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Distributionfree tests against trend and maximum likelihood estimates of ordered parameters

by

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Distributionfree tests against trend and maximum likelihood estimates of ordered parameters

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1. Introduction and summary

It often happens that new statistical methods are developed independently by several authors who are not aware of each others work or even existence. A striking example is WILCOXON's (1945) test for the problem of two samples, a special case of KENDALL's (1943) method of rank correlation which has been independently developed a number of times (one example of this will be cited later in this paper). Inasfar as the work is being done at precisely the same time this is unavoidable but it seems nevertheless useful to give a review from time to time of work being done on specified subjects in different parts of the world.

In this paper such a review is given of a number of publications on the two subjects mentioned in the title. These subjects have been treated separately in the literature so far and the methods of treatment also are different. On the other hand the problems are closely related, which justifies a joint reviewing of both. The contribution of the present author to the methods treated is only slight and confines itself to the theory of tests for trend in a series of probabilities.

No attempt for completeness has been made and all proofs are omitted. With respect to the tests for trend especially only methods derived from KENDALL's theory of rank correlation - or closely related to it - are considered. The maximum likelihood estimation of ordered parameters is a new subject on which only a few papers have so far appeared in print.

2. Distributionfree tests against trend

The general situation under consideration may be described as follows. Given k random variables [1)]

(1) $$\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_k,$$

the hypothesis H_0 that they all have the same probability distribution has to be tested with an upward or downward trend as alternative. The observations available are k independent samples, one for each random variable, the sample corresponding to the variable $\underline{x}_i$ being denoted by

(2) $$\underline{x}_{i,1}, \underline{x}_{i,2}, \ldots, \underline{x}_{i,n_i} \quad (i = 1, 2, \ldots, k; \sum_{i=1}^{k} n_i = n).$$

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1) Random variables will be distinguished from numbers (e.g. from values assumed) by underlining their symbols.

This problem can be solved directly by applying KENDALL's (1943) rank correlation method. For samples of size 1 ($n_1 = n_2 = \cdots = n_k = 1$) this has been done by H.B. MANN (1945) and for samples of all sizes by T.J. TERPSTRA (1952) and (1955-56). The first of the two rankings required for the computation of KENDALL's statistic $\underline{S}$ then consists of k ties of sizes $n_1, n_2, \ldots, n_k$. The second ranking contains the observations (2) in order of i ; it may also contain ties.

The test statistic $\underline{S}$ may be brought in the following form. If $\underline{U}_{i,j}$ denotes the statistic for WILCOXON's (1945) two sample test for the i^{th} and j^{th} sample, i.e. the number of pairs (r,s) with $\underline{x}_{i,r} > \underline{x}_{j,s}$ plus half the number of pairs (r,s) with $\underline{x}_{i,r} = \underline{x}_{j,s}$ and if

(3) $$\underline{V}_{i,j} \overset{def}{=} 2\underline{U}_{i,j} - n_i n_j \text{ , }^{2)}$$

then

(4) $$\underline{S} = \sum_{i=1}^{k} \sum_{j=i+1}^{k} \underline{V}_{i,j} .$$

Small and large values of $\underline{S}$ are critical. $\underline{S}$ is asymptotically normal under general conditions.

In this form however the test has a rather serious drawback which becomes apparent when considering the domain of consistency. This domain proves to depend not only on the probability distributions of the $\underline{x}_i$ - as it should - but also on the ratios n_i/n_j . Consequently alternatives can be indicated where both of the unilateral tests may be made consistent using **different** values of these ratios, so that an upward or a downward trend may be found at will.

This phenomenon, pointed out and illustrated by means of an example by C. **VAN** EEDEN and J. HEMELRIJK (1955), emphasizes the importance of investigating the domain of consistency. The general principle was formulated that the set of alternatives for which a test, involving samples of different sizes, is consistent should not depend on the ratios of these sizes, except if necessary for asymptotic restrictions on these ratios, e.g. their boundedness for $n \to \infty$.

