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MATHEMATICS

A GENERAL CLASS OF DISTRIBUTIONFREE TESTS FOR SYMMETRY CONTAINING THE TESTS OF WILCOXON AND FISHER [1]). I

BY

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1. *Introduction*

In this paper a class of tests for symmetry will be considered which is closely related with the class of two sample tests described in [4] (p. 251). Special cases are e.g. the tests for symmetry of F. WILCOXON (cf. [22], [23], [24], [25]) and of R. A. FISHER [7] (p. 43–47) and the sign test.

In section 2 a description of the tests will be given and in section 3 some properties of the distribution of the test statistic under the hypothesis tested will be proved. In section 4 the relation with the class of two sample tests, described in [4], will be given and in section 5 the consistency of the tests of WILCOXON and FISHER will be investigated. In section 6 a combination of the sign test and the class of tests for symmetry will be given.

All theorems in this paper hold for the case with ties as well as the case without ties.

2. *Description of the tests*

Consider m independent random variables $\mathbf{z}_1, \ldots, \mathbf{z}_m$ [2]) representing a series of observations. In this paper a class of tests is described for the hypothesis H_0 that the probability distributions of $\mathbf{z}_1, \ldots, \mathbf{z}_m$, which need not be identical, are all symmetrical with respect to zero.

The test statistic is defined as follows. The observations which are equal to zero are omitted. Let the remaining observations consist of $\mathbf{a}_i$ times the value $\mathbf{u}_i$ $(i=1, \ldots, k)$, where $0<\mathbf{u}_1< \ldots <\mathbf{u}_k$ and $\mathbf{b}_i$ times the value $-\mathbf{u}_i$ $(i=1, \ldots, \mathbf{k})$. Let further

$$(2.1)\qquad \begin{cases} \mathbf{n}_1 \overset{\text{def}}{=} \sum_{i=1}^{\mathbf{k}} \mathbf{a}_i, \quad \mathbf{n}_2 \overset{\text{def}}{=} \sum_{i=1}^{\mathbf{k}} \mathbf{b}_i, \\ \mathbf{t}_i \overset{\text{def}}{=} \mathbf{a}_i+\mathbf{b}_i \; (i=1,\ldots,\mathbf{k}), \quad \mathbf{n} \overset{\text{def}}{=} \mathbf{n}_1+\mathbf{n}_2 \end{cases}$$

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1) Report SP 54 of the Statistical Department of the Mathematical Centre, Amsterdam.

2) Random variables will be distinguished from numbers (e.g. from the values they take in an experiment) by printing their symbols in bold type.

and

$$r_i \stackrel{\text{def}}{=} \sum_{j=1}^{i} t_j - \tfrac{1}{2}(t_i - 1) \quad (i=1,\ldots,k). \tag{2.2}$$

Then r_i is the rank of the absolute value of the observations which are equal to u_i or $-u_i$ according to increasing size ($i=1, \ldots, k$). The t_i are the sizes of the ties. Finally let $\varphi(u, r)$ be a given function of u and r and let $\varphi_i \stackrel{\text{def}}{=} \varphi(u_i, r_i)$ ($i=1, \ldots, k$). The test is executed under the conditions $k=k$, $t_1=t_1, \ldots, t_k=t_k$, $u_1=u_1, \ldots, u_k=u_k$ ((k, t, u) for short) and the test statistic is

$$T \stackrel{\text{def}}{=} \sum_{i=1}^{k} \varphi_i \cdot (a_i - b_i). \tag{2.3}$$

The distribution of T under the hypothesis H_0 and under the condition (k, t, u) is symmetrical with respect to zero and may be calculated by means of a recursion formula (cf. section 3). Let $\mathrm{P}[T=T|(k, t, u); H_0]$ denote the probability that T assumes the value T under the hypothesis H_0 and under the condition (k, t, u); let further T_α denote the smallest value of T satisfying

$$\mathrm{P}[T \geqq T|(k, t, u); H_0] \leqq \alpha. \tag{2.4}$$

Then the following critical regions are used

$$\begin{cases} Z_l: \ T \leqq -T_\alpha, \\ Z_r: \ T \geqq T_\alpha, \\ Z\ : \ |T| \geqq T_{\frac{1}{2}\alpha}. \end{cases} \tag{2.5}$$

The conditional asymptotic normality of T under the hypothesis H_0 for $n \to \infty$ will be investigated in section 3 (theorem V).

Special cases

If

$$\varphi(u, r) = r \tag{2.6}$$

then we obtain the test statistic T_W of Wilcoxon's test for symmetry

$$T_W = \sum_{i=1}^{k} r_i (a_i - b_i). \tag{2.7}$$

A table of the distribution of T_W under the hypothesis H_0 and under the condition $n=n$ for the untied case and for $n=3(1)20$ may be found in [1] (p. 23–27). The local powerfunction of this test has been investigated by E. L. Lehmann [10] and has been compared with the power functions of the sign test and the tests for symmetry of J. Hemelrijk ([8] and [9]) and N. V. Smirnov [16] by E. Ruist [14].

The test statistic T_F of Fisher's test for symmetry is obtained by substituting u_i for φ_i

$$T_F = \sum_{h=1}^{m} z_h = \sum_{i=1}^{k} u_i (a_i - b_i). \tag{2.8}$$

If φ is independent of u and r, then the test statistic (2.3) reduces to

$$\varphi \sum_{i=1}^{k} (\boldsymbol{a}_i - \boldsymbol{b}_i) = \varphi (\boldsymbol{n}_1 - \boldsymbol{n}_2). \tag{2.9}$$

Thus in this case the test is identical with the sign test.

Remarks

1. WILCOXON uses as a test statistic for his test the sum of the ranks of the positive observations. Denoting this statistic by $\boldsymbol{T}'_W$ we have

$$\boldsymbol{T}'_W = \sum_{i=1}^{k} r_i \, \boldsymbol{a}_i = \tfrac{1}{2} \, \boldsymbol{T}_W + \tfrac{1}{4} \, n \, (n+1). \tag{2.10}$$

Tables of the lefthandsided critical values of $\boldsymbol{T}'_W$ for the untied case may be found in [24] and [25]. These critical values are defined as the values of T'_W which minimize $|\mathrm{P}\,[\boldsymbol{T}'_W \leqq T'_W \,|\, n;\, H_0] - \alpha|$ [3]).

2. By means of the tests described in this paper one may also test the hypothesis H'_0 that the probability distributions of $\boldsymbol{z}_h$ are symmetrical with respect to given points $c_h (h = 1, \ldots, m)$ by applying the test to $\boldsymbol{z}_1 - c_1, \ldots, \boldsymbol{z}_m - c_m$.

3. *Some properties of the distribution of* $\boldsymbol{T}$ *under the hypothesis* H_0 *and under the conditions* $\boldsymbol{k} = k$, $\boldsymbol{t}_1 = t_1, \ldots, \boldsymbol{t}_k = t_k$, $\boldsymbol{u}_1 = u_1, \ldots, \boldsymbol{u}_k = u_k$.

Theorem I:

$$\begin{cases} \mathrm{P}\,[\boldsymbol{T} = T \mid k, t_1, \ldots, t_k, u_1, \ldots, u_k;\, H_0] = \\ = 2^{-t_k} \sum_{\gamma=0}^{t_k} \binom{t_k}{\gamma} \mathrm{P}\,[\boldsymbol{T} = T - (2\gamma - t_k)\, \varphi_k \mid k-1, t_1, \ldots, t_{k-1}, u_1, \ldots, u_{k-1};\, H_0]. \end{cases} \tag{3.1}$$

Proof:

Let E_γ denote the event that the tie of size t_k consists of γ positive and $t_k - \gamma$ negative observations; then

$$\mathrm{P}\,[E_\gamma \mid H_0] = 2^{-t} \binom{t_k}{\gamma} \quad (\gamma = 0, \ldots, t_k). \tag{3.2}$$

If E_γ occurs then the contribution of the observations in the tie of size t_k to the test statistic is

$$\{\gamma - (t_k - \gamma)\}\, \varphi_k = (2\gamma - t_k)\, \varphi_k \quad (\gamma = 0, \ldots, t_k). \tag{3.3}$$

If, on the other hand, (3.3) is the contribution of the observations in the tie of size t_k to $\boldsymbol{T}$ then this tie must contain exactly γ positive and $t_k - \gamma$ negative observations. Thus

$$\begin{cases} \mathrm{P}\,[\boldsymbol{T} = T \mid k, t_1, \ldots, t_k, u_1, \ldots, u_k;\, E_\gamma, H_0] = \\ = \mathrm{P}\,[\boldsymbol{T} = T - (2\gamma - t_k)\varphi_k \mid k-1, t_1, \ldots, t_{k-1}, u_1, \ldots, u_{k-1};\, H_0]. \end{cases} \tag{3.4}$$

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[3]) The tables in [22] and [23] contain mistakes, which have been corrected in [24] and [25].

The recursion formula (3.1) then follows from (3.2), (3.4) and

$$(3.5)\quad \begin{cases} \mathrm{P}[\mathbf{T}=T \mid k, t_1, \ldots, t_k, u_1, \ldots, u_k; H_0] = \\ = \sum_{\gamma=0}^{t_k} \mathrm{P}[E_\gamma \mid H_0]\, \mathrm{P}[\mathbf{T}=T \mid k, t_1, \ldots, t_k, u_1, \ldots, u_k; E_\gamma, H_0]. \end{cases}$$

If $t_i = 1$ for each $i = 1, \ldots, n$ (no ties), (3.1) reduces to

$$(3.6)\quad \begin{cases} 2\mathrm{P}[\mathbf{T}=T \mid n, u_1, \ldots, u_n; H_0] = \\ = \mathrm{P}[\mathbf{T}=T+\varphi_n \mid n-1, u_1, \ldots, u_{n-1}; H_0] + \\ \qquad + \mathrm{P}[\mathbf{T}=T-\varphi_n \mid n-1, u_1, \ldots, u_{n-1}; H_0]. \end{cases}$$

Remarks:

3. The recursion formula (3.1) is analogous to the formula derived by L. J. SMID [15] for the distribution of the test statistic of WILCOXON's two sample test.

4. For the case of WILCOXON's test for symmetry (3.6) reduces to

$$(3.7)\quad 2\mathrm{P}[\mathbf{T}_W=T \mid n; H_0] = \mathrm{P}[\mathbf{T}_W=T-n \mid n-1; H_0] + \mathrm{P}[\mathbf{T}_W=T+n \mid n-1; H_0].$$

This formula may also be found in [19] (p. 15).

