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On slippage tests, 3.  
Two distributionfree slippage tests  
and two tables.

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ON SLIPPAGE TESTS

III. TWO DISTRIBUTIONFREE SLIPPAGE TESTS AND TWO TABLES<sup>1)</sup>

BY

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7. *Slippage tests for the method of  $m$  rankings*

In the well known method of  $m$  rankings due to M. FRIEDMAN (1937) (cf. M. G. KENDALL (1955), chapters 6 and 7)  $m$  “observers” are considered. Each observer ranks  $k$  “objects”. The method of  $m$  rankings enables us to investigate whether the observers agree in their opinion about the objects. For that reason one tests the hypothesis  $H_0$ , which states that the rankings are chosen at random from the collection of all permutations of the numbers  $1, \dots, k$  and that they are independent.

Here we present tests which are powerful especially against the alternative that one of the objects has larger probability than the other ones of being ranked high (or low), whilst the other  $(k-1)$  objects are ranked in a random order. We denote the sums of the  $m$  ranks of each object by

$$(7.1) \quad \mathbf{s}_1, \dots, \mathbf{s}_k, \quad (m \leq \mathbf{s}_i \leq km).$$

Obviously we have

$$(7.2) \quad \sum_{i=1}^k \mathbf{s}_i = \frac{1}{2} mk(k+1).$$

In section 8 the following theorem will be proved.

**Theorem 7.1.** *For each pair  $\mathbf{s}_i, \mathbf{s}_j$  of the variates (7.1) and for every pair of integers  $s_i, s_j$  the following inequality holds under  $H_0$*

$$(7.3) \quad P[\mathbf{s}_i \leq s_i \text{ and } \mathbf{s}_j \leq s_j] \leq P[\mathbf{s}_i \leq s_i] \cdot P[\mathbf{s}_j \leq s_j].$$

So we can apply our approximation method of section 2 for obtaining slippage tests for  $\mathbf{s}_1, \dots, \mathbf{s}_k$ . Because the marginal distributions of the  $\mathbf{s}_i$  are all equal under  $H_0$ , the test statistic for the test against slippage to the right is  $\max \mathbf{s}_i$  and for testing against slippage to the left  $\min \mathbf{s}_i$ . The critical values are determined by the smallest integer  $S_\alpha$  satisfying

$$(7.4) \quad P[\mathbf{s}_i \geq S_\alpha] \leq \alpha/k$$

and the largest integer  $s_\alpha$  satisfying

$$(7.5) \quad P[\mathbf{s}_i \leq s_\alpha] \leq \alpha/k,$$

respectively.

<sup>1)</sup> Parts I and II in Indagationes Mathematicae, 20, 38–55 (1958) and Proc. Kon. Ned. Ak. van Wetensch., 61, Series A, 38–55 (1958).



The distribution of  $\mathbf{s}_i$  is easily seen to be symmetric with respect to the mean value  $\frac{1}{2}m(k+1)$ , so we have

$$(7.6) \quad s_\alpha = m(k+1) - S_\alpha.$$

In section 8 it will be shown that the distribution of  $\mathbf{s}_i$ , under  $H_0$ , reads

$$(7.7) \quad \left\{ \begin{aligned} P[\mathbf{s}_i = n] &= \sum_{x=0}^{\infty} I_{n-kx-m} \binom{m}{x} \binom{n-kx-1}{m-1} (-1)^x k^{-m}, \\ &\quad (i = 1, \dots, k; m \leq n \leq km)^2 \end{aligned} \right.$$

where  $I_y$  is defined by

$$(7.8) \quad \begin{cases} I_y = 0 & \text{if } y < 0, \\ I_y = 1 & \text{if } y \geq 0. \end{cases}$$

The tables of critical values  $s_\alpha$ , presented in section 11, are based on this formula.

