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MATHEMATICS

A LEAST SQUARES INEQUALITY FOR MAXIMUM LIKELIHOOD ESTIMATES OF ORDERED PARAMETERS [1]

BY

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1. *Introduction*

In this paper the results of a further investigation on the maximum likelihood estimates of partially or completely ordered parameters will be given. One of these results is a generalization of the following inequality for the binomial case, which may be found in [1] (p. 644).

If [2])

$$\text{(1.1)} \qquad \mathrm{P}[\mathbf{x}_i=1]=\theta_i,\ \mathrm{P}[\mathbf{x}_i=0]=1-\theta_i \quad (i=1,\ldots,k)$$

and

$$\text{(1.2)} \qquad a_i \overset{\text{def}}{=} \sum_{\gamma=1}^{n_i} x_{i,\gamma},\ b_i \overset{\text{def}}{=} n_i-a_i \qquad (i=1,\ldots,k),$$

then

$$\text{(1.3)} \qquad \sum_{i=1}^{k} n_i\left(y_i-\frac{a_i}{n_i}\right)^2 \geq \sum_{i=1}^{k} n_i\left\{\left(t_i-\frac{a_i}{n_i}\right)^2+(t_i-y_i)^2\right\}$$

for each point $(y_1,\ldots,y_k)\in D$.

The inequality (1.3) is equivalent with

$$\text{(1.4)} \qquad \sum_{i=1}^{k} n_i(t_i-y_i)\left(t_i-\frac{a_i}{n_i}\right)\leqq 0 \qquad \text{for each point } (y_1,\ldots,y_k)\in D.$$

In this paper the inequality (1.4) will be generalized for the case of partially or completely ordered parameters of other probability distributions. The problem will be treated in section 2 and in section 3 some examples will be given.

2. *The problem*

In this paper we suppose that, for each subset M of E, there exists a pair of values (i,j) with

$$\text{(2.1)} \qquad \begin{cases} 1.\ \ i\in M,\ j\in \overline{M}, \\ 2.\ \ \alpha_{i,j}\neq 0. \end{cases}$$

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[1]) Report SP 60 of the Statistical Department of the Mathematical Centre, Amsterdam.

[2]) The notation in this paper is the same as the one used in [2] and [3].

This may be supposed without any loss of generality for if there exists a subset M of E not satisfying this condition then (cf. theorem IV in [2]) the estimates $t_1, \ldots, t_k$ may be found by separately maximizing $\sum_{i \in M} L_i(y_i)$ in the domain

$$(2.2) \qquad D_1: \begin{cases} \alpha_{i,j}(y_i - y_j) \leqq 0 \\ y_i \in I_i \end{cases} \qquad (i, j \in M)$$

and $\sum_{i \in \bar{M}} L_i(y_i)$ in the domain

$$(2.3) \qquad D_2: \begin{cases} \alpha_{i,j}(y_i - y_j) \leqq 0 \\ y_i \in I_i \end{cases} \qquad (i, j \in \bar{M})$$ [3]).

Let J_i be the set of all values of y for which $F_i(x_i \mid y)$ is a distribution function $(i = 1, \ldots, k)$; we suppose J_i to be an interval. Let further for any subset M of E

$$(2.4) \qquad J_M \overset{\text{def}}{=} \bigcap_{i \in M} J_i.$$

In this paper we suppose that the following condition is satisfied.