Theorem II:

$$(3.8)\quad \mathscr{E}(X^{\mathbf{T}} \mid (k, t, u); H_0) = 2^{-n} \prod_{i=1}^{k} (X^{\varphi_i} + X^{-\varphi_i})^{t_i}.$$

Proof:

From (2.3) it follows that

$$(3.9)\quad \mathbf{T} = \sum_{i=1}^{k} (2\mathbf{a}_i - t_i)\, \varphi_i$$

and from (3.9) and the fact that $\mathbf{a}_1, \ldots, \mathbf{a}_k$ are distributed independently follows

$$(3.10)\quad \mathscr{E}(X^{\mathbf{T}} \mid (k, t, u); H_0) = \prod_{i=1}^{k} \mathscr{E}(X^{\varphi_i(2\mathbf{a}_i - t_i)} \mid t_i, u_i; H_0).$$

Further $\mathbf{a}_i$ possessing a binomial probability distribution with parameters $(t_i, \frac{1}{2})$, we have

$$(3.11)\quad \mathscr{E}(X^{\varphi_i(2\mathbf{a}_i - t_i)} \mid t_i, u_i; H_0) = 2^{-t_i}(X^{\varphi_i} + X^{-\varphi_i})^{t_i} \qquad (i = 1, \ldots, k).$$

From (3.10) and (3.11) then follows

$$(2.12)\quad \mathscr{E}(X^{\mathbf{T}} \mid (k, t, u); H_0) = 2^{-n} \prod_{i=1}^{k} (X^{\varphi_i} + X^{-\varphi_i})^{t_i}.$$

Remark:

5. (3.8) may also be deduced from the recursion formula. From (3.1) it follows that

$$(3.13)\quad \begin{cases} \mathscr{E}(X^{\mathbf{T}} \mid k, t_1, \ldots, t_k, u_1, \ldots, u_k; H_0) = \\ = 2^{-t_k} \sum_{\gamma=0}^{t_k} \binom{t_k}{\gamma} X^{\varphi_k(2\gamma - t_k)}\, \mathscr{E}(X^{\mathbf{T}} \mid k-1, t_1, \ldots, t_{k-1}, u_1, \ldots, u_{k-1}; H_0) = \\ = 2^{-t_k}(X^{\varphi_k} + X^{-\varphi_k})^{t_k}\, \mathscr{E}(X^{\mathbf{T}} \mid k-1, t_1, \ldots, t_{k-1}, u_1, \ldots, u_{k-1}; H_0) \end{cases}$$

and (3.8) follows from (3.13).

Now let $\varkappa_\nu$ denote the ν-th cummulant of the distribution of $\boldsymbol{T}$ under the hypothesis H_0 and under the condition (k, t, u), i.e. $\varkappa_\nu$ is the coefficient of $\tau^\nu/\nu!$ in the expansion of $\ln \mathscr{E}(e^{\tau \boldsymbol{T}} | (k, t, u); H_0)$. Then we have [4])

Theorem III:

$$\varkappa_{2\nu+1} = 0 \quad (\nu = 0, 1, \ldots) \tag{3.14}$$

and

$$\varkappa_{2\nu} = \frac{2^{2\nu}(2^{2\nu}-1) B_{2\nu}}{2\nu} \sum_{i=1}^{k} t_i \varphi_i^{2\nu} \quad (\nu = 1, 2, \ldots), \tag{3.15}$$

where $B_{2\nu}$ are Bernoulli's numbers.

Proof:

From (3.8) it follows that

$$\mathscr{E}(e^{\tau \boldsymbol{T}} | (k, t, u); H_0) = \prod_{i=1}^{k} (ch\, \tau \varphi_i)^{t_i}, \tag{3.16}$$

thus

$$\ln \mathscr{E}(e^{\tau \boldsymbol{T}} | (k, t, u); H_0) = \sum_{i=1}^{k} t_i \ln ch\, \tau \varphi_i. \tag{3.17}$$

Further we have

$$\ln ch\, x = \int_0^x th\, u\, du \tag{3.18}$$

and

$$th\, u = \sum_{\nu=1}^{\infty} \frac{2^{2\nu}(2^{2\nu}-1) B_{2\nu}}{(2\nu)!} u^{2\nu-1}, \tag{3.19}$$

thus

$$\ln ch\, x = \sum_{\nu=1}^{\infty} \frac{2^{2\nu}(2^{2\nu}-1) B_{2\nu}}{(2\nu)!} \frac{x^{2\nu}}{2\nu}. \tag{3.20}$$

From (3.17) and (3.20) then follows

$$\ln \mathscr{E}(e^{\tau \boldsymbol{T}} | (k, t, u); H_0) = \sum_{\nu=1}^{\infty} \frac{\tau^{2\nu}}{(2\nu)!} \frac{2^{2\nu}(2^{2\nu}-1) B_{2\nu}}{2\nu} \sum_{i=1}^{k} t_i \varphi_i^{2\nu}. \tag{3.21}$$

Thus the coefficient of $\frac{\tau^{2\nu+1}}{(2\nu+1)!}$ is

$$\varkappa_{2\nu+1} = 0 \quad (\nu = 0, 1, \ldots) \tag{3.22}$$

and the coefficient of $\frac{\tau^{2\nu}}{(2\nu)!}$ is

$$\varkappa_{2\nu} = \frac{2^{2\nu}(2^{2\nu}-1) B_{2\nu}}{2\nu} \sum_{i=1}^{k} t_i \varphi_i^{2\nu} \quad (\nu = 1, 2, \ldots). \tag{3.23}$$

From (3.22) it follows that the distribution of $\boldsymbol{T}$ under the hypothesis H_0 and under the condition (k, t, u) is symmetrical and

$$\varkappa_1 = \mathscr{E}(\boldsymbol{T} | (k, t, u); H_0) = 0. \tag{3.24}$$

4) Cf. also D. van Dantzig [2] (Chapter VI) for the cummulants of the distribution of the test statistic of Wilcoxon's two sample test for the untied case.

From (3.23) it follows that

$$\varkappa_2 = \sigma^2(T \mid (k, t, u); H_0) = \sum_{i=1}^{k} t_i \varphi_i^2. \tag{3.25}$$

Special cases

From (3.25) and (2.7) it follows that

$$\sigma^2(T_W \mid (k, t); H_0) = \sum_{i=1}^{k} t_i r_i^2 = \frac{n^3 - \sum_{i=1}^{k} t_i^3 + 3n(n+1)^2}{12} \tag{3.26}$$

and if $t_i = 1$ for each $i = 1, \ldots, n$ then (3.26) reduces to

3.27) $\sigma^2(T_W \mid n; H_0) = \frac{1}{6} n(n+1)(2n+1)$ (cf. [24] and [25]) [5]).

Further it follows from (3.24) and (2.10)

(3.28) $\mathscr{E}(T'_W \mid (k, t); H_0) = \frac{1}{4} n(n+1)$ (cf. [22], [23], [24] and [25]).

A table of σ^2 and σ according to formula (3.27) may be found in [1] (p. 30).

From (2.8) and (3.25) follows

$$\sigma^2(T_F \mid (k, t, u); H_0) = \sum_{i=1} t_i u_i^2 = \sum_{h=1}^{m} z_h^2. \tag{3.29}$$

In the following theorem a necessary and sufficient condition will be given for a constant difference between the successive values T assumes under the condition (k, t, u).

The smallest value T assumes is

$$T_{min} = \sum_{\varphi_i < 0} t_i \varphi_i - \sum_{\varphi_i > 0} t_i \varphi_i = -\sum_{i=1}^{k} t_i |\varphi_i|, \tag{3.30}$$

thus

$$T - T_{min} = \sum_{i=1}^{k} \varphi_i (a_i - b_i) + \sum_{i=1}^{k} t_i |\varphi_i| = 2 \sum_{\varphi_i < 0} |\varphi_i| b_i + 2 \sum_{\varphi_i > 0} |\varphi_i| a_i. \tag{3.31}$$

Now let $|\varphi_1|, \ldots, |\varphi_k|$ consist of k' different values $\varphi'_1, \ldots, \varphi'_{k'}$ with

$$0 < \varphi'_1 < \ldots < \varphi'_{k'} \tag{3.32}$$

and let

$$\begin{cases} t'_i \overset{\text{def}}{=} \sum\limits_{\substack{j \\ |\varphi_j| = \varphi_i'}} t_j, \\ a'_i \overset{\text{def}}{=} \sum\limits_{\substack{j \\ \varphi_j = \varphi'}} a_j + \sum\limits_{\substack{j \\ \varphi_j = -\varphi_i'}} b_j, \qquad (i = 1, \ldots, k') \\ b'_i \overset{\text{def}}{=} t'_i - a'_i, \end{cases} \tag{3.33}$$

then $T - T_{min}$ may be written in the form

$$T' \overset{\text{def}}{=} T - T_{min} = 2 \sum_{i=1}^{k'} \varphi'_i a'_i. \tag{3.34}$$

[5]) The fomulae for the variance of T'_W in [22] and [23] are in error, but have been corrected in [24] and [25].

Theorem IV: *The difference d between the successive values $\mathbf{T}$ assumes under the conditions $\mathbf{k}=k$, $\mathbf{t}_1=t_1, \ldots, \mathbf{t}_k=t_k$, $\mathbf{u}_1=u_1, \ldots, \mathbf{u}_k=u_k$ is constant if and only if for $i=2, \ldots, k'$*

$$\text{(3.35)} \qquad \begin{cases} 1. & \varphi_i' \text{ is a multiple of } \varphi_1' \text{ and} \\ 2. & \varphi_i' \leqq \sum_{j=1}^{i-1} t_j' \varphi_j' + \varphi_1'. \end{cases}$$

This difference then equals $2\varphi_1'$.

Proof:

It will be proved that the difference between the successive values $\mathbf{T}'$ assumes under the condition (k, t, u) is constant if and only if (3.35) is satisfied.

From (3.34) it follows that

$$\text{(3.36)} \qquad \begin{cases} 1. & \text{if } a_i'=0 \text{ for each } i=1,\ldots,k', \text{ then } T'=0, \\ 2. & \text{if } a_1'=\sum_{i=1}^{k'} a_i'=1, \text{ then } T'=2\varphi_1' \end{cases}$$

and from (3.32) and (3.36) it follows that $\mathbf{T}'$ does not assume values between 0 and $2\varphi_1'$, i.e. $d=2\varphi_1'$ if d exists.

Further if T_0' is a value $\mathbf{T}'$ assumes, then $\mathbf{T}'$ also assumes the values $T_0'+2\varphi_i'$ and (or) $T_0'-2\varphi_i'$. Thus a necessary condition for the existence of d is that $2\varphi_i'$ is a multiple of $2\varphi_1'$ $(i=2, \ldots, k')$. Further if (3.35.1) is satisfied then all values $\mathbf{T}'$ assumes are multiples of $2\varphi_1'$.

Now suppose that (3.35.1) is satisfied then it will be proved that (3.35.2) is a necessary and sufficient condition for the occurrence of *all* multiples of $2\varphi_1'$ between $T_{min}'=0$ and $T_{max}'=2\sum_{i=1}^{k'} t_i' \varphi_i'$.