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2) Thus $\underline{V}_{i,j} = \sum_{r=1}^{n_i} \sum_{s=1}^{n_j} \operatorname{sgn}(\underline{x}_{i,r} - \underline{x}_{j,s})$, where $\operatorname{sgn} z = 1, 0, -1$ respectively if $z > 0$, $= 0$ or < 0 respectively. The symbol $\overset{def}{=}$ serves to indicate that the lefthand side of the equation is defined by the righthand side.

A small change in the test statistic is sufficient to overcome this difficulty and in his second paper (1955-56) TERPSTRA worked out the theory for the statistic

(5) $$\underline{S}' \stackrel{\text{def}}{=} \sum_{i=1}^{k} \sum_{j=i+1}^{k} \frac{V_{i,j}}{n_i n_j}.$$

Under H_0 the expected value of $\underline{S}'$ is 0. If the pooled observations consist of h ties of sizes $t_1, t_2, \dots, t_h$ $\left(\sum_{\nu=1}^{h} t_\nu = n\right)$ and if for $\mu = 2,3$ and $n \geq 3$

(6) $$T_\mu \stackrel{\text{def}}{=} 1 - \frac{1}{n^{!\mu}} \sum_{\nu=1}^{h} t_\nu^{!\mu}$$ 3)

and

(7) $$T_{2,3} \stackrel{\text{def}}{=} \tfrac{1}{2} T_2 - \tfrac{1}{3} T_3,$$

then the conditional variance of $\underline{S}'$, given the sizes of ties, is given by

(8) $$\sigma^2\{\underline{S}' | t_1, t_2, \dots, t_h; H_0\} = \tfrac{1}{3} T_3 \sum_{i=1}^{k} \frac{(k+1-2i)^2}{n_i} + T_{2,3}\left\{\left(\sum_{i=1}^{k} \frac{1}{n_i}\right)^2 - \sum_{i=1}^{k} \frac{1}{n_i^2}\right\}.$$

Moreover the asymptotic normality of $\underline{S}'$ given the sizes of the ties was proved under the following conditions:

(9) For $n \to \infty$ all n_i remain positive and bounded;

(10) $$\limsup_{n\to\infty} \max\left\{\frac{t_1}{n}, \frac{t_2}{n}, \dots, \frac{t_h}{n}\right\} < 1.$$

This means, that the number of samples, k, tends to infinity. The asymptotic normality has also been proved (but the proof has not yet been published) for k remaining bounded and $n \to \infty$; the conditions are then

(11) $$\limsup_{n\to\infty} \max\left\{\frac{n_1}{n}, \frac{n_2}{n}, \dots, \frac{n_k}{n}\right\} < 1$$

and (10).

If

(12) $$\varepsilon_{i,j} \stackrel{\text{def}}{=} P[\underline{x}_i > \underline{x}_j] - P[\underline{x}_i < \underline{x}_j]$$

the test is consistent if for $n \to \infty$

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3) $a^{!b} \stackrel{\text{def}}{=} \frac{a!}{(a-b)!}$.

(13) $$k^{-2} n^{\frac{1}{2}} \sum_{i=1}^{k} \sum_{j=i+1}^{k} \varepsilon_{i,j} \to \pm \infty$$

provided that the conditions (10) and (11) are satisfied.[4)]
The proof of this theorem has not yet been published.

The efficiency of the test has not yet been investigated.

Remarks

2.1

If all n_i are 1 the rank correlation test treated by MANN (1945) is obtained. His result for the consistency coincides with (13). The change from $\underline{s}$ to $\underline{s}'$ does not affect this special case of the test. MANN also gave a theorem on the unbiasedness of the onesided tests.

2.2

Inasfar as TERPSTRA's test is called a test against trend a trend should be said to be present if (13) is satisfied. His (1955 - 1956) paper also describes two other generalisations of KENDALL's (1943) and (1948) rank correlation methods.