(2.5) Condition: *For each M with $J_M \neq 0$ the function $L_M(z)$ is strictly unimodal in J_M.* [4])

Let w_M be the value of z which maximizes $L_M(z)$ in J_M and let w_i denote the value of y which maximizes $L_i(y)$ in $J_i (i = 1, \ldots, k)$. Then if I_i is the interval (c_i, d_i) and if (cf. [3], section 2) v_M is the value of z which maximizes $L_M(z)$ in I_M

$$(2.6) \qquad \begin{cases} v_M = w_M & \text{if} \quad \max\limits_{i \in M} c_i \leqq w_M \leqq \min\limits_{i \in M} d_i, \\ v_M = \max\limits_{i \in M} c_i & \text{if} \quad w_M < \max\limits_{i \in M} c_i, \\ v_M = \min\limits_{i \in M} d_i & \text{if} \quad w_M > \min\limits_{i \in M} d_i. \end{cases}$$

Now let E_0 be a subset of E with

$$(2.7) \qquad \begin{cases} 1. \quad t_i \neq v_i & \text{for each } i \in E_0, \\ 2. \quad t_i = v_i & \text{for each } i \in \bar{E}_0, \end{cases}$$

then

Lemma I: *The estimates $t_1, \ldots, t_k$ may also be found by separately maximizing $\sum_{i \in E_0} L_i(y_i)$ in the domain*

$$(2.8) \qquad D': \begin{cases} \alpha_{i,j}(y_i - y_j) \leqq 0 \\ y_i \in I_i \end{cases} \qquad (i, j \in E_0)$$

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[3]) In the definitions of the domains D_1, D_2, D', D'' and D'_ν (cf. (2.8), (2.9) and (2.21)) the coordinates which are not mentioned may assume any values.

[4]) If $J_i = I_i$ for each $i \in E$ then this condition is identical with condition (4.3) in [2].

and $\sum_{i \in \bar{E}_0} L_i(y_i)$ *in the domain*

$$D'': \; y_i \in I_i \;\; (i \in \bar{E}_0).^{5)} \qquad (2.9)$$

Proof:

The function $\sum_{i \in \bar{E}_0} L_i(y_i)$ attains its maximum in D'' for $y_i = v_i = t_i$ $(i \in \bar{E}_0)$. Further the function $L(y_1, \ldots, y_k)$ attains its maximum under the conditions $y_i = t_i (i \in \bar{E}_0)$ in D for $y_i = t_i (i = 1, \ldots, k)$, i.e. the function $\sum_{i \in \bar{E}_0} L_i(t_i) + \sum_{i \in E_0} L_i(y_i)$ attains its maximum in D for $y_i = t_i (i = 1, \ldots, k)$. Thus $\sum_{i \in E_0} L_i(y_i)$ attains its maximum in D' for $y_i = t_i (i \in E_0)$.

Now let M_0 be a subset of E_0 with

$$t_i = t_j \text{ for each pair of values } (i, j) \in M_0. \qquad (2.10)$$

Theorem I: *If* $\delta_1, \ldots, \delta_k$ *satisfy*

$$(2.11) \quad \begin{cases} 1. & \sum_{i \in M_0} \delta_i (w_{M_0} - w_i) = 0 \textit{ for each } M_0 \subset E_0 \textit{ satisfying } (2.10), \\ 2. & \delta_i > 0 \textit{ for each } i \in E, \end{cases}$$

then the function

$$Q = Q\,(y_1, \ldots, y_k) \stackrel{\text{def}}{=} \sum_{i=1}^{k} \delta_i\,(y_i - w_i)^2 \qquad (2.12)$$

attains its minimum in D *for* $y_i = t_i$ $(i = 1, \ldots, k)$.

Proof:

If M is a subset of E then $-\sum_{i \in M} \delta_i (z - w_i)^2$ is a strictly unimodal function of z; thus, analogous to theorem I in [2], $-Q$ possesses a unique maximum in D, i.e. Q possesses a unique minimum in D.

Every term of the function $\sum_{i \in \bar{E}_0} \delta_i (y_i - w_i)^2$ (and thus the function itself) attains its minimum in D'' (cf. (2.9)) for $y_i = v_i = t_i$ $(i \in \bar{E}_0)$. Lemma I then implies that it is sufficient to prove that the minimum of

$$\sum_{i \in E_0} \delta_i (y_i - w_i)^2$$

in D' (cf. (2.8)) coincides with the maximum of $\sum_{i \in E_0} L_i(y_i)$ in D'.