#### 8. Proofs of the results of section 7

First we shall prove theorem 7.1. We suppose that both  $s_i$  and  $s_j$  are lying between  $m$  and  $km$ , because otherwise (7.3) obviously holds with the equality sign. For  $m=1$  we have

$$(8.1) \quad \begin{cases} P[\mathbf{s}_i \leq s_i \text{ and } \mathbf{s}_j \leq s_j | m=1] = \frac{s_i s_j - \min(s_i, s_j)}{k(k-1)}, \\ P[\mathbf{s}_i \leq s_i | m=1] = \frac{s_i}{k}, \\ P[\mathbf{s}_j \leq s_j | m=1] = \frac{s_j}{k}, \end{cases}$$

so in that case (7.3) is true. Now let us suppose that (7.3) is true for  $m$  observers, then we have

$$(8.2) \quad \left\{ \begin{aligned} &P[\mathbf{s}_i \leq s_i \text{ and } \mathbf{s}_j \leq s_j | m+1] = \\ &= \sum_{a \neq b} P[\mathbf{s}_i \leq s_i - a \text{ and } \mathbf{s}_j \leq s_j - b | m] \cdot P[\text{the } i\text{-th object has rank } a \text{ and the } j\text{-th object rank } b \text{ in the } (m+1)\text{-st ranking}] = \\ &= \sum_{a \neq b} P[\mathbf{s}_i \leq s_i - a \text{ and } \mathbf{s}_j \leq s_j - b | m] \cdot \frac{1}{k(k-1)} \leq \\ &\leq \sum_{a \neq b} P[\mathbf{s}_i \leq s_i - a | m] \cdot P[\mathbf{s}_j \leq s_j - b | m] \cdot \frac{1}{k(k-1)} = \\ &= \sum_{a=1}^k P[\mathbf{s}_i \leq s_i - a | m] \cdot \frac{1}{k} \cdot \sum_{b=1}^k P[\mathbf{s}_j \leq s_j - b | m] \cdot \frac{1}{k} + \\ &+ \frac{1}{k^2(k-1)} \sum_{a=1}^k P[\mathbf{s}_i \leq s_i - a | m] \cdot \sum_{b=1}^k P[\mathbf{s}_j \leq s_j - b | m] + \end{aligned} \right.$$

<sup>2)</sup> We owe this formula to Mr. A. BENARD, Statistical Department of the Mathematical Centre.

$$(8.2) \quad \left\{ \begin{aligned} & -\frac{1}{k(k-1)} \sum_{a=1}^k P[\mathbf{s}_i \leq s_i - a | m] \cdot P[\mathbf{s}_j \leq s_j - a | m] = \\ & = P[\mathbf{s}_i \leq s_i | m+1] \cdot P[\mathbf{s}_j \leq s_j | m+1] + \\ & -\frac{1}{k(k-1)} \sum_{a=1}^k \left\{ P[\mathbf{s}_i \leq s_i - a | m] - \frac{\sum_{b=1}^k P[\mathbf{s}_i \leq s_i - b | m]}{k} \right\} \cdot \\ & \cdot \left\{ P[\mathbf{s}_j \leq s_j - a | m] - \frac{\sum_{b=1}^k P[\mathbf{s}_j \leq s_j - b | m]}{k} \right\} \leq \\ & \leq P[\mathbf{s}_i \leq s_i | m+1] \cdot P[\mathbf{s}_j \leq s_j | m+1]. \end{aligned} \right.$$

So theorem 7.1 is proved by induction.

Formula 7.7 can be proved in the following way:

$k^m P[\mathbf{s}_i = n | m]$  = the number of partitions of  $n$  into  $m$  positive integers, no one being larger than  $k$  (different permutations of the same integers are counted as different partitions).

Thus

$k^m P[\mathbf{s}_i = n | m]$  = coefficient of  $z^n$  in  $(z + z^2 + \dots + z^k)^m$  = coefficient of  $z^{n-m}$  in  $\left(\frac{1-z^k}{1-z}\right)^m$  = coefficient of  $z^{n-m}$  in

$$\begin{aligned} & \sum_{x=0}^{\infty} \binom{m}{x} (-1)^x z^{kx} \sum_{r=0}^{\infty} \binom{m+r-1}{r} z^r = \\ & = \sum_{x=0}^{\infty} I_{n-kx-m} \binom{m}{x} \binom{n-kx-1}{m-1} (-1)^x, \end{aligned}$$

which proves (7.7).