We first prove that (3.35.2) is a necessary condition. Consider for any fixed value of i the following two cases

$$\text{(3.37)} \qquad \begin{cases} 1. & a_j'=0 \text{ for each } j=i,\ldots,k', \\ 2. & a_j' \geqq 1 \text{ for at least one value of } j=i,\ldots,k'. \end{cases}$$

These two cases are mutually exclusive and one of the two must occur. The greatest value $\mathbf{T}'$ assumes in case (3.37.1) is $2\sum_{j=1}^{i-1} t_j' \varphi_j'$ and the smallest value in case (3.37.2) is $2\varphi_i'$. Thus if $2\varphi_i' > 2\sum_{j=1}^{-1} t_j' \varphi_j'$ then no values between these two can be assumed by $\mathbf{T}'$. This means that the difference

$$2\varphi_i' - 2\sum_{j=1}^{i-1} t_j' \varphi_j'$$

should not be larger than $d=2\varphi_1'$, or

$$\text{(3.38)} \qquad \varphi_i' \leqq \sum_{j=1}^{i-1} t_j' \varphi_j' + \varphi_1' \quad (i=2,\ldots,k').$$

The sufficiency of condition (3.35.2) will be proved by induction.

Suppose that it has been proved, for a certain value of i, that in case (3.37.1) the difference between the successive values T' assumes are constant and equal to $2\varphi_1'$. Then T' assumes in this case the values

$$2l\varphi_1' \qquad (l=0,\dots,\frac{1}{\varphi_1'}\sum_{j=1}^{i-1} t_j'\varphi_j'). \tag{3.39}$$

For $i=1$ this is true. Further the contribution of the "tie" of size t_i' to T' equals

$$2h\varphi_i' \qquad (h=0,\dots,t_i'). \tag{3.40}$$

Thus if $a_j'=0$ for each $j=i+1, \dots, k'$ then T' assumes the values

$$2h\varphi_i'+2l\varphi_1' \begin{cases} (l=0,\dots,\frac{1}{\varphi_1'}\sum_{j=1}^{i-1} t_j'\varphi_j', \\ h=0,\dots,t_i'). \end{cases} \tag{3.41}$$

For each possible value of h and l these values are multiples of $2\varphi_1'$ and for any fixed value of h, say h_0, the difference between these values for $l=0, \dots, \frac{1}{\varphi_1'}\sum_{j=1}^{i-1} t_j'\varphi_j'$ is constant and equal to $2\varphi_1'$. Thus it remains to be proved that no gap can arise by raising h from, say, h_0 to h_0+1. The smallest value T' assumes if $h=h_0+1$ is $2(h_0+1)\varphi_i'$ and the greatest value of T' if $h=h_0$ is $2h_0\varphi_i'+2\sum_{j=1}^{i-1} t_j'\varphi_j'$. Thus if

$$\varphi_i' \leqq \sum_{j=1}^{i-1} t_j'\varphi_j'+\varphi_1' \qquad (i=2,\dots,k') \tag{3.42}$$

then no gap arises if h is raised from h_0 to h_0+1, i.e. (3.35.2) is a sufficient condition for the occurrence of all multiples of $2\varphi_1'$ between 0 and $2\sum_{i=1}^{k} t_i'\varphi_i'$.

Special case

For WILCOXON's test for symmetry we have $\varphi(u,r)=r$ and

$$0<r_1<\dots<r_k. \tag{3.43}$$

Condition (3.35) reduces in this case to

$$\begin{cases} 1. \quad t_i-(-1)^i \text{ is a multiple of } t_1+1 \text{ and} \\ 2. \quad t_i \leqq \{\sum_{j=1}^{i-1} t_j\}^2-\sum_{j=2}^{i-1} t_j \end{cases} \tag{3.44}$$

for $i=2, \dots, k$ and this condition is e.g. satisfied if

$$\begin{cases} t_{2i+1}=t_1 \text{ for } i=1,\dots,\left[\frac{k-1}{2}\right] \text{ and} \\ t_{2i} \quad =1 \text{ for } i=1,\dots,\left[\frac{k}{2}\right]. \end{cases} \tag{3.45}$$

The difference d then equals t_1+1.

A special case of (3.45) is the case that $t_i = 1$ for each $i = 1, \ldots, k$. Then $d = 2$.

In order to prove the conditional asymptotic normality of the distribution of $\boldsymbol{T}$ under the hypothesis H_0 we consider a sequence $\{\mathbf{z}_\lambda\}$ $(\lambda = 1, 2, \ldots)$ of independent random variables (cf. section 2). Let

$$\pi_\lambda \stackrel{\text{def}}{=} \mathrm{P}\,[\mathbf{z}_\lambda \neq 0], \tag{3.46}$$

then if

$$\sum_{\lambda=1}^{\infty} \pi_\lambda = \infty \tag{3.47}$$

the sequence $\{\mathbf{z}_\lambda\}$ has, according to the BOREL CANTELLI lemma (cf. e.g. W. FELLER [6] p. 155), probability one of containing infinitely many elements $\neq 0$. Thus omitting the elements which on observation assume the value 0 an infinite sequence remains.

Let the non zero values assumed by $|\mathbf{z}_1|, \ldots, |\mathbf{z}_\lambda|$ consist of $t_{i,\lambda}$ times the value $\mathbf{u}_{i,\lambda}$ $(i = 1, \ldots, \mathbf{k}_\lambda)$, where $\mathbf{u}_{1,\lambda} < \ldots < \mathbf{u}_{\mathbf{k}_\lambda,\lambda}$ and let

$$\mathbf{r}_{i,\lambda} \stackrel{\text{def}}{=} \sum_{j=1}^{i} \mathbf{t}_{j,\lambda} - \tfrac{1}{2}\,(\mathbf{t}_{i,\lambda} - 1) \qquad (i = 1, \ldots, \mathbf{k}_\lambda). \tag{3.48}$$

Let further

$$\varphi_{i,\lambda} \stackrel{\text{def}}{=} \varphi\,(\mathbf{u}_{i,\lambda}, \mathbf{r}_{i,\lambda}) \tag{3.49}$$

and let $\boldsymbol{T}_\lambda$ denote the test statistic for $\mathbf{z}_1, \ldots, \mathbf{z}_\lambda$. Finally let

$$\mathbf{n}_\lambda \stackrel{\text{def}}{=} \sum_{i=1}^{\mathbf{k}_\lambda} \mathbf{t}_{i,\lambda} \tag{3.50}$$

and

$$\sigma_{0,\lambda}^2 \stackrel{\text{def}}{=} \sigma^2\,(\boldsymbol{T}_\lambda \,|\, k_\lambda, t_{1,\lambda}, \ldots, t_{k_\lambda,\lambda}, u_{1,\lambda}, \ldots, u_{k_\lambda,\lambda}; H_0). \tag{3.51}$$

Then $\mathbf{n}_\lambda \to \infty$ with λ except for a probability 0.

Theorem V: *Let $\{k_\lambda\}$, $\{n_\lambda\}$ and $\{t_{1,\lambda}\}, \ldots, \{t_{k_\lambda,\lambda}\}$ be sequences of non negative integers with $n_\lambda = \sum_{i=1}^{k_\lambda} t_{i,\lambda}$ $(\lambda = 1, 2, \ldots)$ and $n_\lambda \to \infty$ for $\lambda \to \infty$; let further $\{u_{1,\lambda}\}, \ldots, \{u_{k_\lambda,\lambda}\}$ be sequences of numbers with $0 < u_{1,\lambda} < \ldots < u_{k_\lambda,\lambda}$ $(\lambda = 1, 2, \ldots)$ then if*

$$\lim_{\lambda\to\infty} \frac{\sum_{i=1}^{k_\lambda} t_{i,\lambda}\,\varphi_{i,\lambda}^4}{\left\{\sum_{i=1}^{k_\lambda} t_{i,\lambda}\,\varphi_{i,\lambda}^2\right\}^2} = 0 \tag{3.52}$$

the random variable $\boldsymbol{T}_\lambda/\sigma_{0,\lambda}$ is, under the hypothesis H_0 and under the conditions $\mathbf{k}_\lambda = k_\lambda$, $\mathbf{t}_{1,\lambda} = t_{1,\lambda}, \ldots, \mathbf{t}_{k_\lambda,\lambda} = t_{k_\lambda,\lambda}$, $\mathbf{u}_{1,\lambda} = u_{1,\lambda}, \ldots, \mathbf{u}_{k_\lambda,\lambda} = u_{k_\lambda,\lambda}$ for $\lambda \to \infty$ asymptotically normally distributed with mean 0 and variance 1.

Proof: The notation will be simplified by omitting the index λ. From (3.14) it follows that it is sufficient to prove that

$$\lim_{\lambda\to\infty}\frac{\varkappa_{2\nu}}{\sigma_0^{2\nu}}=0 \quad \text{for } \nu>1 \tag{3.53}$$

and from (3.15) and (3.25) follows

$$\frac{\varkappa_{2\nu}}{\sigma_0^{2\nu}}=\frac{2^{2\nu}(2^{2\nu}-1)B_{2\nu}}{2\nu}\frac{\sum_{i=1}^{k}t_i\varphi_i^{2\nu}}{\{\sum_{i=1}^{k}t_i\varphi_i^2\}^{\nu}}\leqq\frac{2^{2\nu}(2^{2\nu}-1)B_{2\nu}}{2\nu}\frac{\sum_{i=1}^{k}t_i\varphi_i^4}{\{\sum_{i=1}^{k}t_i\varphi_i^2\}^2} \quad \text{for } \nu>1. \tag{3.54}$$

Thus

$$\lim_{\lambda\to\infty}\frac{\varkappa_{2\nu}}{\sigma_0^{2\nu}}\leqq\frac{2^{2\nu}(2^{2\nu}-1)B_{2\nu}}{2\nu}\lim_{\lambda\to\infty}\frac{\sum_{i=1}^{k}t_i\varphi_i^4}{\{\sum_{i=1}^{k}t_i\varphi_i^2\}^2}=0 \quad \text{for } \nu>1. \tag{3.55}$$

Special cases

For WILCOXON's test for symmetry we have

$$\varphi_i=r_i<2n \qquad (i=1,\dots,k), \tag{3.56}$$

thus

$$\sum_{i=1}^{k}t_i\varphi_i^4<(2n)^4\sum_{i=1}^{k}t_i=2^4\cdot n^5 \tag{3.57}$$

and (cf. (3.26))

$$\sum_{i=1}^{k}t_i\varphi_i^2\geqq\tfrac{1}{4}n^3. \tag{3.58}$$

From (3.57) and (3.58) then follows

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{k}t_i\varphi_i^4}{\{\sum_{i=1}^{k}t_i\varphi_i^2\}^2}=0 \tag{3.59}$$

and (3.52) then follows from the fact that n tends to infinity with λ.