2.3

G. ELVING and J.H. WHITLOCK (1950) treated the combination of a number of tests of MANN's type into an overall test using the sum of the values of $\underline{s}$ obtained for a number of rankings. They investigated the efficiency of the method for a common linear trend (and found the asymptotical value $\frac{3}{\pi}$ for equal numbers of observations in each ranking). For unequal numbers their test does not obey the general principle outlined above, if more general alternatives than they considered are also permitted.

3. A special case: generalised tests against trend in probabilities

A special case of practical importance arises if the random variables $\underline{x}_i$ $(i = 1, 2, \ldots, k)$ represent dichotomies, i.e. if their probability distributions are given by

(14) $$P[\underline{x}_i = 1] = p_i \quad ; \quad P[\underline{x}_i = 0] = q_i = 1 - p_i .$$

The hypothesis tested is then

(15) $$H_0 : p_1 = p_2 = \cdots = p_k$$

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4) Taking the fact into account, that the sizes of the ties are random variables, condition (11) should be satisfied with probability 1.

and the test statistic $\underline{S}'$ assumes the form

(16) $$\underline{S}' = \sum_{i=1}^{k}\sum_{j=i+1}^{k}\frac{\underline{a}_i\underline{b}_j-\underline{a}_j\underline{b}_i}{n_i n_j} = \sum_{i=1}^{k}\sum_{j=i+1}^{k}\left(\frac{\underline{a}_i}{n_i}-\frac{\underline{a}_j}{n_j}\right) = \sum_{i=1}^{k}(k+1-2i)\frac{\underline{a}_i}{n_i},$$

with

(17) $$\underline{a}_i \overset{\text{def}}{=} \sum_{\kappa=1}^{n_i}\underline{x}_{i,\kappa} \;;\; \underline{b}_i \overset{\text{def}}{=} n_i-\underline{a}_i \quad (i=1,2,\ldots,k).$$ [5]

This case has been considered in a generalised form.by C. VAN EEDEN and J. HEMELRIJK (1955) and J. HEMELRIJK (1955). The test-statistic used was

(18) $$\underline{W} \overset{\text{def}}{=} \sum_{i=1}^{k} g_i\frac{\underline{a}_i}{n_i},$$

where the g_i are weights satisfying the relations

(19) $$\sum_{i=1}^{k} g_i = 0 \;,\; \sum_{i=1}^{k}|g_i| = 1,$$

but to be chosen freely otherwise.[6]

With

(20) $$\underline{t}_1 \overset{\text{def}}{=} \sum_{i=1}^{k}\underline{a}_i \;,\; \underline{t}_2 \overset{\text{def}}{=} n-\underline{t}_1$$

the conditional expectation and variance of $\underline{W}$ for $\underline{t}_1 = t_1$, under H_0 are given by

(21) $$\mathcal{E}\{\underline{W}\,|\,t_1; H_0\} = 0$$

and

(22) $$\sigma^2\{\underline{W}\,|\,t_1; H_0\} = \frac{t_1 t_2}{n(n-1)}\sum_{i=1}^{k}\frac{g_i^2}{n_i}.$$

The conditional distribution of $\underline{W}$ may be approximated by means of a normal distribution with the same mean and variance as $\underline{W}$ and, with small and large values of $\underline{W}$ critical, this leads to a conditional test. It can, however, also be formulated as an unconditional test by introducing

(23) $$\underline{s}^2 \overset{\text{def}}{=} \sigma^2\{\underline{W}\,|\,\underline{t}_1; H_0\} = \frac{\underline{t}_1\underline{t}_2}{n(n-1)}\sum_{i=1}^{k}\frac{g_i^2}{n_i}$$

and using

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5) Thus $\underline{a}_i$ is e.g. the number of successes in the i^{th} series of trials if the probability of a succes is p_i $(i=1,2,\ldots,k)$.

6) The relations (19) are only imposed for the sake of convenience but do not restrict the choice of the weights essentially.