#### 9. A distribution free $k$ -sample slippage test

We consider the independent variates

$$(9.1) \quad \mathbf{u}_1, \dots, \mathbf{u}_k,$$

which have, under  $H_0$ , the same continuous distribution function. From the  $i^{\text{th}}$  population we have  $t_i$  independent observations  $\mathbf{u}_{ij}$  ( $j = 1, \dots, t_i$ ). We want to test  $H_0$  against the alternatives

$$(9.2) \quad H_{1i} \left\{ \begin{aligned} & P[\mathbf{u}_i > \mathbf{u}_j] > \frac{1}{2} \quad (j \neq i), \\ & \mathbf{u}_j \quad (j = 1, \dots, i-1, i+1, \dots, k) \text{ follow the same distribution,} \end{aligned} \right.$$

for one unknown value of  $i$  and

$$(9.3) \quad H_{2i} \left\{ \begin{aligned} & P[\mathbf{u}_i > \mathbf{u}_j] < \frac{1}{2} \quad (j \neq i), \\ & \mathbf{u}_j \quad (j = 1, \dots, i-1, i+1, \dots, k) \text{ follow the same distribution.} \end{aligned} \right.$$

Now the following test procedure is proposed. If all observations  $\mathbf{u}_{ij}$  ( $i = 1, \dots, k$ ;  $j = 1, \dots, t_i$ ) are ranked, we denote by  $\mathbf{T}_i$  the sum of the ranks of the observations  $\mathbf{u}_{ij}$  ( $j = 1, \dots, t_i$ ). As  $\mathbf{T}_i$  is a linear function of WILCOXON'S test statistic applied to the  $i^{\text{th}}$  sample and the other  $k-1$



samples together, its distribution function under  $H_0$  is known (cf. H. B. MANN and D. R. WHITNEY (1947)). So for each set of values  $T_1, \dots, T_k$  we can, under  $H_0$ , compute

$$(9.4) \quad q_i = P[T_i \geq T_i].$$

Now, when testing  $H_0$  against  $H_{1,i}$ ,  $H_0$  is rejected when  $\min q_i \leq \alpha/k$ . A similar procedure is followed for slippage to the left. In the next section we shall prove the inequality

$$(9.5) \quad P[T_i \geq T_i \text{ and } T_j \geq T_j] \leq P[T_i \geq T_i] \cdot P[T_j \geq T_j],$$

so the limits, between which the level of significance may vary, are known also in this case.

Let now for every fixed  $i$  the hypothesis  $H_{1,i}$  be

$$\begin{cases} P[u_i > u_j] > \frac{1}{2} & (j \neq i), \\ u_j & (j = 1, \dots, i-1, i+1, \dots, k), \end{cases} \text{ follow the same distribution.}$$

Put

$$P[T_i|H_0] \stackrel{\text{def}}{=} P[T_i \geq T_i|H_0].$$

This probability still depends on  $t_1, \dots, t_k$ .

In the same way as in sections 3 and 5 we consider the decision procedure  $\delta$ :

Decide that  $H_0$  is true if

$$P[T_j|H_0] > \frac{\alpha}{k} \text{ for } j=1, \dots, k.$$

Decide that  $H_{1,i}$  is true if  $j$  is the smallest integer such that

$$P[T_j|H_0] \leq \frac{\alpha}{k} \text{ and } P[T_l|H_0] \geq P[T_j|H_0], \quad l \neq j.$$

We prove in the next section

**Theorem 9.1.** *If  $H_{1,i}$  is true, the probability of a correct decision with the procedure  $\delta$  tends to 1 if  $t_1 \rightarrow \infty, \dots, t_k \rightarrow \infty$  such that*

$$\liminf \frac{t_i}{\sum t_l} > 0 \quad (i=1, \dots, k).$$

Another test for the  $k$ -sample slippage problem was proposed by F. MOSTELLER (1948) (cf. also F. MOSTELLER and J. W. TUKEY (1950)) who uses as test statistic the number of observations of the sample with the largest observation which exceed all observations of all other samples. A comparison of the power of both tests with respect to some alternatives of practical interest seems desirable.