In the case of FISHER's test for symmetry we have

$$\varphi(u,r)=u \tag{3.60}$$

and thus

$$\frac{\sum_{i=1}^{k}t_i\varphi_i^4}{\{\sum_{i=1}^{k}t_i\varphi_i^2\}^2}=\frac{\sum_{h=1}^{m}z_h^4}{\{\sum_{h=1}^{m}z_h^2\}^2}. \tag{3.61}$$

Thus in this case (3.52) is identical with

$$\lim_{\lambda\to\infty}\frac{\sum_{h=1}^{\lambda}z_h^4}{\{\sum_{h=1}^{\lambda}z_h^2\}^2}=0. \tag{3.62}$$

Now suppose there exists a random variable $\boldsymbol{X}$ with [6])

$$(3.63)\quad \begin{cases} 1.\ P[|\mathbf{z}_\lambda| \geqq X | \mathbf{z}_\lambda \neq 0] \leqq P[|\boldsymbol{X}| \geqq X] \text{ for each } X \geqq 0 \text{ and each } \lambda, \\ 2.\ \mathscr{E}\boldsymbol{X}^2 < \infty, \end{cases}$$

then (cf. e.g. M. LOÈVE [11], p. 242)

$$(3.64)\qquad \lim_{\lambda\to\infty} \frac{1}{n_\lambda^2} \sum_{h=1}^{\lambda} \mathbf{z}_h^4 = 0 \quad \text{except for a probability } 0$$

and

$$(3.65)\qquad \lim_{\lambda\to\infty} \frac{1}{n_\lambda} \sum_{h=1}^{\lambda} \{\mathbf{z}_h^2 - \mathscr{E}(\mathbf{z}_h^2 \,|\, \mathbf{z}_h \neq 0)\} = 0 \quad \text{except for a probability } 0.$$

Thus if moreover

$$(3.66)\qquad \liminf_{\lambda\to\infty} \frac{1}{n_\lambda} \sum_{h=1}^{\lambda} \mathscr{E}(\mathbf{z}_h^2 \,|\, \mathbf{z}_h \neq 0) > 0$$

then

$$(3.67)\qquad \lim_{\lambda\to\infty} \frac{\sum_{h=1}^{\lambda} \mathbf{z}_h^4}{\{\sum_{h=1}^{\lambda} \mathbf{z}_h^2\}^2} = 0 \quad \text{except for a probability } 0.$$

If the distributions of $\mathbf{z}_\lambda$ under the condition $\mathbf{z}_\lambda \neq 0$ are, for $\lambda = 1, 2, \ldots$, identical then the conditions (3.63) and (3.66) reduce to

$$(3.68)\qquad \mathscr{E}(\mathbf{z}_\lambda^2 | \mathbf{z}_\lambda \neq 0) < \infty.$$

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[6]) This result we owe to Mr. J. TH. RUNNENBURG.

A GENERAL CLASS OF DISTRIBUTIONFREE TESTS FOR SYMMETRY CONTAINING THE TESTS OF WILCOXON AND FISHER *). II

BY

CONSTANCE VAN EEDEN AND A. BENARD

(Communicated by Prof. D. VAN DANTZIG at the meeting of March 30, 1957)

4. *The relation with the class of two sample tests described in* [4]

From (2.3) it follows that T may be written in the form

$$T = 2 \sum_{i=1}^{k} \varphi_i a_i - \sum_{i=1}^{k} \varphi_i t_i = 2t^* - \sum_{i=1}^{k} \varphi_i t_i, \tag{4.1}$$

where t^* is the test statistic for the two sample problem defined in [4] (p. 251) applied to the positive observations as the first sample and the absolute values of the negative observations as the second sample.

Further if (cf. e.g. [4] p. 252)

$$\tilde{t}^* \overset{\text{def}}{=} t^* - \mathscr{E}(t^* \mid (k,t,u), n_1; H_0) = t^* - \frac{n_1}{n} \sum_{i=1}^{k} \varphi_i t_i, \tag{4.2}$$

then

$$T = 2\tilde{t}^* + \frac{2}{n} (n_1 - \tfrac{1}{2} n) \sum_{i=1}^{k} \varphi_i t_i. \tag{4.3}$$

Thus the test statistic T is a combination of the statistic t^* for the two sample problem and the statistic n_1 of the sign test.

Special cases

For WILCOXON's test for symmetry we obtain from (4.3)

$$T_W = \tilde{W} + (n+1)(n_1 - \tfrac{1}{2} n), \tag{4.4}$$

with

$$\tilde{W} \overset{\text{def}}{=} W - n_1 n_2, \tag{4.5}$$

where W is the test statistic of WILCOXON's two sample test [7]).

In the case of FISHER's test for symmetry we have

$$T_F = 2\tilde{t}_P + \frac{2}{n} (n_1 - \tfrac{1}{2} n) \sum_{h=1}^{m} |z_h|, \tag{4.6}$$

where t_P is E. J. G. PITMAN's test statistic for the two sample problem [13].

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*) Report SP 54 of the Statistical Department of the Mathematical Centre, Amsterdam.

[7]) The test statistic of WILCOXON's two sample test for the samples $x_1, \ldots, x_{n_1}$ and $y_1, \ldots, y_{n_2}$ is defined here as twice the number of pairs (x_i, y_j) with $x_i > y_j$, increased by the number of pairs (x_i, y_j) with $x_i = y_j$ $(i = 1, \ldots, n_1;\ j = 1, \ldots, n_2)$ (cf. [20]).

Remark

6. Other tests for symmetry may e.g. be obtained by choosing for $\mathbf{t}^*$ the test statistic of the two sample tests of M. E. TERRY [18] or B. L. VAN DER WAERDEN [21], i.e. by taking

$$\varphi_i = \frac{1}{t_i} \sum_{\gamma=1}^{t_i} \mathscr{E} \mathbf{Z}_{n, s_i+\gamma} \quad (i=1, \ldots, k) \tag{4.7}$$

or

$$\varphi_i = \frac{1}{t_i} \sum_{\gamma=1}^{t_i} \Psi\left(\frac{s_i+\gamma}{n+1}\right) \quad (i=1, \ldots, k), \tag{4.8}$$

with

$$s_i \stackrel{\text{def}}{=} \sum_{j=1}^{i-1} t_j \quad (i=1, \ldots, k) \tag{4.9}$$

and where $\mathscr{E} \mathbf{Z}_{n,r}$ is the expectation of the r-th order statistic of a random sample of size n from a standard normal distribution and $\Psi(x)$ is defined by

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Psi(x)} e^{-\frac{1}{2}u^2} du = x. \tag{4.10}$$

Further the hypothesis H_0 implies, under the conditions (k, t, u) and $\mathbf{n}_1 = n_1$ the hypothesis H_0'' that the positive observations are a random sample without replacement taken from the absolute values of all observations (cf. [9] p. 71 and [5] p. 307). The mean and variance of $\mathbf{T}$ under the hypothesis H_0 and under the condition (k, t, u) thus also follow from the formulae for the mean and variance of $\mathbf{t}^*$ under the hypothesis H_0'' (cf. e.g. [4] p. 252).

From (4.3) it follows

$$\mathscr{E}(\mathbf{T} \mid (k, t, u), n_1; H_0) = \frac{2}{n}(n_1 - \tfrac{1}{2}n) \sum_{i=1}^{k} \varphi_i t_i \tag{4.11}$$

and

$$\sigma^2(\mathbf{T} \mid (k, t, u), n_1; H_0) = \frac{4 n_1 n_2}{n(n-1)} \left\{ \sum_{i=1}^{k} t_i \varphi_i^2 - \frac{1}{n} \left(\sum_{i=1}^{k} t_i \varphi_i\right)^2 \right\}. \tag{4.12}$$

From (4.11) and (4.12) then follows

$$\begin{cases} \mathscr{E}(\mathbf{T} \mid (k, t, u); H_0) = \mathscr{E}\{\mathscr{E}(\mathbf{T} \mid (k, t, u), \mathbf{n}_1; H_0) \mid (k, t, u); H_0\} = \\ = \frac{2}{n} \sum_{i=1}^{k} t_i \varphi_i \, \mathscr{E}(\mathbf{n}_1 - \tfrac{1}{2}n \mid (k, t, u); H_0) = 0 \qquad \text{(cf. (3.24))} \end{cases} \tag{4.13}$$

and

$$\begin{cases} \sigma^2(\mathbf{T} \mid (k, t, u); H_0) = \\ = \sigma^2\{\mathscr{E}(\mathbf{T} \mid (k, t, u), \mathbf{n}_1; H_0) \mid (k, t, u); H_0\} + \\ + \mathscr{E}\{\sigma^2(\mathbf{T} \mid (k, t, u), \mathbf{n}_1; H_0) \mid (k, t, u); H_0\} = \\ = \frac{4}{n^2} \{\sum_{i=1}^{k} t_i \varphi_i\}^2 \sigma^2(\mathbf{n}_1 \mid (k, t, u); H_0) + \\ + \frac{4}{n(n-1)} \{\sum_{i=1}^{k} t_i \varphi_i^2 - \frac{1}{n} (\sum_{i=1}^{k} t_i \varphi_i)^2\} \mathscr{E}(\mathbf{n}_1 \mathbf{n}_2 \mid (k, t, u); H_0) = \\ = \frac{1}{n} \{\sum_{i=1}^{k} t_i \varphi_i\}^2 + \{\sum_{i=1}^{k} t_i \varphi_i^2 - \frac{1}{n} (\sum_{i=1}^{k} t_i \varphi_i)^2\} = \sum_{i=1}^{k} t_i \varphi_i^2 \quad \text{(cf. (3.25))}. \end{cases} \tag{4.14}$$

5. *The consistency of the tests of* WILCOXON *and* FISHER

In this section the consistency of the tests for symmetry of WILCOXON and FISHER will be investigated.

We again consider the sequence $\{\mathbf{z}_\lambda\}$ and an alternative hypothesis H stating that the distributions of $\mathbf{z}_\lambda$ under the condition $\mathbf{z}_\lambda \neq 0$ are, for $\lambda = 1, 2, \ldots,$ identical. Let $\mathbf{x}_{1,\lambda}, \ldots, \mathbf{x}_{n_{1,\lambda},\lambda}$ denote the positive observations and $\mathbf{y}_{1,\lambda}, \ldots, \mathbf{y}_{n_{2,\lambda},\lambda}$ the absolute values of the negative observations, with $\mathbf{n}_{1,\lambda} + \mathbf{n}_{2,\lambda} = \mathbf{n}_\lambda$. Let further

(5.1) $$\begin{cases} p \overset{\text{def}}{=} \mathrm{P}\,[\mathbf{z}_\lambda > 0 \mid \mathbf{z}_\lambda \neq 0] \quad (\lambda = 1, 2, \ldots), \\ q \overset{\text{def}}{=} 1 - p. \end{cases}$$

We first prove the consistency of WILCOXON's test. Let

(5.2) $$\theta \overset{\text{def}}{=} \mathrm{P}\,[\mathbf{x}_\lambda > \mathbf{y}_\mu] - \mathrm{P}\,[\mathbf{x}_\lambda < \mathbf{y}_\mu] \quad (\lambda, \mu = 1, 2, \ldots),$$

then we have

Lemma I:

(5.3) $$\mu_W \overset{\text{def}}{=} \mathscr{E}\,(\mathbf{T}_W \mid n; H) = n\,(n-1)\,pq\theta + n\,(n+1)\,(p - \tfrac{1}{2}).$$

Proof: From (4.4) it follows that

(5.4) $$\mathbf{T}_W = \sum_{i=1}^{\mathbf{n}_1} \sum_{j=1}^{\mathbf{n}_2} sgn\,(\mathbf{x}_i - \mathbf{y}_j) + (n+1)\,(\mathbf{n}_1 - \tfrac{1}{2}n),$$

where

(5.5) $$sgn\ z = \begin{cases} 1 \text{ if } z > 0 \\ 0 \text{ if } z = 0 \\ -1 \text{ if } z < 0. \end{cases}$$

From (5.4) follows

(5.6) $$\mathscr{E}(\mathbf{T}_W | n, n_1; H) = n_1\, n_2 \theta + (n+1)(n_1 - \tfrac{1}{2}n),$$

thus

(5.7) $$\begin{cases} \mathscr{E}(\mathbf{T}_W | n; H) = \mathscr{E}\{\mathscr{E}(\mathbf{T}_W | n, \mathbf{n}_1; H) | n; H\} = \\ \quad = \theta \mathscr{E}(\mathbf{n}_1 \mathbf{n}_2 | n; H) + (n+1)\mathscr{E}(\mathbf{n}_1 - \tfrac{1}{2}n | n; H) = \\ \quad = n(n-1)pq\theta + n(n+1)(p - \tfrac{1}{2}). \end{cases}$$

Lemma II:

(5.8) $$\sigma_W^2 \overset{\text{def}}{=} \sigma^2\,(\mathbf{T}_W \mid n; H) = \mathrm{O}\,(n^3)$$

and the coefficient of n^3 *in* (5.8) *is* $\leq \frac{13}{16}$.