We first prove this (by induction [6])) under the following stronger condition for $\delta_1, \ldots, \delta_k$

$$(2.13) \quad \begin{cases} 1. & \sum_{i \in M} \delta_i (w_M - w_i) = 0 \text{ for each } M \subset E_0, \\ 2. & \delta_i > 0 \text{ for each } i \in E_0. \end{cases}$$

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[5]) Cf. footnote 3.

[6]) Cf. the proof of theorem I in [2], which runs along the same ways.

Let $M_\nu (\nu=1, \ldots, N)$ be subsets of E_0 with

$$(2.14)\quad \begin{cases} 1. & \bigcup_{\nu=1}^{N} M_\nu = E_0, \\ 2. & M_{\nu_1} \cap M_{\nu_2} = 0 \text{ for each pair of values } (\nu_1, \nu_2) \text{ with } \nu_1 \neq \nu_2, \\ 3. & I_{M_\nu} \neq 0 \text{ for each } \nu = 1, \ldots, N. \end{cases}$$

Let further (cf. (4.6) in [2])

$$(2.15)\qquad G_N \overset{\text{def}}{=} \prod_{\nu=1}^{N} I_{M_\nu}$$

and (cf. (4.7) and (4.8) in [2])

$$(2.16)\qquad \begin{cases} Q_{M_\nu}(z) \overset{\text{def}}{=} \sum_{i \in M_\nu} \delta_i (z - w_i)^2, \\ Q'(z_1, \ldots, z_N) \overset{\text{def}}{=} \sum_{\nu=1}^{N} Q_{M_\nu}(z_\nu) \end{cases}$$

and (cf. (4.9) in [2])

$$(2.17)\qquad D'_{N,s} \overset{\text{def}}{=} D' \cap G_N,$$

where s denotes the number of essential restrictions defining D'. Then the function $Q_{M_\nu}(z)$ attains its minimum in the interval $(-\infty, +\infty)$ for

$$(2.18)\qquad z = \frac{\sum_{i \in M_\nu} \delta_i w_i}{\sum_{i \in M_\nu} \delta_i} = w_{M_\nu} \qquad \text{(cf. (2.13))}$$

and the fact that $-Q_{M_\nu}(z)$ is strictly unimodal in the interval $(-\infty, +\infty)$ then entails that $Q_{M_\nu}(z)$ attains its minimum in I_{M_ν} for $z = v_M$ (cf. (2.6)). The minimum of Q' in G_N thus coincides with the maximum of L' in $G_N = D'_{N,0}$.

Now suppose that it has been proved that the minimum of Q' in $D'_{N,s}$ coincides with the maximum of L' in $D'_{N,s}$ for each $s \leqq s_0$, for each partition $M_1, \ldots, M_N$ of E_0 satisfying (2.14) and for each N.

We then prove that the same holds for s_0+1 essential restrictions. Consider, for a given partition $M_1, \ldots, M_N$ satisfying (2.14) a domain D'_{N,s_0+1} and the domain D'_{N,s_0} which is obtained by omitting one of the essential restrictions defining D'_{N,s_0+1}. Let this be the restriction $z_{\nu_1} \leqq z_{\nu_2}$. Then $D'_{N,s_0+1} \subset D'_{N,s_0}$. The minimum of Q' in D'_{N,s_0} coincides with the maximum of L' in D'_{N,s_0} in (say) the point $(z_1^0, \ldots, z_N^0)$ and the following two cases may be distinguished:

1. $z_{\nu_1}^0 \leqq z_{\nu_2}^0$; then $(z_1^0, \ldots, z_N^0) \in D'_{N,s_0+1}$. Thus in this case the minimum of Q' in D'_{N,s_0+1} coincides with the maximum of L' in D'_{N,s_0+1}.