#### 10. Proof of the inequality (9.5) and of theorem 9.1 <sup>3)</sup>

For definiteness we take in (9.5)  $i=1, j=2$ . We also take  $k=3$ . This

<sup>3)</sup> The proofs in this section were found by Mr. H. KESTEN, then working in the Statistical Department of the Mathematical Centre.



is no restriction on the generality as pooling of the 3rd, 4th, ... and  $k$ th sample does not affect

$$P[T_1|H_0], P[T_2|H_0] \text{ or } P[T_1, T_2|H_0] \stackrel{\text{def}}{=} P[\mathbf{T}_1 \geq T_1 \text{ and } \mathbf{T}_2 \geq T_2|H_0].$$

Put now

$$(10.1) \quad t \stackrel{\text{def}}{=} t_1 + t_2 + t_3$$

and define

$$\begin{aligned} P_{n_1, n_2, n_3}[T_i] &\stackrel{\text{def}}{=} P[T_i|H_0] \text{ if } t_1 = n_1, t_2 = n_2, t_3 = n_3. \\ P_{n_1, n_2, n_3}[T_i, 1] &\stackrel{\text{def}}{=} P[\mathbf{T}_i \geq T_i \text{ and the largest element belongs to sample} \\ &\quad \text{number 1}|H_0] \text{ if } t_1 = n_1, t_2 = n_2, t_3 = n_3. \\ P_{n_1, n_2, n_3}[T_i|1] &\stackrel{\text{def}}{=} \text{the conditional probability of } \mathbf{T}_i \geq T_i \text{ under } H_0, \text{ given} \\ &\quad \text{that the largest element belongs to sample number 1} \\ &\quad \text{if } t_1 = n_1, t_2 = n_2, t_3 = n_3. \end{aligned}$$

In the same way we define

$$P_{n_1, n_2, n_3}[T_i, T_j], P_{n_1, n_2, n_3}[T_i, T_j, 1] \text{ and } P_{n_1, n_2, n_3}[T_i, T_j|1]$$

for the events  $\{\mathbf{T}_i \geq T_i \text{ and } \mathbf{T}_j \geq T_j\}$ .

We shall prove (9.5) by induction with respect to  $n_1 + n_2 + n_3$ . So we have to prove

$$(10.2) \quad P_{n_1, n_2, n_3}[T_1, T_2] \leq P_{n_1, n_2, n_3}[T_1] \cdot P_{n_1, n_2, n_3}[T_2].$$

Clearly (10.2) holds for  $n_1 + n_2 + n_3 = 2$  (we take  $\mathbf{T}_i = 0$  with probability 1 when  $t_i = 0$ ). Now suppose (10.2) holds if  $n_1 + n_2 + n_3 \leq t - 1$ . We have

$$(10.3) \quad P_{t_1, t_2, t_3}[T_1, T_2] = \sum_{i=1}^3 \frac{t_i}{t} P_{t_1, t_2, t_3}[T_1, T_2|i].$$

For the first term of the sum in the right hand member we get

$$(10.4) \quad \left\{ \begin{aligned} P_{t_1, t_2, t_3}[T_1, T_2|1] &= P_{t_1-1, t_2, t_3}[T_1-t, T_2] \leq \\ &\quad \text{(according to our assumption)} \\ &\leq P_{t_1-1, t_2, t_3}[T_1-t] \cdot P_{t_1-1, t_2, t_3}[T_2] = P_{t_1, t_2, t_3}[T_1|1] \cdot P_{t_1, t_2, t_3}[T_2|1]. \end{aligned} \right.$$

In the same way, it can be derived that

$$(10.5) \quad P_{t_1, t_2, t_3}[T_1, T_2|2] \leq P_{t_1, t_2, t_3}[T_1|2] \cdot P_{t_1, t_2, t_3}[T_2|2].$$

Further

$$(10.6) \quad \left\{ \begin{aligned} P_{t_1, t_2, t_3}[T_1, T_2|3] &= P_{t_1, t_2, t_3-1}[T_1, T_2] \leq P_{t_1, t_2, t_3-1}[T_1] \cdot P_{t_1, t_2, t_3-1}[T_2] = \\ &= P_{t_1, t_2, t_3}[T_1|3] \cdot P_{t_1, t_2, t_3}[T_2|3]. \end{aligned} \right.$$

So, combining (10.3), (10.4), (10.5) and (10.6) we find, dropping the subscripts

$$(10.7) \quad P[T_1, T_2] \leq \sum_{i=1}^3 \frac{t_i}{t} P[T_1|i] \cdot P[T_2|i] = \sum_{i=1}^3 P[T_1|i] \cdot P[T_2|i].$$