Proof: We have

(5.9) $$\sigma^2(\mathbf{T}_W | n; H) = \sigma^2\{\mathscr{E}(\mathbf{T}_W | n, \mathbf{n}_1; H) | n; H\} + \mathscr{E}\{\sigma^2(\mathbf{T}_W | n, \mathbf{n}_1; H) | n; H\}.$$

From (5.6) it follows that

(5.10) $$\sigma^2\{\mathscr{E}(\mathbf{T}_W | n, \mathbf{n}_1; H) | n; H\} = \sigma^2\{\mathbf{n}_1\, \mathbf{n}_2 \theta + (n+1)(\mathbf{n}_1 - \tfrac{1}{2}n) | n; H\} = \mathrm{O}(n^3)$$

and the coefficient of n^3 in (5.10) is

(5.11) $$pq(\theta + 1 - 2pq\theta)^2.$$

Further (cf. (5.4))

$$\sigma^2(\mathbf{T}_W \mid n, n_1; H) = \sigma^2(\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} sgn\,(\mathbf{x}_i - \mathbf{y}_j) \mid n, n_1; H) \tag{5.12}$$

and from D. J. STOKER ([17], p. 67–68) it follows that

$$\sigma^2(\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} sgn\,(\mathbf{x}_i - \mathbf{y}_j) \mid n, n_1; H) \leqq n_1 n_2 (n+1), \tag{5.13}$$

thus

$$\mathscr{E}\{\sigma^2(\mathbf{T}_W | n, \mathbf{n}_1; H) | n; H\} \leqq n(n^2-1)pq. \tag{5.14}$$

Thus

$$\sigma^2(\mathbf{T}_W | n; H) = O(n^3) \tag{5.15}$$

and the coefficient of n^3 in (5.15) is

$$\leqq pq(\theta+1-2pq\theta)^2 + pq \leqq \tfrac{13}{16}\ ^{8)}. \tag{5.16}$$

Theorem VI: *If* (3.47) *is satisfied then the test for symmetry of* WILCOXON *based on the critical region* Z (*cf.* (2.5)) *is, for* $\lambda \to \infty$, *consistent for the class of alternative hypotheses*

$$|p - \tfrac{1}{2} + pq\theta| > 0. \tag{5.17}$$

The tests based on the critical regions Z_l *and* Z_r *respectively are consistent for the classes of alternative hypotheses*

$$p - \tfrac{1}{2} + pq\theta < 0 \tag{5.18}$$

and

$$p - \tfrac{1}{2} + pq\theta > 0 \tag{5.19}$$

respectively and not consistent for the classes of alternative hypotheses

$$p - \tfrac{1}{2} + pq\theta > 0 \tag{5.20}$$

and

$$p - \tfrac{1}{2} + pq\theta < 0 \tag{5.21}$$

respectively.

All tests of WILCOXON *mentioned are, for sufficiently small* α, *not consistent for the class of alternative hypotheses*

$$p - \tfrac{1}{2} + pq\theta = 0. \tag{5.22}$$

Proof: [9]) The index λ will be omitted. Let

$$\begin{cases} 1. & \mathbf{s}_W^2 \stackrel{\text{def}}{=} \sigma^2(\mathbf{T}_W \mid n, \mathbf{t}_1, \ldots, \mathbf{t}_k; H_0), \\ 2. & c_1^2 \stackrel{\text{def}}{=} \tfrac{1}{4} n (n+1)^2 \\ 3. & c_2^2 \stackrel{\text{def}}{=} \tfrac{1}{6} n (n+1)(2n+1), \end{cases} \tag{5.23}$$

---

[8]) If $p = \frac{1}{2}$ and $\theta = 1$ then

$$pq(\theta + 1 - 2pq\theta)^2 = \tfrac{9}{16}$$

and (cf. [17]. p. 67–68)

$$\sigma^2(\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} sgn\,(\mathbf{x}_i - \mathbf{y}_j) \mid n, n_1; H) = n_1 n_2 (n-2).$$

Thus in this case the coefficient of n^3 in (5.15) equals $\frac{13}{16}$.

[9]) Cf. also D. VAN DANTZIG [3] for the proof of the consistency of WILCOXON's two sample test.

then

$$c_1^2 \leqq s_W^2 \leqq c_2^2. \tag{5.24}$$

We first consider the case that

$$p - \tfrac{1}{2} + pq\theta < 0. \tag{5.25}$$

For the test based on Z_l we have (cf. lemma I and II)

$$\begin{cases} \lim\limits_{\lambda\to\infty} \mathrm{P}\,[\boldsymbol{T}_W \notin \boldsymbol{Z}_l \mid n; H] = \lim\limits_{\lambda\to\infty} \mathrm{P}\,[\boldsymbol{T}_W > -\xi_\alpha \boldsymbol{s}_W \mid n; H] \leqq \\ \leqq \lim\limits_{\lambda\to\infty} \mathrm{P}\left[\frac{\boldsymbol{T}_W - \mu_W}{\sigma_W} > -\frac{\xi_\alpha c_2 + \mu_W}{\sigma_W} \mid n; H\right], \end{cases} \tag{5.26}$$

where ξ_α is defined by

$$\frac{1}{\sqrt{2\pi}} \int_{\xi_\alpha}^{\infty} e^{-\frac{1}{2}x^2}\, dx = \alpha. \tag{5.27}$$

From (5.23), (5.25), lemma II and the fact that n tends to infinity with λ it follows that $-\frac{\xi_\alpha c_2 + \mu_W}{\sigma_W}$ is positive for sufficiently large λ; thus according to the inequality of Bienaymé–Tchebychef

$$\lim_{\lambda\to\infty} \mathrm{P}\,[\boldsymbol{T}_W \notin \boldsymbol{Z}_l \mid n; H] \leqq \lim_{\lambda\to\infty} \frac{\sigma_W^2}{(\xi_\alpha c_2 + \mu_W)^2} = 0. \tag{5.28}$$

Thus the test based on the critical region Z_l is, for $\lambda \to \infty$, consistent for the class of alternative hypotheses (5.25).

If

$$p - \tfrac{1}{2} + pq\theta > 0 \tag{5.29}$$

then

$$\begin{cases} \lim\limits_{\lambda\to\infty} \mathrm{P}\,[\boldsymbol{T}_W \in \boldsymbol{Z}_l \mid n; H] \leqq \lim\limits_{\lambda\to\infty} \mathrm{P}\,[\boldsymbol{T}_W \leqq -\xi_\alpha c_1 \mid n; H] \leqq \\ \leqq \lim\limits_{\lambda\to\infty} \frac{\sigma_W^2}{(\xi_\alpha c_1 + \mu_W)^2} = 0, \end{cases} \tag{5.30}$$

$-\frac{\xi_\alpha c_1 + \mu_W}{\sigma_W}$ being negative for sufficiently large λ. Thus the test based on Z_l is, for $\lambda \to \infty$, not consistent for the class of alternative hypotheses (5.29).

Finally if

$$p - \tfrac{1}{2} + pq\theta = 0 \tag{5.31}$$

then

$$\begin{cases} \lim\limits_{\lambda\to\infty} \mathrm{P}\,[\boldsymbol{T}_W \in \boldsymbol{Z}_l \mid n, H] \leqq \lim\limits_{\lambda\to\infty} \mathrm{P}\left[\frac{\boldsymbol{T}_W - \mu_W}{\sigma_W} \leqq -\frac{\xi_\alpha c_1 + \mu_W}{\sigma_W} \mid n; H\right] \leqq \\ \leqq \lim\limits_{\lambda\to\infty} \left(\frac{\sigma_W}{\xi_\alpha c_1}\right)^2. \end{cases} \tag{5.32}$$

Thus if

$$\xi_\alpha > \lim_{\lambda\to\infty} \frac{\sigma_W}{c_1} \tag{5.33}$$

then the test based on Z_l is, for $\lambda \to \infty$, not consistent for the class of alternative hypotheses (5.31) and from (5.23) and lemma II follows

(5.34) $$\lim_{\lambda\to\infty} \frac{\sigma_W}{c_1} \leqq \sqrt{3{,}25} = 1{,}80.$$

The proofs for the tests based on Z_r and Z are analogous.

Theorem VII: *If the distributions of* $\mathbf{z}_1, \ldots, \mathbf{z}_m$ *are identical and symmetrical with respect to* a *then*

(5.35) $$\begin{cases} 1.\ p-\frac{1}{2}+pq\theta=0 \text{ if } a=0, \\ 2.\ (p-\frac{1}{2}+pq\theta)a>0 \text{ if } a\neq 0. \end{cases}$$

Proof: Let

(5.36) $$H(z) \overset{\text{def}}{=} \mathrm{P}\,[\mathbf{z}_h \leqq z]$$

and (cf. (3.46))

(5.37) $$\pi \overset{\text{def}}{=} \mathrm{P}\,[\mathbf{z}_h \neq 0].$$

Then (cf. (5.1))

(5.38) $$p=\frac{1}{\pi}\int_0^{\infty} dH(z), \qquad q=\frac{1}{\pi}\int_{-\infty}^{0-} dH(z).\ ^{10)}$$

If $a=0$ then $p=\frac{1}{2}$ and $\theta=0$, thus

(5.39) $$p-\tfrac{1}{2}+pq\theta=0 \text{ if } a=0.$$

Now consider the case that $a>0$; then $p\geqq\frac{1}{2}$. From the fact that the distribution of $\mathbf{z}_h$ is symmetrical with respect to a it follows that

(5.40) $$q=\frac{1}{\pi}\int_{2a}^{\infty} dH(z).$$

If further

(5.41) $$F(x) \overset{\text{def}}{=} \mathrm{P}\,[\mathbf{x}_i \leqq x], \qquad G(y) \overset{\text{def}}{=} \mathrm{P}\,[\mathbf{y}_j \leqq y]$$

then

(5.42) $$dF(x)=\frac{dH(x)}{p}, \qquad F(x)=\frac{1}{p}\int_0^x dH(u)$$

and from the symmetry of the distribution of $\mathbf{z}_h$ with respect to a it follows that