(24) $$\underline{V} \stackrel{\text{def}}{=} \frac{\underline{W}}{\underline{S}}$$

as test statistic. Under H_0 the statistic $\underline{V}$ is asymptotically normal with mean zero and variance 1 under either of the following pairs of conditions for $n \to \infty$

(25) $$\begin{cases} 1.\ \left(\sum_{i=1}^{k} \frac{g_i^2}{n_i}\right)^{-\frac{1}{2}r} \cdot n^{\frac{1}{2}r-1} \cdot \sum_{i=1}^{k} \frac{g_i^r}{n_i^{r-1}} = O(1) & \text{for each integer } r>2 \\ \text{and} \\ 2.\ \sum_{i=1}^{k} n_i p_i \to \infty,\ \sum_{i=1}^{k} n_i q_i \to \infty \end{cases}$$

or

(26) $$\begin{cases} 1.\ \left(\sum_{i=1}^{k} \frac{g_i^2}{n_i}\right)^{-1} \max_i \frac{g_i^2}{n_i^2} = o(1) \\ \text{and} \\ 2.\ \frac{\sum_{i=1}^{k} n_i q_i}{\sum_{i=1}^{k} n_i p_i} = O(1) \quad \text{and} \quad \frac{\sum_{i=1}^{k} n_i p_i}{\sum_{i=1}^{k} n_i q_i} = O(1). \end{cases}$$

Small and large values of $\underline{V}$ are critical.
The test is <u>consistent</u> if (and under certain conditions only if)

(27) $$\liminf_{n\to\infty} \left|\sum_{i=1}^{k} g_i p_i\right| > 0.$$

<u>Remarks</u>

<u>3.1</u>

The alternatives (27) include more possibilities than trend only. This depends on the choice of the weights g_i .

Taking

(28) $$g_i = 2\,\frac{k+1-2i}{K} \qquad (i=1,2,\ldots,k)$$

with

(29) $$K \stackrel{\text{def}}{=} \begin{cases} k^2 & \text{if } k \text{ is even,} \\ k^2-1 & \text{if } k \text{ is odd,} \end{cases}$$

the statistic $\underline{W}$ reduces to $\underline{S}'$ applied to the present problem. The test is then consistent if

(30) $$\liminf_{n\to\infty} K^{-1}\left|\sum_{i=1}^{k}\sum_{j=i+1}^{k}(p_i-p_j)\right| > 0,$$

which may serve as a definition for trend in probabilities.

Considerable simplifications of the conditions (25) and (26) occur if either $n_i = m$ for all i or k is bounded.

3.2

With $n_i = m$ for all i and the weights of (28) the test reduces to a straightforward application of WILCOXON's (1945) test for two samples. The two samples are then represented by the successes on one hand and the failures on the other and the values of the index i for these two groups take the place of the observations to which WILCOXON's test has to be applied. For the case $m=1$ this has already been pointed out by J.B.S. HALDANE and C.A.B. SMITH (1948), who developed WILCOXON's test independently in solving this problem. B.M. BENNETT (1956) also considered this problem -- using previous results of WILCOXON and H.B. MANN and D.R. WHITNEY (1947) - and developed the generating function of the test statistic as a function of $p_1, p_2, \ldots, p_k$. He also investigated the power of the test for the case of a linear trend in $\tau_i \overset{\text{def}}{=} \ln p_i/q_i$.

3.3

For different values of n_i the problem as treated by P. ARMITAGE (1955) who used $\underset{\sim}{S}$ instead of $\underset{\sim}{S}'$. His method results from formula (18) by taking g_i proportional to $(k+1-2i)n_i$. It does, therefore, not satisfy the principle that the domain of consistency should not depend on the ratios of the n_i.

3.4

In J. HEMELRIJK (1955) an attempt is made to develop a method for comparing trends in two series of probabilities by means of a test for their equality. No general solution is obtained but for specially designed experiments a test for this purpose is given.