2. $z_{\nu_1}^0 > z_{\nu_2}^0$; then (cf. theorem II in [2]) Q' attains its minimum (and L' its maximum) in D'_{N,s_0+1} for $z_{\nu_1} = z_{\nu_2}$. The domain D'_{N,s_0+1} reduces, with $z_{\nu_1} = z_{\nu_2}$, to a domain $D''_{N-1,s_0'}$ with $s_0' \leqq s_0$ and the minimum of Q' under the condition $z_{\nu_1} = z_{\nu_2}$ in $D''_{N-1,s_0'}$ coincides with the maximum of L' under

the condition $z_{\nu_1}=z_{\nu_2}$ in $D'_{N-1,s_0'}$. Thus if $\delta_i(i \in E_0)$ satisfy (2.13) then the minimum of Q' in $D'_{N,s}$ coincides with the maximum of L' in $D'_{N,s}$. This holds for each N, i.e. it holds for $N=k'$, if k' is the number of elements of E_0. Thus if $\delta_i(i \in E_0)$ satisfy (2.13) then the minimum of $\sum_{i\in E_0} \delta_i(y_i-w_i)^2$ in $D'_{k',s}=D'$ coincides with the maximum of $\sum_{i\in E_0} L_i(y_i)$ in D'.

We now prove the theorem under condition (2.11). Let $E_\nu(\nu=1, \ldots, K)$ be subsets of E_0 with

$$(2.19)\quad \begin{cases} 1. \quad \bigcup_{\nu=1}^{K} E_\nu = E_0, \\ 2. \quad t_i < t_j \text{ for each pair of values } (i, j) \text{ with } i \in E_{\nu_1}, \\ \qquad\qquad j \in E_{\nu_2}\ (\nu_1<\nu_2;\ \nu_1, \nu_2=1, \ldots, K), \\ 3. \quad t_i=t_j \text{ for each pair of values } (i, j) \in E_\nu(\nu=1, \ldots, K), \end{cases}$$

then (2.11) is identical with

$$(2.20)\quad \begin{cases} 1. \quad \sum_{i\in M} \delta_i(w_M-w_i)=0 \quad \text{for each } M \subset E_\nu\ (\nu=1, \ldots, K), \\ 2. \quad \delta_i>0 \quad \text{for each } i \in E. \end{cases}$$

Further it follows from lemma II in [3] and theorem IV in [2] that the maximum of $\sum_{i\in E_0} L_i(y_i)$ in D' may also be found by maximizing, for $\nu=1, \ldots, K$, $\sum_{i\in E_\nu} L_i(y_i)$ in the domain [7])

$$(2.21)\qquad D'_\nu: \begin{cases} \alpha_{i,j}(y_i-y_j) \leqq 0 \\ y_i \in I_i \end{cases} \qquad (i,j \in E_\nu).$$

Further the fact that $\delta_1, \ldots, \delta_k$ satisfy (2.20) entails that the minimum of $\sum_{i\in E_\nu} \delta_i(y_i-w_i)^2$ in D'_ν coincides with the maximum of $\sum_{i\in E_\nu} L_i(y_i)$ in $D'_\nu(\nu=1, \ldots, K)$. This proves the theorem under condition (2.11).

Remark 1:

In the proof of theorem I the fact has been used that the maximum of L in D coincides with the maximum of L in the domain

$$(2.22)\qquad B \overset{\text{def}}{=} D'' \cap \bigcap_{\nu=1}^{K} D'_\nu.\ ^{8)}$$

The same holds for the minimum of Q.

Theorem II: *If* $\delta_1, \ldots, \delta_k$ *satisfy* (2.11) *and if* $t_i \neq w_i$ *for at least one value of* $i \in E$ *then the ellipsoid*

$$(2.23)\qquad \sum_{i=1}^{k} \delta_i\left(y_i-\frac{t_i+w_i}{2}\right)^2=\sum_{i=1}^{k} \delta_i\left(\frac{t_i-w_i}{2}\right)^2$$

touches the domains B *(and* D*) in the point* $(t_1, \ldots, t_k)$.

