}_i(0^k X\, 1^{r-s})\}\, \varphi_k\Big(\sum_1^s{}^i \lambda_i (1 - X_i)\Big) \tag{6.8}$$

and for $k = s+1$

$$\bar{f}_{s+1}(X) = \sum_1^{s+1}{}^i \bar{f}_i(0^r)\, \varphi_{s+1}\Big(\sum_1^s{}^i \lambda_i\, (1 - X_i)\Big). \tag{6.9}$$

From (6.8) and (6.9)

$$\left\{\begin{aligned} X_k \bar{f}_k(X) &= \sum_1^s{}^i \{\bar{f}_i(0^{k-1} X\, 1^{r-s}) - \bar{f}_i(0^k X\, 1^{r-s})\} + \\ &+ \sum_1^{s+1}{}^i \bar{f}_i(0^r)\Big\{\varphi_{s+1}\Big(\sum_1^s{}^j \lambda_j - \sum_k^s{}^j \lambda_j X_j\Big) - \varphi_{s+1}\Big(\sum_1^s{}^j \lambda_j - \sum_{k+1}^s{}^j \lambda_j X_j\Big)\Big\}\varphi_k\Big(\sum_1^s{}^i \lambda_i (1 - X_i)\Big). \end{aligned}\right. \tag{6.10}$$

Equation (6.10) is the analogue of (5.2), while the analogue of (5.4) is (for $|X_i| \leqslant 1$, $i \in \{1, \ldots, s\}$ and $|U_j| \leqslant 1$, $j \in \{1, \ldots, k-1\}$ and $k \in \{2, \ldots, s\}$)

$$\frac{\bar{f}_k(X\, 1^{r-s})}{\varphi_k(\lambda(1 - p(X, 1^{r-s})))} = \frac{\bar{f}_k(U^{(k)} X\, 1^{r-s})}{\varphi_k(\lambda(1 - p(U^{(k)}, X, 1^{r-s})))} \tag{6.11}$$

and (5.3) can be written

(6.12) $$\bar{f}_k(1^{k-1}X1^{r-k})=\bar{f}_k(1^r)\,\psi_k(\lambda_k(1-X_k))\,\varphi_k(\lambda_k(1-X_k))$$

for $k\in\{1,\ldots,s\}$. Therefore the moments of the waitingtime distribution can be found as in section 5 for $k\in\{1,\ldots,s\}$. One obtains

(6.13) $$\bar{f}_k(1^r)=\frac{\lambda_k\mu_{s+1}^{(1)}}{1-\sum_1^s{}^i\,\lambda_i\mu_i^{(1)}+\mu_{s+1}^{(1)}\sum_1^s{}^i\,\lambda_i}\qquad\text{for } k\in\{1,\ldots,s\},$$

(6.14) $$\bar{f}_{s+1}(1^r)=\frac{1-\sum_1^s{}^i\,\lambda_i\mu_i^{(1)}}{1-\sum_1^s{}^i\,\lambda_i\mu_i^{(1)}+\mu_{s+1}\sum_1^s{}^i\,\lambda_i},$$

(6.15) $$\sum_1^{s+1}{}^i\,\bar{f}_i(0^r)=\frac{1-\sum_1^s{}^i\,\lambda_i\mu_i^{(1)}}{1-\sum_1^s{}^i\,\lambda_i\mu_i^{(1)}+\mu_{s+1}\sum_1^s{}^i\,\lambda_i},$$

(6.16) $$\mathscr{E}\mathbf{w}_k=\frac{\sum_1^s{}^i\,\lambda_i\mu_i^{(2)}+\dfrac{\mu_{s+1}^{(2)}}{\mu_{s+1}^{(1)}}\,(1-\sum_1^{r_s}{}^i\,\lambda_i\mu_i^{(1)})}{2\,(1-\sum_1^{k-1}{}^i\,\lambda_i\mu_i^{(1)})\,(1-\sum_1^k{}^i\,\lambda_i\mu_i^{(1)})}\qquad\text{for } k\in\{1,\ldots,s\},$$

which is COBHAM's formula (cf. [3]).

In addition one can prove, that

$$\mathscr{E}\mathbf{w}_k=\infty\quad\text{for}\quad k\in\{s+1,\ldots,r\}.$$

The authors wish to thank Prof. Dr. D. VAN DANTZIG for suggesting the use of his "method of collective marks" and for his valuable advice and criticism, which helped to give the paper its final form.

Mathematical Centre, Amsterdam

REFERENCES

1. S. BOCHNER, W. T. MARTIN, Several complex variables, (Princeton University Press, Princeton, 1948)
2. A. COBHAM, Priority assignment in waiting line problems, J. Op. Res. Soc. of America, 2, 70–76 (1954).
3. A. COBHAM, Priority assignment—a correction, J. Op. Res. Soc. of America, 3, 547–547 (1955).
4. R.E. COX, Traffic flow in an exponential delay system with priority categories, Proc. Inst. El. Eng., 815–818 (Nov. 1955).
5. D. VAN DANTZIG, Kadercursus Statistiek, Hoofdstuk 2, Stencilled notes of lectures held at the University of Amsterdam, (1947).
6. ———, Sur la méthode des fonctions génératrices, Colloques internationaux du centre national de la recherche scientifique 13, 29–45 (1948).
7. ———, Chaînes de Markof dans les ensembles abstraits et applications aux processus avec régions absorbantes et au problème des boucles, Ann. de l'Inst. H. Poincaré, 14 (facs. 3) 145–199 (1955).
8. W. FELLER, An introduction to probability theory and its applications (Wiley & Sons, Inc. New York, 1950).

9. F. G. FOSTER, On the stochastic matrices associated with certain queuing processes, Ann. Math. Stat. 24, 355–360 (1953).
10. D. G. KENDALL, Some Problems in the theory of queues, J. Roy. Stat. Soc. B13, 151–185 (1951).
11. P. LEVY, Théorie de l'addition des variables aléatoires (Gauthier-Villars, Paris, 1937).
12. M. D. MOUSTAFA, Input-output Markov processes, Proc. Kon. Ned. Ak. v. Wet. A 60, Indagationes Mathematicae, 19, 112–118 (1957).
13. E. C. TITCHMARSH, The Theory of Functions (Oxford University Press, Second Edition, 1939).
14. D. V. WIDDER, The Laplace transform (Princeton University Press, Princeton, 1946).

Note added in proof.

If we compare equation (5.8) with equation (49) of L. TAKÁCS paper "Investigation of waiting time problems by reduction to Markov processes", Acta Mathematica Acad. Sc. Hung. VI, 101—129 (1955), it turns out that (5.8) can be regarded as a special case of (49) and therefore z_k can be considered as the LAPLACE transform of a (proper) distribution function. Lemma 2.2 and lemma 5.3 now become obvious.