3.5

TERPSTRA's test of the foregoing section could be generalised to a test for "weighted" trend in the same way by changing $\underset{\sim}{S}'$ from (5) into

$$(31) \qquad \underset{\sim}{S}'' \overset{\text{def}}{=} \sum_{i=1}^{k} \sum_{j=i+1}^{k} g_{i,j} \frac{\underset{\sim}{V}_{i,j}}{n_i n_j}.$$

The consistency of the test would then depend on the val[illegible]
$\sum_{i=1}^{k} \sum_{j=i+1}^{k} g_{i,j} \varepsilon_{i,j}$ instead of on the sum of the $\varepsilon_{i,j}$. The tran[illegible] tion from $\underset{\sim}{S}''$ to $\underset{\sim}{W}$ (cf. (18)) would then be obtained by putt[illegible]

$$(32) \qquad g_i = \sum_{j=i+1}^{k} g_{i,j} - \sum_{j=1}^{i-1} g_{i,j} \qquad (i=1,2,\ldots,k).$$

4. Maximum likelihood estimates of completely or partially ordered parameters

The problem considered in this and the next section is the following. The random variables $\underline{x}_i$ from (1) have distribution functions

$$F_i(x_i \mid \theta_i) \overset{\text{def}}{=} P[\underline{x}_i \leq x_i \mid \theta_i] \quad (i = 1, 2, \ldots, k), \tag{33}$$

where $\theta_1, \theta_2, \ldots, \theta_k$ are unknown parameters. Information about the θ_i is available in two forms. In the first place they are confined to given intervals $\mathfrak{J}_i$:

$$\theta_i \in \mathfrak{J}_i \quad (i = 1, 2, \ldots, k) \tag{34}$$

which together form a part

$$G \overset{\text{def}}{=} \prod_{i=1}^{k} \mathfrak{J}_i \tag{35}$$

of the parameterspace, i.e. for every $\theta_i \in \mathfrak{J}_i$ the function $F_i(x_i \mid \theta_i)$ must be a probability distribution.

In the second place a partial (or complete) ordering is given for the θ_i , i.e. a number of (non contradictory) relations of the form $\theta_i \leq \theta_j$ are given.

These data together delimit a convex domain D in the k - dimensional parameterspace (with $D \subset G$) and the problem is to find the maximum likelihood estimates $t_1, t_2, \ldots, t_k$ within D based on k samples as given by (2).[7)]

This is the formulation of the problem as given by CONSTANCE VAN EEDEN (1957a). The same problem has been posed and under certain additional conditions solved by H.D. BRUNK (1955), without data of the form (32). His formulation of the problem is rather different but in a forthcoming paper Miss VAN EEDEN (1957b) proves that not only is the problem identical with the one treated by her but also is his solution valid under the weaker additional condition (to be mentioned later) which she uses for her solution.

Whereas BRUNK gives an explicit formula for the maximum likelihood estimates t_i - which leads, however, to cumbersome calculations - Miss VAN EEDEN has developed theorems and lemma's which do not contain an explicit formula for the t_i , but which greatly simplify the computation of these estimates.

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7) The underlinings being omitted. We consider here the computation of the estimates from given observations, not the properties of the estimators as random variables.

In the following some of these theorems will be described and BRUNK's formula will be given in the notation of Miss VAN EEDEN's papers.

We first state the condition which Miss VAN EEDEN imposes on the distributions of the $\underline{x}_i$. Let $F_i(x_i | \theta_i)$ stand for the probability density of $\underline{x}_i$ if the distribution is continuous and for $P[\underline{x}_i = x_i | \theta_i]$ if it is discrete. Let further (cf. (2))

$$L_i(\theta_i) \stackrel{\text{def}}{=} \sum_{\alpha=1}^{n_i} \ln F_i(x_{i,\alpha} | \theta_i) \tag{35}$$

and, if M is a subset of the numbers $1, 2, \ldots, k$, let

$$L_M(\theta) \stackrel{\text{def}}{=} \sum_{i \in M} L_i(\theta) \tag{36}$$

for any real number θ.