[7]) Cf. footnote 3.

[8]) The domain D is independent of $t_1, \ldots, t_k$, but B depends on these estimates.

Proof:

If $t_i \neq w_i$ for at least one value of $i \in E$ then $(t_1, \ldots, t_k)$ is a borderpoint of B and $(w_1, \ldots, w_k) \notin B$. Further, if $0 < \beta \leqq 1$, then

$$\{\beta w_1 + (1-\beta)t_1, \ldots, \beta w_k + (1-\beta)t_k\} \notin B.$$

This may be seen as follows. From lemma I in [2] it follows that

$$\sum_{i=1}^{k} L_i\{\beta w_i + (1-\beta)\, t_i\}$$

is a monotone increasing function of β in the interval $0 \leqq \beta \leqq 1$. Thus if $0 < \beta \leqq 1$ then

$$\sum_{i=1} L_i\{\beta w_i + (1-\beta)t_i\} > \sum_{i=1}^{k} L_i(t_i).$$

The fact that L attains its maximum in B in the point $(t_1, \ldots, t_k)$ then implies $\{\beta w_1 + (1-\beta)t_1, \ldots, \beta w_k + (1-\beta)t_k\} \notin B$.

Now let

$$\text{(2.24)} \qquad \begin{cases} y_i' \overset{\text{def}}{=} \sqrt{\delta_i}\left(y_i - \dfrac{t_i + w_i}{2}\right), \\ w_i' \overset{\text{def}}{=} \sqrt{\delta_i}\left(w_i - \dfrac{t_i + w_i}{2}\right) = \sqrt{\delta_i}\,\dfrac{w_i - t_i}{2}, \qquad (i = 1, \ldots, k) \\ t_i' \overset{\text{def}}{=} \sqrt{\delta_i}\left(t_i - \dfrac{t_i + w_i}{2}\right) = -\sqrt{\delta_i}\,\dfrac{w_i - t_i}{2}, \end{cases}$$

then (2.23) reduces to

$$\text{(2.25)} \qquad \sum_{i=1}^{k} y_i'^2 = \sum_{i=1}^{k} t_i'^2 \left(= \sum_{i=1}^{k} w_i'^2 \right)$$

and B reduces to a domain B'. Further $(t_1', \ldots, t_k')$ is a borderpoint of B', $(w_1, \ldots, w_k) \notin B'$ and, for each β with $0 < \beta \leqq 1$,

$$\{\beta w_1' + (1-\beta)t_1', \ldots, \beta w_k' + (1-\beta)t_k'\} \notin B'.$$

From (2.24) follows

$$\text{(2.26)} \qquad \sum_{i=1}^{k} \delta_i\,(y_i - w_i)^2 = \sum_{i=1}^{k} (y_i' - w_i')^2.$$

From theorem I and remark 1 then follows that $\sum_{i=1}^{k} (y_i' - w_i')^2$ attains its minimum in B' in the point $(t_1', \ldots, t_k')$, thus the sphere (2.25) touches B' in $(t_1', \ldots, t_k')$; i.e. the ellipsoid (2.23) touches B in $(t_1, \ldots, t_k)$.

We now prove the following lemma:

Lemma II: *Let C be a convex domain and S a point on its boundary. Let K_S be an ellipsoid touching C on the outside in S and let the diameter of K_S, passing through S, intersect K_S in a point U. Let further Y be a point inside C or on its boundary and K_Y an ellipsoid with diameter YU, with axes parallel to those of K_S and with the length of the axes proportional to those of K_S. Then S lies inside or on K_Y.*

Proof:

We apply a linear transformation such that K_S reduces to a sphere K'_S; then K_Y reduces to a sphere K'_Y, C to a convex domain C', S to a point S' on the boundary of C' and Y to a point Y' inside or on the boundary of C'. The sphere K'_S touches C' in S' and it may easily be seen that S' lies inside or on K'_Y.