(5.43) $$dG(y)=\frac{dH(y+2a)}{q}, \qquad G(y)=\frac{1}{q}\int_{2a}^{2a+y} dH(u).$$

---

10) Here we define

$$\int_{z_1}^{z_2} dH(z) \overset{\text{def}}{=} \mathrm{P}\,[z_1 < \mathbf{z} \leqq z_2]$$

and

$$\int_{z_1}^{z_2-} dH(z) \overset{\text{def}}{=} \mathrm{P}\,[z_1 < \mathbf{z} < z_2].$$

If $q>0$ then

$$(5.44)\quad \begin{cases} \theta = \mathrm{P}[\mathbf{x}_i > \mathbf{y}_j] - \mathrm{P}[\mathbf{x}_i < \mathbf{y}_j] > \mathrm{P}[\mathbf{x}_i > \mathbf{y}_j + 2a] - \mathrm{P}[\mathbf{x}_i < \mathbf{y}_j + 2a] = \\ = \frac{1}{pq}\{\int_{2a}^{\infty} dH(x) \int_{2a}^{x} dH(u) - \int_{0}^{\infty} dH(x+2a) \int_{0}^{x+2a} dH(u)\} \end{cases}$$

and from (5.44) follows

$$(5.45)\quad \begin{cases} pq\theta > \int_{2a}^{\infty} dH(x) \int_{2a}^{x} dH(u) - \int_{0}^{\infty} dH(x+2a) \int_{0}^{x+2a} dH(u) = \\ = \int_{2a}^{\infty} dH(x) \int_{2a}^{x} dH(u) - \int_{2a}^{\infty} dH(x) \int_{0}^{x} dH(u) = \\ = \int_{2a}^{\infty} dH(x) \int_{2a}^{\infty} dH(u) - \int_{2a}^{\infty} dH(x) \int_{0}^{\infty} dH(u) = \\ = \pi^2 q^2 - \pi^2 pq = \pi^2 q(q-p). \end{cases}$$

Thus if $q>0$ then

$$(5.46)\quad \begin{cases} p-\tfrac{1}{2}+pq\theta > p-\tfrac{1}{2}+\pi^2 q(q-p) = (p-q)(\tfrac{1}{2}-\pi^2 q) \geqq \\ \geqq (p-q)(\tfrac{1}{2}-q) = \tfrac{1}{2}(p-q)^2 \geqq 0. \end{cases}$$

Further if $q=0$ then $p=1$ and

$$(5.47)\quad p-\tfrac{1}{2}+pq\theta = p-\tfrac{1}{2} > 0.$$

Thus $p-\frac{1}{2}+pq\theta$ is positive if a is positive.
The proof for $a<0$ is analogous.

From the theorems VI and VII it follows that if the distributions of $\mathbf{z}_\lambda$ are, for $\lambda=1, 2, \ldots$, identical and symmetrical with respect to a then WILCOXON's test for symmetry based on the critical region Z is, for $\lambda \to \infty$, consistent for the class of alternative hypotheses

$$(5.48)\quad a \neq 0.$$

The tests based on Z_l and Z_r respectively are consistent for the classes of alternative hypotheses

$$(5.49)\quad a<0$$

and

$$(5.50)\quad a>0$$

respectively and not consistent for the classes of alternative hypotheses

$$(5.51)\quad a>0$$

and

$$(5.52)\quad a<0$$

respectively.

We now consider FISHER's test for symmetry.

Theorem VIII: *If* (3.47) *is satisfied and if*

$$(5.53)\quad \mathscr{E}(\mathbf{z}_\lambda^2 | \mathbf{z}_\lambda \neq 0) < \infty$$

then FISHER*'s test for symmetry based on the critical region Z is, for $\lambda \to \infty$, consistent for the class of alternative hypotheses*

(5.54) $$\mathscr{E}(\mathbf{z}_\lambda | \mathbf{z}_\lambda \neq 0) \neq 0.$$

The tests based on the critical regions Z_l and Z_r respectively are consistent for the classes of alternative hypotheses

(5.56) $$\mathscr{E}(\mathbf{z}_\lambda | \mathbf{z}_\lambda \neq 0) < 0$$

and

(5.57) $$\mathscr{E}(\mathbf{z}_\lambda | \mathbf{z}_\lambda \neq 0) > 0$$

respectively and not consistent for the classes of alternative hypotheses

(5.58) $$\mathscr{E}(\mathbf{z}_\lambda | \mathbf{z}_\lambda \neq 0) > 0$$

and

(5.59) $$\mathscr{E}(\mathbf{z}_\lambda | \mathbf{z}_\lambda \neq 0) < 0$$

respectively.

All tests of FISHER *mentioned are, for sufficiently small α, not consistent for the class of alternative hypotheses*

(5.60) $$\mathscr{E}(\mathbf{z}_\lambda | \mathbf{z}_\lambda \neq 0) = 0.$$

Proof: The index λ is omitted.

We have

(5.61) $$\mu_F \overset{\text{def}}{=} \mathscr{E}(\boldsymbol{T}_F | n; H) = n \mathscr{E}(\mathbf{z} | \mathbf{z} \neq 0)$$

and

(5.62) $$\sigma_F^2 \overset{\text{def}}{=} \sigma^2(\boldsymbol{T}_F | n; H) = n \sigma^2(\mathbf{z} | \mathbf{z} \neq 0).$$

We first consider the case that

(5.63) $$\mathscr{E}(\mathbf{z} | \mathbf{z} \neq 0) < 0.$$

Let

(5.64) $$\mathbf{s}_F^2 \overset{\text{def}}{=} \sigma^2(\boldsymbol{T}_F | n, \mathbf{t}_1, \ldots, \mathbf{t}_k, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_k; H_0) = \sum_{h=1}^{m} \mathbf{z}_h^2,$$

then we have for each $\delta > 0$

(5.65) $$\begin{cases} \lim\limits_{\lambda\to\infty} \mathrm{P}\,[\boldsymbol{T}_F \notin \boldsymbol{Z}_l | n; H] = \lim\limits_{\lambda\to\infty} \mathrm{P}\,[\boldsymbol{T}_F > -\xi_\alpha \mathbf{s}_F | n; H] = \\ = \lim\limits_{\lambda\to\infty} \mathrm{P}\,[\boldsymbol{T}_F > -\xi_\alpha \mathbf{s}_F \text{ and } |(1/n)\,\mathbf{s}_F^2 - \mathscr{E}(\mathbf{z}^2 | \mathbf{z} \neq 0)| < \delta | n; H] + \\ + \lim\limits_{\lambda\to\infty} \mathrm{P}\,[\boldsymbol{T}_F > -\xi_\alpha \mathbf{s}_F \text{ and } |(1/n)\,\mathbf{s}_F^2 - \mathscr{E}(\mathbf{z}^2 | \mathbf{z} \neq 0)| \geqq \delta | n; H] \leqq \\ \leqq \lim\limits_{\lambda\to\infty} \mathrm{P}\,[\boldsymbol{T}_F > -\xi_\alpha \sqrt{n\{\mathscr{E}(\mathbf{z}^2 | \mathbf{z} \neq 0) + \delta\}} | n; H] + \\ + \lim\limits_{\lambda\to\infty} \mathrm{P}\,[|(1/n)\,\mathbf{s}_F^2 - \mathscr{E}(\mathbf{z}^2 | \mathbf{z} \neq 0)| \geqq \delta | n; H]. \end{cases}$$

Further it follows from (5.53) (cf. also (3.65)) that the second term in the right hand member of (5.65) is zero; thus according to the inequality of BIENAYMÉ–TCHEBYCHEF we have

$$(5.66)\quad \begin{cases} \lim\limits_{\lambda\to\infty} \mathrm{P}\,[\boldsymbol{T}_F \notin \boldsymbol{Z}_l \,|\, n; H] \leqq \\ \leqq \lim\limits_{\lambda\to\infty} \mathrm{P}\left[\dfrac{\boldsymbol{T}_F-\mu_F}{\sigma_F} > -\dfrac{\xi_\alpha \sqrt{n\{\mathscr{E}(\mathbf{z}^2|\mathbf{z}\neq 0)+\delta\}}+\mu_F}{\sigma_F} \,\middle|\, n; H\right] \leqq \\ \leqq \lim\limits_{\lambda\to\infty} \dfrac{\sigma_F}{[\xi_\alpha \sqrt{n\{\mathscr{E}(\mathbf{z}^2|\mathbf{z}\neq 0)+\delta\}}+\mu_F]^2} = 0, \end{cases}$$

$-\dfrac{\xi_\alpha \sqrt{n\{\mathscr{E}(\mathbf{z}^2|\mathbf{z}\neq 0)+\delta\}}+\mu_F}{\sigma_F}$ being positive for sufficiently large λ. Thus the test based on Z_l is, for $\lambda \to \infty$, consistent for the class of alternative hypotheses (5.63).

In an analogous way it may be proved (cf. also the proof of theorem VI) that the test based on Z_l is not consistent for the class of alternative hypotheses

$$(5.67)\quad \mathscr{E}(\mathbf{z}|\mathbf{z}\neq 0) > 0.$$

Finally if

$$(5.68)\quad \mathscr{E}(\mathbf{z}|\mathbf{z}\neq 0) = 0$$

then we have (cf. (5.65) and (5.66)), for $0<\delta<\mathscr{E}(\mathbf{z}^2|\mathbf{z}\neq 0)$,

$$(5.69)\quad \begin{cases} \lim\limits_{\lambda\to\infty} \mathrm{P}\,[\boldsymbol{T}_F \in \boldsymbol{Z}_l \,|\, n; H] \leqq \\ \leqq \lim\limits_{\lambda\to\infty} \mathrm{P}\left[\dfrac{\boldsymbol{T}_F-\mu_F}{\sigma_F} \leqq -\dfrac{\xi_\alpha \sqrt{n\{\mathscr{E}(\mathbf{z}^2|\mathbf{z}\neq 0)-\delta\}}}{\sigma_F}\right] \leqq \\ \leqq \lim\limits_{\lambda\to\infty} \dfrac{\sigma_F^2}{\xi_\alpha^2\, n\{\mathscr{E}(\mathbf{z}^2|\mathbf{z}\neq 0)-\delta\}} = \dfrac{\mathscr{E}(\mathbf{z}^2|\mathbf{z}\neq 0)}{\xi_\alpha^2\{\mathscr{E}(\mathbf{z}^2|\mathbf{z}\neq 0)-\delta\}}. \end{cases}$$

Thus if

$$(5.70)\quad \xi_\alpha > \sqrt{\frac{\mathscr{E}(\mathbf{z}^2|\mathbf{z}\neq 0)}{\mathscr{E}(\mathbf{z}^2|\mathbf{z}\neq 0)-\delta}}$$

then the test based on Z_l is not consistent for the class of alternative hypotheses (5.68).

The proofs for the tests based on Z_r and Z are analogous.