Then, if (cf (34))

$$\vartheta_M \stackrel{\text{def}}{=} \bigcap_{i \in M} \vartheta_i, \tag{37}$$

the condition is, that for each M with ϑ_M not empty a value $\theta^* \in \vartheta_M$ exists such that for all pairs of values $\theta, \theta' \in \vartheta_M$ with θ' between θ and θ^* the function L_M satisfies

$$L_M(\theta) < L_M(\theta') < L_M(\theta^*). \tag{38}$$

Under this condition the likelihood has a unique maximum and the following theorem holds.

If from the relations defining the ordering of the θ_i one is omitted, e.g. the relation $\theta_{j_1} \leq \theta_{j_2}$, and if $t'_1, t'_2, \ldots, t'_k$ are the maximum likelihood estimates under the remaining relations, then

$$t_i = t'_i \ (i = 1, 2, \ldots, k) \qquad \text{if} \quad t'_{j_1} \leq t'_{j_2} \tag{39}$$

and

$$t_{j_1} = t_{j_2} \qquad \text{if} \quad t'_{j_1} > t'_{j_2}, \tag{40}$$

where $t_i \ (i = 1, 2, \ldots, k)$ is the maximum likelihood estimate of θ_i satisfying all relations.

This makes it possible to eliminate one of the relations if the problem can be solved under the remaining ones. In the first case, (39), the relation can be omitted altogether, in the second case, (40), further calculations can be based on the original relations supplemented by $\theta_{j_1} = \theta_{j_2}$.[8]

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8) This relation need not be true, but that does not affect th validity of the resulting estimates.

Applying this theorem step by step the estimates $t_1, t_2, \dots, t_k$ can always be found but in many cases this procedure is rather laborious and short cuts are often provided by the following theorem.

If a new relation between two of the parameters, e.g. $\theta_{j_1} \leq \theta_{j_2}$ is added to the existing relations and if $t'_1, t'_2, \dots, t'_k$ are the maximum likelihood estimates satisfying all relations, including this new one, then

(41) $$t_i = t'_i \ (i = 1, 2, \dots, k) \qquad \text{if } t'_{j_1} < t'_{j_2}$$

and

(42) $$t_{j_1} \geqq t_{j_2} \qquad \text{if } t'_{j_1} = t'_{j_2}.$$

Thus in the first case the relation may be added and in the second case this holds for the reverse relation $\theta_{j_1} \geqq \theta_{j_2}$.

In this way a partial ordering can be made complete (in one or more steps) and for a complete ordering the calculations are much simpler.

The following two lemmas, which are special cases of the two theorems mentioned, are also very useful. Let the indices of the θ_i be chosen such that for $i < j$ either no relation is given for θ_i and θ_j or the relation is $\theta_i \leqq \theta_j$. Let further $v_i \ (i = 1, 2, \dots, k)$ denote the maximum likelihood estimates of θ_i in G, i.e. only taking (34) into account, but not the ordering of the θ_i.

If then, for a pair (i, j) with $i < j$, the relation $\theta_i \leqq \theta_j$ is given and the other relations between the parameters satisfy the following conditions: for h between i and j no relation exists between θ_h and either θ_i or θ_j and for h outside the interval (i, j) either no relations or the same relations hold between the pairs (θ_h, θ_i) and (θ_h, θ_j); if in this situation $v_i > v_j$, then $t_i = t_j$.

The relation $\theta_i \leqq \theta_j$ is then eliminated in a much simpler way then the first theorem provides, for in order to apply this lemma only the v_i have to be calculated and for these explicit formulas are usually available.

The second lemma is concerned with an equally simple method for introducing new relations.

If, for a pair (i, j) with $i < j$, no relation between θ_i and θ_j is given and if the relations between the parameters satisfy the following conditions: for $h < i$ θ_h either stands in the same relation to θ_i and θ_j or it stands in no relation to θ_i, and for $h > j$ θ_h either has the same relation with θ_i and θ_j or it stands in no relation to θ_j; if in that situation $v_i \leqq v_j$ then $t_i \leqq t_j$ and the relation $\theta_i \leqq \theta_j$ may be added.[9)]

9) Cf. footnote.[8)]

These lemmas usually greatly facilitate the computation of the estimates $t_1, t_2, \dots, t_k$.