Theorem III: *If* $\delta_1, \ldots, \delta_k$ *satisfy* (2.11) *then*

(2.27) $$\sum_{i=1}^{k} \delta_i(t_i - w_i)(t_i - Y_i) \leqq 0 \text{ for each point } (Y_1, \ldots, Y_k) \in B.$$

Proof:

If $t_i = w_i$ for each $i \in E$ then (2.11) reduces to

(2.28) $$\delta_i > 0 \text{ for each } i \in E.$$

Then the theorem is immediately clear.

If $t_i \neq w_i$ for at least one value of $i \in E$ then (cf theorem II) the ellipsoid (2.23) touches B in the point $(t_1, \ldots, t_k)$. Thus if $(Y_1, \ldots, Y_k)$ is a point in B then it follows from lemma II that $(t_1, \ldots, t_k)$ lies inside or on the ellipsoid

(2.29) $$\sum_{i=1}^{k} \delta_i \left(y_i - \frac{w_i + Y_i}{2}\right)^2 = \sum_{i=1}^{k} \delta_i \left(\frac{w_i - Y_i}{2}\right)^2,$$

i.e. $t_1, \ldots, t_k$ satisfy

(2.30) $$\sum_{i=1}^{k} \delta_i \left(t_i - \frac{w_i + Y_i}{2}\right)^2 \leq \sum_{i=1}^{k} \delta_i \left(\frac{w_i - Y_i}{2}\right)^2$$

and (2.30) is identical with

(2.31) $$\sum_{i=1}^{k} \delta_i (t_i - w_i)(t_i - Y_i) \leqq 0.$$

Further it follows from the foregoing that the following theorem holds.

Theorem IV: *If* $\delta_1, \ldots, \delta_k$ *satisfy* (2.11) *then there exists exactly one point* $(y_1, \ldots, y_k) \in B$ *satisfying the inequalities*

(2.32) $$\sum_{i=1}^{k} \delta_i (y_i - w_i)(y_i - Y_i) \leqq 0 \qquad (Y_1, \ldots, Y_k) \in B.$$

Thus if $\delta_1, \ldots, \delta_k$ satisfy (2.11) and are independent of $t_1, \ldots, t_k$ then the estimates $t_1, \ldots, t_k$ may also be found by minimizing $Q(y_1, \ldots, y_k)$ in D or by solving the inequalities (2.32) with $(Y_1, \ldots, Y_k) \in D$.

3. Examples

If (cf. section 1)

(3.1) $$\mathrm{P}[\mathbf{x}_i = 1] = \theta_i, \ \mathrm{P}[\mathbf{x}_i = 0] = 1 - \theta_i \quad (i = 1, \ldots, k)$$

and

(3.2) $$a_i \overset{\text{def}}{=} \sum_{\gamma=1}^{n_i} x_{i,\gamma}, \quad b_i \overset{\text{def}}{=} n_i - a_i \qquad (i = 1, \ldots, k),$$

then

(3.3) $$w_M = \frac{\sum_{i \in M} a_i}{\sum_{i \in M} n_i} = \frac{\sum_{i \in M} n_i w_i}{\sum_{i \in M} n_i}.$$

Thus if $\delta_i = n_i$ $(i = 1, \ldots, k)$, then $\delta_1, \ldots, \delta_k$ satisfy (2.11) and are independent of $w_1, \ldots, w_k$; i.e. the estimates $t_1, \ldots, t_k$ may also be found by minimizing

(3.4) $$Q(y_1, \ldots, y_k) = \sum_{i=1}^{k} n_i (y_i - w_i)^2$$

in D and $t_1, \ldots, t_k$ satisfy

(3.5) $$\sum_{i=1}^{k} n_i (t_i - w_i)(t_i - Y_i) \leqq 0 \qquad \text{for each point } (Y_1, \ldots, Y_k) \in B.$$