Remark

7. If

$$(5.71)\quad \begin{cases} \mu_1 \overset{\text{def}}{=} \mathscr{E}(\mathbf{x}_{i,\lambda}) = \mathscr{E}(\mathbf{z}_\lambda \,|\, \mathbf{z}_\lambda > 0), \\ \mu_2 \overset{\text{def}}{=} \mathscr{E}(\mathbf{y}_{j,\lambda}) = -\mathscr{E}(\mathbf{z}_\lambda \,|\, \mathbf{z}_\lambda < 0) \end{cases}$$

then (cf. (5.1))

$$(5.72)\quad \mathscr{E}(\mathbf{z}_\lambda|\mathbf{z}_\lambda\neq 0) = p\mu_1 - q\mu_2.$$

Thus $\mathscr{E}(\mathbf{z}_\lambda|\mathbf{z}_\lambda\neq 0) \gtreqless 0$ is identical with

$$(5.73)\quad p \gtreqless \frac{\mu_2}{\mu_1+\mu_2}.$$

A GENERAL CLASS OF DISTRIBUTIONFREE TESTS FOR SYMMETRY CONTAINING THE TESTS OF WILCOXON AND FISHER [1]). III

BY

CONSTANCE VAN EEDEN AND A. BENARD

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6. *A combination of the class of tests for symmetry and the sign test*

In this section a class of tests for the hypothesis H_0 will be described which is a combination of the sign test and the class of tests for symmetry.

Let $n_{1,\alpha}$ denote the smallest integer satisfying

$$\mathrm{P}[\boldsymbol{n}_1 \geqq n_{1,\alpha} | n; H_0] \leqq \alpha, \tag{6.1}$$

then the following critical regions are used (cf. (2.4) and (2.5))

$$\begin{cases} Z_l': n_1 \leqq n - n_{1,\alpha_1} \text{ and (or) } T \leqq -T_{\alpha_2}, \\ Z_r': n_1 \geqq n_{1,\alpha_1} \text{ and (or) } T \geqq T_{\alpha_2}, \\ Z': |n_1 - \tfrac{1}{2}n| \geqq n_{1,\frac{1}{2}\alpha_1} - \tfrac{1}{2}n \text{ and (or) } |T| \geqq T_{\frac{1}{2}\alpha_2}. \end{cases} \tag{6.2}$$

Now let

$$\begin{cases} \varepsilon_1 \overset{\text{def}}{=} \mathrm{P}\,[\boldsymbol{n}_1 \geqq n_{1,\alpha_1} \,|\, n; H_0], \\ \varepsilon_2 \overset{\text{def}}{=} \mathrm{P}\,[\boldsymbol{T} \geqq T_{\alpha_2} \,|\, (k, t, u)\,;\, H_0] \end{cases} \tag{6.3}$$

and let ε denote the size of the critical region Z_r', then

$$\begin{cases} \varepsilon = \varepsilon_1 + (1-\varepsilon_1)\, \mathrm{P}\,[\boldsymbol{T} \geqq T_{\alpha_2} \,|\, \boldsymbol{n}_1 < n_{1,\alpha_1}, (k, t, u)\,; H_0] = \\ = \varepsilon_1 + (1-\varepsilon_1) \displaystyle\sum_{i=0}^{n_{1,\alpha_1}-1} \frac{2^{-n}\binom{n}{i}}{1-\varepsilon_1}\, \mathrm{P}\,[\boldsymbol{T} \geqq T_{\alpha_2} \,|\, \boldsymbol{n}_1 = i, (k, t, u)\,; H_0]. \end{cases} \tag{6.4}$$

Analogous formulae hold for the other onesided and the twosided test.

Thus, $\boldsymbol{T} - \frac{2}{n}(n_1 - \frac{1}{2}n) \sum_{i=1}^{k} t_i \varphi_i$ possessing under the hypothesis H_0 and under the conditions (k, t, u) and $\boldsymbol{n}_1 = n_1$ the same probability distribution as the statistic $2\tilde{\boldsymbol{t}}^*$ for the two sample problem under the hypothesis H_0'' (cf. section 4), ε may be calculated from (6.4) for each α_1, α_2 and n by means of tables of the distribution of $\boldsymbol{t}^*$.

Special case

For WILCOXON's test for symmetry we have (cf. (4.4))

$$2\boldsymbol{t}^* = \boldsymbol{W} \tag{6.5}$$

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1) Report SP 54 of the Statistical Department of the Mathematical Centre, Amsterdam.

and tables of the distribution of **W** under the hypothesis H_0 may e.g. be found in [20].

On the other hand the critical regions Z_l', Z_r' and Z' are not uniquely determined by ε and n. One may now proceed e.g. in one of the following two ways.

1. Suppose one wants to test the hypothesis H_0 by means of the combination of the class of tests for symmetry and the sign test with level of significance α. Then for each $\varepsilon_1 < \alpha$ let $\varepsilon_{2,max}$ denote the largest value of ε_2 satisfying $\varepsilon \leqq \alpha$. This value may be found from (6.4). Further, for this value $\varepsilon_{2,max}$ of ε_2, let $\varepsilon_{1,max}$ denote the largest value of ε_1 satisfying $\varepsilon \leqq \alpha$. Of these pairs $(\varepsilon_{1,max}, \varepsilon_{2,max})$ choose the one with the smallest difference.
If two pairs of values have the same value of $|\varepsilon_{1,max} - \varepsilon_{2,max}|$ then choose the pair with the largest value of ε.

2. Take $\alpha_1 = \alpha_2$ and choose the largest value of $\alpha_1 = \alpha_2 \leqq \alpha$ satisfying $\varepsilon \leqq \alpha$.

These two procedures do not always give the same critical values, but if they give different results then in general the first procedure gives a larger value of ε. Further it will be clear that the two procedures are asymptotically, for $n \to \infty$, identical.

Special case

A table of the critical values of Z_r' for the combination of Wilcoxon's test for symmetry and the sign test for the untied case calculated according to the first method described above, may be found in [1] (p. 31), for $n = 5(1)20$ and $\alpha = 0{,}005$; $0{,}01$; $0{,}025$ and $0{,}05$.

In the following an approximation to α will be given for large values of n. First we prove the following theorems.

Theorem IX: *If $\varkappa_{s,r}$ $(s=0, 1, \ldots; r=0, 1, \ldots, s+r>0)$ are the cumulants of the simultaneous probability distribution of* **T** *and* $\mathbf{n}_1 - \frac{1}{2}n$ *under the hypothesis H_0 and under the condition (k, t, u), then*

$$\varkappa_{s,2\nu+1-s} = 0 \qquad (\nu \geqq 0, 0 \leqq s \leqq 2\nu+1) \tag{6.6}$$

and

$$\varkappa_{s,2\nu-s} = \frac{2^s(2^{2\nu}-1)B_{2\nu}}{2\nu} \sum_{i=1}^{k} t_i \varphi_i^s \qquad (\nu > 0, 0 \leqq s \leqq 2\nu). \tag{6.7}$$

Proof: In the same way as in section 3 we find

$$\tag{6.8} \begin{cases} \ln \mathscr{E}\,(e^{\tau_1 \mathbf{T} + \tau_2(\mathbf{n}_1 - \frac{1}{2}n)} \mid (k, t, u); H_0) = \sum_{i=1}^{k} t_i \ln ch\,(\tau_1 \varphi_i + \frac{1}{2}\tau_2) = \\ = \sum_{\nu=1}^{\infty} \frac{2^{2\nu}(2^{2\nu}-1)B_{2\nu}}{(2\nu)!\,2\nu} \sum_{i=1}^{k} t_i (\tau_1 \varphi_i + \frac{1}{2}\tau_2)^{2\nu} = \\ = \sum_{\nu=1}^{\infty} \frac{(2^{2\nu}-1)B_{2\nu}}{2\nu} \sum_{s=0}^{2\nu} \frac{\tau_1^s \tau_2^{2\nu-s}}{s!\,(2\nu-s)!}\, 2^s \sum_{i=1}^{k} t_i \varphi_i^s. \end{cases}$$

Thus the coefficient of $\dfrac{\tau_1^s \tau_2^{2\nu+1-s}}{s!(2\nu+1-s)!}$ is

(6.9) $$\varkappa_{s,2\nu+1-s} = 0 \qquad (\nu \geqq 0,\ 0 \leqq s \leqq 2\nu+1)$$

and the coefficient of $\dfrac{\tau_1^s \tau_2^{2\nu-s}}{s!(2\nu-s)!}$ is

(6.10) $$\varkappa_{s,2\nu-s} = \frac{2^s(2^{2\nu}-1)B_{2\nu}}{2\nu} \sum_{i=1}^{k} t_i \varphi_i^s \qquad (\nu > 0,\ 0 \leqq s \leqq 2\nu).$$

From (6.10) it follows that

(6.11) $$\varkappa_{2,0} = \sigma^2(T \mid (k, t, u); H_0) = \sum_{i=1}^{k} t_i \varphi_i^2 \qquad \text{(cf. (3.25))},$$

(6.12) $$\varkappa_{0,2} = \sigma^2(n_1 \mid n; H_0) = \tfrac{1}{4} n$$

and

(6.13) $$\varkappa_{1,1} = cov(T, n_1 \mid (k, t, u; H_0) = \tfrac{1}{4} \sum_{i=1}^{k} t_i \varphi_i.$$

Thus the correlation coefficient of T and n_1 under the hypothesis H_0 and under the condition (k, t, u) is

(6.14) $$\varrho(T, n_1 \mid (k, t, u); H_0) = \frac{\sum_{i=1}^{k} t_i \varphi_i}{2\sqrt{n \sum_{i=1}^{k} t_i \varphi_i^2}}.$$

In order to prove the conditional asymptotic normality of the simultaneous distribution of T and n_1 under the hypothesis H_0 we again consider the sequence $\{z_\lambda\}$ (cf. section 3).

Theorem X: *If $\{k_\lambda\}$ and $\{t_{1,\lambda}\}, \ldots, \{t_{k_\lambda,\lambda}\}$ are sequences of non negative integers with $n_\lambda = \sum_{i=1}^{k_\lambda} t_{i,\lambda}$, and $n_\lambda \to \infty$ for $\lambda \to \infty$, if $\{u_{1,\lambda}\}, \ldots, \{u_{k_\lambda,\lambda}\}$ are sequences of numbers with $0 < u_{1,\lambda} < \ldots < u_{k_\lambda,\lambda}$, if (3.47) and (3.52) are satisfied and if moreover*

(6.15) $$\varrho \overset{\text{def}}{=} \lim_{\lambda\to\infty} \frac{\sum_{i=1}^{k_\lambda} t_{i,\lambda} \varphi_{i,\lambda}}{2\sqrt{n_\lambda \sum_{i=1}^{k_\lambda} t_{i,\lambda} \varphi_{i,\lambda}^2}}$$

exists and is in absolute value < 1 then the random variables

(6.16) $$\frac{T_\lambda}{\sigma_{0,\lambda}} \quad \text{and} \quad \frac{n_{1,\lambda} - \frac{1}{2} n_\lambda}{\frac{1}{2}\sqrt{n_\lambda}}$$

possess, under the hypothesis H_0 and under the conditions

$$k_\lambda = k_\lambda,\ t_{1,\lambda} = t_{1,\lambda}, \ldots, t_{k_\lambda,\lambda} = t_{k_\lambda,\lambda},\ u_{1,\lambda} = u_{1,\lambda}, \ldots, u_{k_\lambda,\lambda} = u_{k_\lambda,\lambda}$$

asymptotically, for $\lambda \to \infty$, a two dimensional normal probability distribution with zero means, variances 1 and correlation coefficient ϱ.