We have

$$(10.8) \quad P[T_1|2] = P[T_1|3] = P[T_1|2 \text{ or } 3]$$

and similarly with 1 and 2 interchanged, and

$$(10.9) \quad \begin{cases} P[T_1] \cdot P[T_2] = \left\{ \frac{t_1}{t} P[T_1|1] + \frac{t_2+t_3}{t} P[T_1|2 \text{ or } 3] \right\} \cdot \\ \cdot \{P[T_2, 1] + P[T_2, 2 \text{ or } 3]\}. \end{cases}$$

From (10.7), (10.8) and (10.9) we see that it is sufficient to prove

$$(10.10) \quad \begin{cases} \sum_{i=1}^3 P[T_1|i] \cdot P[T_2, i] = P[T_1|1] \cdot P[T_2, 1] + P[T_1|2] \cdot P[T_2, 2 \text{ or } 3] \leq \\ \leq \left\{ \frac{t_1}{t} P[T_1|1] + \frac{t_2+t_3}{t} P[T_1|2 \text{ or } 3] \right\} \{P[T_2, 1] + P[T_2, 2 \text{ or } 3]\} \end{cases}$$

or its equivalent

$$(10.11) \quad \{P[T_1|1] - P[T_1|2]\} \left\{ \frac{t_2+t_3}{t} P[T_2, 1] - \frac{t_1}{t} P[T_2, 2 \text{ or } 3] \right\} \leq 0.$$

But the inequality

$$(10.12) \quad P[T_1|1] \geq P[T_1|2]$$

holds as can be seen in the following way. (10.12) is equivalent to

$$(10.13) \quad t_1 P[T_1, 2] \leq t_2 P[T_1, 1].$$

Consider now a ranking which gives  $T_1$  and 2 (i.e. the largest element belongs to the 2nd sample and  $T_1 \geq T_1$ ) and interchange the last element with every element of the first sample. This gives  $t_1$  different rankings with  $T_1$  and 1. In this way we get each ranking with  $T_1$  and 1 at most  $t_2$  times, because in a ranking with  $T_1$  and 1 the last element can be interchanged with at most  $t_2$  different elements of the second sample. This proves (10.13) and thus (10.12). Interchanging 1 and 2 in (10.12) we find

$$(10.14) \quad P[T_2|2] \geq P[T_2|1].$$

(10.11) and thus (10.2) is an immediate consequence of (10.12) and (10.14). This completes the proof of (9.5).

We now turn to the proof of theorem 9.1. Let  $H_{1,1}$  be true. If  $t_i \rightarrow \infty$  ( $i=1, \dots, k$ ) such that

$$\liminf \frac{t_1}{\sum_{i=1}^k t_i} > 0 \text{ and } \liminf \frac{\sum_{i=1}^k t_i - t_1}{\sum_{i=1}^k t_i} > 0,$$

we know that Wilcoxon's test comparing sample 1 with the other samples pooled is consistent. This means

$$(10.15) \quad \lim_{t_i \rightarrow \infty} P[P[T_1] \leq \eta | H_{1,1}] = 1 \quad \text{for every } \eta (0 \leq \eta \leq 1)$$



or the exceedance probability found in the first sample converges to 0 in probability (cf. D. VAN DANTZIG (1951)).

In a similar way as in D. VAN DANTZIG (1951) we find, if

$$(10.16) \quad p \stackrel{\text{def}}{=} P[u_1 > u_j | H_{1,1}] > \frac{1}{2} \\ E(\mathbf{T}_j | H_0) = \frac{1}{2} t_j (\sum t_i - t_j) + \frac{1}{2} t_j (t_j + 1)$$

and

$$(10.17) \quad E(\mathbf{T}_j | H_{1,1}) = \frac{1}{2} t_j (\sum t_i - t_j - t_1) + (1-p) t_j t_1 + \frac{1}{2} t_j (t_j + 1) < E(\mathbf{T}_j | H_0).$$

Further

$$(10.18) \quad \sigma^2(\mathbf{T}_j | H_{1,1}) \leq 3\sigma^2(\mathbf{T}_j | H_0).$$

From (10.15) we have

$$(10.19) \quad \lim_{t_i \rightarrow \infty} P[P[\mathbf{T}_j] \leq P[\mathbf{T}_1] | H_{1,1}] \leq \lim_{t_i \rightarrow \infty} P[P[\mathbf{T}_j] \leq \eta | H_{1,1}]$$

for every  $\eta (0 \leq \eta \leq 1)$ .