The formula of BRUNK is based on a number of definitions. In the first place, let v_M, where M is a subset of the numbers $1,2,\dots,k$, be that value of θ in ϑ_M (cf. (37)) for which $L_M(\theta)$ (cf. (36)) is maximized.

Let S_i be a subset of the numbers $1,2,\dots,k$ containing i and all those values h for which it is given that $\theta_h \leq \theta_i$ and T_i the subset containing i and all values h for which $\theta_h \geq \theta_i$ is given. Let further for any given subset N of the numbers $1,2,\dots,k$

$$S \stackrel{\text{def}}{=} \bigcup_{i \in N} S_i \quad \text{and} \quad T \stackrel{\text{def}}{=} \bigcup_{i \in N} T_i , \tag{43}$$

then the maximum likelihood estimates of $\theta_1, \theta_2, \dots, \theta_k$ are

$$t_i = \underset{\substack{T \\ i \in T \cap S}}{\operatorname{Max}} \; \underset{S}{\operatorname{Min}} \; v_{T \cap S} \qquad (i = 1,2,\dots,k), \tag{44}$$

where the Max Min is taken over all T and S corresponding with the different possible choices for N, with the restriction that $T \cap S$ must contain the number i.

Remarks

4.1

In the case of a complete ordering matters are greatly simplified. For every pair ($i, i+1$) we then have $\theta_i \leq \theta_{i+1}$ and if then $v_i > v_{i+1}$ it follows at once that $t_i = t_{i+1}$.

4.2

It is not necessary that the distributions of all $\underline{x}_i$ are of the same form. E.g. part of them may be normal (with mean unknown and given variance) and others may be Poisson-distributions with unknown mean. In practice, however, they will often be of the same form and only differ in the value of the unknown parameters θ_i. If this is the case samples, for which it is found in the course of the computations that $t_i = t_j$ must simply be pooled.

4.3

Both authors mentioned give a theorem on the consistency of the estimates.

5. A special case: ordered probabilities

The situation of section 3, where the θ_i are probabilities, is also of interest here. The observations are then $\underline{a}_1, \underline{a}_2, \dots, \underline{a}_k$ (cf. (17)), the numbers of successes in $n_1, n_2, \dots, n_k$ independent trials with $p_1, p_2, \dots, p_k$ as probabilities of succes. This case has no special features except its simplicity. If (cf.(34)) the interval $\mathcal{J}_i \supseteq (0,1)$ for all i, the maximum likelihood estimates v_i are a_i/n_i and for the case of a complete ordering (44) reduces to

$$t_i = \max_{1 \leq r \leq i} \min_{i \leq s \leq k} \frac{a_r + \dots + a_s}{n_r + \dots + n_s}. \tag{45}$$

This case was first treated by MIRIAM AYER, H.D. BRUNK e.a. (1955). Miss VAN EEDEN (1956) also treated this special case before the general one, including the partial ordering. Numerical examples of the use of her theorems and lemmas are included in her paper.

For constant k the estimators are in this case always consistent if all $n_i \to \infty$.

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Résumé

Toutefois des nouvelles méthodes statistiques sont développées indépendamment par plusieurs auteurs, qui ne connaissent pas les traveaux l'un de l'autre. Pour cette raison il est utile de donner de temps en temps des sommaires de diverses articles de différents auteurs sur le même sujet. Cela a été fait dans cette article pour deux sujets; des tests pour une tendance de variables aléatoires et l'estimation "maximum likelihood" de paramètres ordonnées partiellement ou complètement. Le cas special de probabilités inconnus est aussi traité.

Les articles, qui ont été considérés, peuvent être trouvés à la fin de l'article.