If $\mathbf{x}_i$ possesses a normal distribution with mean θ_i and variance σ_i^2 $(i = 1, \ldots, k)$, where σ_i^2/σ_j^2 is known for each pair of values (i, j) then

(3.6) $$w_M = \frac{\sum_{i \in M} \frac{1}{\sigma_i^2} \sum_{\gamma=1}^{n_i} x_{i,\gamma}}{\sum_{i \in M} \frac{n_i}{\sigma_i^2}} = \frac{\sum_{i \in M} \frac{n_i w_i}{\sigma_i^2}}{\sum_{i \in M} \frac{n_i}{\sigma_i^2}},$$

thus $\delta_i = n_i/\sigma_i^2$ $(i = 1, \ldots, k)$ satisfies (2.11); i.e. the estimates $t_1, \ldots, t_k$ may also be found by minimizing [9])

(3.7) $$Q(y_1, \ldots, y_k) = \sum_{i=1}^{k} \frac{n_i}{\sigma_i^2} (y_i - w_i)^2$$

in D and $t_1, \ldots, t_k$ satisfy the inequalities

(3.8) $$\sum_{i=1}^{k} \frac{n_i}{\sigma_i^2} (t_i - w_i)(t_i - Y_i) \leqq 0 \qquad \text{for each point } (Y_1, \ldots, Y_k) \in B.$$

In the same way it may be proved that $\delta_i = n_i$ $(i = 1, \ldots, k)$ satisfies (2.11) if

1. $\mathbf{x}_i$ possesses a normal distribution with known mean μ_i and variance $\theta_i (i = 1, \ldots, k)$,
2. $\mathbf{x}_i$ possesses an exponential distribution

(3.9) $$\mathrm{P}[\mathbf{x}_i \leqq x] = 1 - e^{-\frac{x}{\theta_i}} \qquad (i = 1, \ldots, k).$$

In all these cases the estimates $t_1, \ldots, t_k$ are the ordinary least squares estimates in D.

If on the other hand $\mathbf{x}_i$ possesses a rectangular distribution "between"

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[9]) This also follows from

$$L(y_1, \ldots, y_k) = -\tfrac{1}{2} \sum_{i=1}^{k} n_i \ln 2\pi\sigma_i^2 - \tfrac{1}{2} \sum_{i=1}^{k} \frac{\sum_{\gamma=1}^{n_i} (x_{i,\gamma} - y_i)^2}{\sigma_i^2} =$$

$$= -\tfrac{1}{2} \sum_{i=1}^{k} n_i \ln 2\pi\sigma_i^2 - \tfrac{1}{2} \sum_{i=1}^{k} \frac{\sum_{\gamma=1}^{n_i} (x_{i,\gamma} - w_i)^2}{\sigma_i^2} - \tfrac{1}{2} \sum_{i=1}^{k} \frac{n_i}{\sigma_i^2} (y_i - w_i)^2.$$

0 and $\theta_i (i=1, \ldots, k)$ then

$$w_M = \max_{i \in M} \max_{1 \leq \gamma \leq n_i} x_{i,\gamma} = \max_{i \in M} w_i . \tag{3.10}$$

Thus in this case there are no numbers $\delta_1, \ldots, \delta_k$ satisfying (2.11).

Note added in proof

If v'_M is the value of z which maximizes $Q_M(z)$ in I_M then the theorems I – IV also hold if $\delta_1, \ldots, \delta_k$ are chosen in such a way that

$$\begin{cases} 1. & \max_T v'_{T \cap E_\nu} = \min_S v'_{S \cap E_\nu} = v_{E_\nu} \quad (\nu = 1, \ldots, K), \\ 2. & \delta_i > 0 \text{ for each } i \in E. \end{cases}$$

The proof, which is based on formula (2.5) in [3] will be given in a following paper.

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