Proof: The index λ is omitted.

It is sufficient to prove that

$$\lim_{\lambda\to\infty} \frac{\varkappa_{s,2\nu-s}}{(\varkappa_{2,0})^{s/2}(\varkappa_{0,2})^{\nu-(s/2)}} = 0 \quad \text{for } \nu > 1 \text{ and } 0 \leq s \leq 2\nu. \tag{6.17}$$

From (6.7), (6.11) and (6.12) it follows that

$$\frac{\varkappa_{s,2\nu-s}}{(\varkappa_{2,0})^{s/2}(\varkappa_{0,2})^{\nu-(s/2)}} = \frac{2^{2\nu}(2^{2\nu}-1)B_{2\nu}}{2\nu} \frac{\sum_{i=1}^{k} t_i \varphi_i^s}{(\sum_{i=1}^{k} t_i \varphi_i^2)^{s/2} n^{\nu-(s/2)}}. \tag{6.18}$$

If $\nu-(s/2)=0$ then

$$\frac{\sum_{i=1}^{k} t_i \varphi_i^s}{(\sum_{i=1}^{k} t_i \varphi_i^2)^{s/2} n^{\nu-(s/2)}} = \frac{\sum_{i=1}^{k} t_i \varphi_i^{2\nu}}{(\sum_{i=1}^{k} t_i \varphi_i^2)^{\nu}} \leq \frac{\sum_{i=1}^{k} t_i \varphi_i^4}{(\sum_{i=1}^{k} t_i \varphi_i^2)^2}. \tag{6.19}$$

From (6.18), (6.19) and (3.52) then follows

$$\lim_{\lambda\to\infty} \frac{\varkappa_{s,2\nu-s}}{(\varkappa_{2,0})^{s/2}(\varkappa_{0,2})^{\nu-(s/2)}} = 0 \quad \text{for } \nu-(s/2)=0. \tag{6.20}$$

If $\nu-(s/2)>0$ then

$$\frac{\sum_{i=1}^{k} t_i \varphi_i^s}{(\sum_{i=1}^{k} t_i \varphi_i^2)^{s/2} n^{\nu-(s/2)}} \leq \frac{1}{n^{\nu-(s/2)}}. \tag{6.21}$$

From (6.18), (6.21) and the fact that n tends to infinity with λ then follows

$$\lim_{\lambda\to\infty} \frac{\varkappa_{s,2\nu-s}}{(\varkappa_{2,0})^{s/2}(\varkappa_{0,2})^{\nu-(s/2)}} = 0 \quad \text{for } \nu-(s/2) > 0. \tag{6.22}$$

Special case

For WILCOXON's test for symmetry condition (3.52) is satisfied (cf. (3.59)). Further the correlation coefficient of $\boldsymbol{T}_W$ and $\boldsymbol{n}_1$ under the hypothesis H_0 and under the condition (k, t, u) is

$$\varrho(\boldsymbol{T}_W, \boldsymbol{n}_1 \mid (k,t,u); H_0) = \frac{\sum_{i=1}^{k} t_i r_i}{2\sqrt{n \sum_{i=1}^{k} t_i r_i^2}} = \frac{1}{\sqrt{1 + \frac{n^3 - \sum_{i=1}^{k} t_i^3}{3n(n+1)^2}}}. \tag{6.23}$$

Thus in this case the limit (6.15) exists and is in absolute value <1 if $\lim_{\lambda\to\infty} \sum_{i=1}^{k_\lambda} \frac{t_{i,\lambda}^3}{n_\lambda^3}$ exists and is <1.

From theorem X it follows that, for Z_r' and $\alpha_1=\alpha_2=\alpha'$, α may be approximated by

$$\alpha \approx 2\alpha' - \frac{1}{2\pi\sqrt{1-r^2}} \int_{\xi_{\alpha'}}^{\infty} \int_{\xi_{\alpha'}}^{\infty} e^{-\frac{1}{2}\frac{x^2+y^2-2rxy}{1-r^2}} dx\, dy, \tag{6.24}$$

where

$$r \stackrel{\text{def}}{=} \frac{\sum_{i=1}^{k} t_i \varphi_i}{2\sqrt{n \sum_{i=1}^{k} t_i \varphi_i^2}}. \tag{6.25}$$

Analogous formulae hold for the other onesided and for the twosided test.

Thus an approximation to α may be found by means of a table of the two dimensional normal distribution with correlation coefficient r (cf. e.g. [12], p. 52–57). Table 1 contains this approximation for the onesided test for some values of α' and r.

TABLE 1
Approximation to α for some values of α' and r

r \ α'	0,005	0,01	0,025	0,05
0,85	0,008	0,015	0,037	0,072
0,90	0,007	0,015	0,035	0,068
0,95	0,007	0,013	0,032	0,063

Further an approximation to α' may be found from (6.24) for given values of α and r; table 2 contains this approximation for the onesided test.

TABLE 2
Approximation to α' for some values of α and r

r \ α	0,01	0,025	0,05
0,85	0,0064	0,0165	0,034
0,90	0,0068	0,0175	0,036
0,95	0,0075	0,0193	0,040

Special case

For WILCOXON's test for symmetry we have

$$r_W = \frac{1}{\sqrt{1 + \dfrac{n^3 - \sum_{i=1}^{k} t_i^3}{3n(n+1)^2}}} \geqq \tfrac{1}{2}\sqrt{3} = 0{,}866. \tag{6.26}$$

In [1] (p. 32–33) a table is given of the approximate critical values of Z'_r for the combination of the sign test and WILCOXON's test for symmetry for $n = 21(1)100$, $\alpha = 0{,}01$; $0{,}025$; $0{,}05$ and $r_W = 0{,}85$ (i.e. for $\alpha' = 0{,}0064$; $0{,}0165$; $0{,}034$).

In order to prove the consistency of the combination of the sign test and WILCOXON's (respectively FISHER's) test for symmetry we again consider the sequence $\{z_\lambda\}$ and an alternative hypothesis H stating that the distributions of z_λ, under the condition $z_\lambda \neq 0$ are, for $\lambda = 1, 2, \ldots,$

identical. Then it follows from the theorems VI and VIII and the properties of the sign test that the following theorems hold.

Theorem XI: *If (3.47) is satisfied then the combination of the sign test and* WILCOXON*'s test for symmetry based on the critical region Z' is, for $\lambda \to \infty$, consistent for the class of alternative hypotheses*

$$p \neq \tfrac{1}{2} \text{ and (or) } \theta \neq 0 \tag{6.27}$$

and, for sufficiently small α, not consistent for the class of alternative hypotheses

$$p = \tfrac{1}{2},\ \theta = 0. \tag{6.28}$$

The test based on Z'_l is, for $\lambda \to \infty$, consistent for the classes of alternatives

$$\begin{cases} 1. & p < \tfrac{1}{2}, \\ 2. & p \geqq \tfrac{1}{2},\ p - \tfrac{1}{2} + pq\theta < 0, \end{cases} \tag{6.29}$$

not consistent for the class of alternatives

$$p \geqq \tfrac{1}{2},\ p - \tfrac{1}{2} + pq\theta > 0 \tag{6.30}$$

and, for sufficiently small α, not consistent for the class of alternatives

$$p \geqq \tfrac{1}{2},\ p - \tfrac{1}{2} + pq\theta = 0. \tag{6.31}$$

The test based on Z'_r is, for $\lambda \to \infty$, consistent for the classes of alternatives

$$\begin{cases} 1. & p > \tfrac{1}{2}, \\ 2. & p \leqq \tfrac{1}{2},\ p - \tfrac{1}{2} + pq\theta > 0, \end{cases} \tag{6.32}$$

not consistent for the class of alternatives

$$p \leqq \tfrac{1}{2},\ p - \tfrac{1}{2} + pq\theta < 0 \tag{6.33}$$

and, for sufficiently small α, not consistent for the class of alternatives

$$p \leqq \tfrac{1}{2},\ p - \tfrac{1}{2} + pq\theta = 0. \tag{6.34}$$

Theorem XII: *If (3.47) is satisfied and if*

$$\mathscr{E}(\mathbf{z}_\lambda^2 \mid \mathbf{z}_\lambda \neq 0) < \infty \tag{6.35}$$

then the combination of the sign test and FISHER*'s test for symmetry based on the critical region Z' is, for $\lambda \to \infty$, consistent for the class of alternative hypothesis*

$$\mu_1 \neq \mu_2 \text{ and (or) } p \neq \tfrac{1}{2} \tag{6.36}$$

and, for sufficiently small α not consistent for the class of alternatives

$$\mu_1 = \mu_2,\ p = \tfrac{1}{2}. \tag{6.37}$$

The test based on Z'_l is, for $\lambda \to \infty$, consistent for the classes of alternatives

$$\begin{cases} 1. & p < \tfrac{1}{2}, \\ 2. & p \geqq \tfrac{1}{2},\ p\mu_1 - q\mu_2 < 0, \end{cases} \tag{6.38}$$

not consistent for the class of alternatives

$$p \geqq \tfrac{1}{2},\ p\mu_1 - q\mu_2 > 0 \tag{6.39}$$

and, for sufficiently small α, not consistent for the class of alternatives

(6.40) $$p \geqq \tfrac{1}{2},\ p\mu_1 - q\mu_2 = 0.$$

The test based on Z'_r is, for $\lambda \to \infty$, consistent for the classes of alternatives

(6.41) $$\begin{cases} 1. & p > \tfrac{1}{2} \\ 2. & p \leqq \tfrac{1}{2},\ p\mu_1 - q\mu_2 > 0 \end{cases}$$

not consistent for the class of alternatives

(6.42) $$p \leqq \tfrac{1}{2},\ p\mu_1 - q\mu_2 < 0$$

and, for sufficiently small α, not consistent for the classes of alternatives

(6.43) $$p \leqq \tfrac{1}{2},\ p\mu_1 - q\mu_2 = 0.$$

The combination of the sign test and the class of tests for symmetry has two advantages

1. If n_1 falls in the critical region then the test statistic T need not be computed,
2. The tests are consistent for a larger class of alternatives than the class of tests for symmetry.

Remark

8. The combination of the sign test and WILCOXON's test for symmetry is analogous to the test for symmetry of HEMELRIJK (cf. [9], p. 69–81), which is based on $\boldsymbol{n}_1$ and the test statistic $\boldsymbol{W}$ of WILCOXON's two sample test (cf. section 4). The critical regions differ only slightly from the ones given here, but the computations are more complicated. The two sided test of HEMELRIJK is consistent for the same class of alternatives as the two sided test described in this section, but other critical regions are also given, which are consistent for other alternatives, e.g. for $p < \frac{1}{2}$, for $\theta < 0$, etc.

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