As the limit distribution under  $H_0$  of  $\frac{\mathbf{T}_j - E(\mathbf{T}_j | H_0)}{\sigma(\mathbf{T}_j | H_0)}$  is normal with mean 0 and unit variance (10.19) leads to

$$(10.20) \quad \left\{ \begin{aligned} \lim_{t_i \rightarrow \infty} P[P[\mathbf{T}_j] \leq \eta | H_{1,1}] &= \lim_{t_i \rightarrow \infty} P\left[\frac{\mathbf{T}_j - E(\mathbf{T}_j | H_0)}{\sigma(\mathbf{T}_j | H_0)} \geq \xi_\eta | H_{1,1}\right] \leq \\ &\leq \lim_{t_i \rightarrow \infty} P\left[\frac{\mathbf{T}_j - E(\mathbf{T}_j | H_{1,1})}{\sigma(\mathbf{T}_j | H_{1,1})} \geq \sqrt{3} \xi_\eta | H_{1,1}\right] \leq \frac{1}{3\xi_\eta^2} \end{aligned} \right.$$

where  $\xi_\eta$  is defined by

$$\frac{1}{\sqrt{2\pi}} \int_{\xi_\eta}^{\infty} e^{-\frac{x^2}{2}} dx = \eta.$$

(10.20) is valid for every  $\eta (0 \leq \eta \leq 1)$  and as  $\xi_\eta \rightarrow \infty (\eta \rightarrow 0)$  (10.19) combined with (10.20) gives

$$(10.21) \quad \lim_{t_i \rightarrow \infty} P[P[\mathbf{T}_j] \leq P[\mathbf{T}_1] | H_{1,1}] = 0.$$

If  $H_{1,1}$  is true the probability of correct decision is

$$(10.22) \quad \left\{ \begin{aligned} &P[P[\mathbf{T}_1] \leq \frac{\alpha}{k} \text{ and } P[\mathbf{T}_1] < P[\mathbf{T}_j] \text{ for } j \neq 1 | H_{1,1}] \geq \\ &\geq P[P[\mathbf{T}_1] \leq \frac{\alpha}{k} | H_{1,1}] - \sum_{j=2}^k P[P[\mathbf{T}_j] \leq P[\mathbf{T}_1] | H_{1,1}]. \end{aligned} \right.$$

(10.15) and (10.21) show that the probability of a correct decision converges to 1, which proves theorem 9.1.

#### 11. Tables of critical values for the Poisson distribution and for the method of $m$ rankings

Table 11.1 gives critical values for the test for Poisson variates against slippage to the right if  $H_0$  is:  $p_1 = p_2 = \dots = p_k$ . The critical values for



$\max z_i$  as test statistic are given for the values 0.05 (the upper numbers) and 0.01 (the lower numbers) of  $\alpha$ . Owing to the discontinuous character of the binomial distribution the true level of significance will generally be less, and very often considerably less, than  $\alpha$ . Therefore approximated true levels of significance (i.e.  $\alpha'$  cf. (2.17)) are shown also. The exact values satisfy inequality (2.13). The table was constructed with the help of a table of the binomial distribution. This can also be done for critical values for the test against slippage to the left.

Table 11.2 gives critical values for specified  $\alpha$  for the method of  $m$  rankings, when testing against slippage to the left with  $\min s_i$  as test statistic. If this critical value is equal to 1, the critical value  $r$  at the same level of significance for the test against slippage to the right is given by  $r = m(k+1) - 1$ . As in table 11.1 the approximated true levels of significance ( $\alpha'$ ) are also given.

## 12. Acknowledgements

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<sup>1)</sup> Formerly at the Statistical Department of the Mathematical Centre at Amsterdam, now at Cornell University, Ithaca, N.Y.

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TABLE 11.1

Critical values for the slippage test to the right in the Poisson-case with  $H_0: \mu_1 = \mu_2 = \dots = \mu_k$ .  
 Test statistic:  $\max z_i$ . Approximate significance level 0.05 (upper values) and 0.01 (lower values).  
 The approximated true level of significance is written behind the critical value. Number of  
 observations  $k$ , sum of the observations  $n$

$n \backslash k$	2	3	4	5	6	7	8	9	10
2	— —	— —	— —	— —	— —	— —	— —	— —	— —
3	— —	— —	— —	3 0.040	3 0.028	3 0.020	3 0.016	3 0.012	3 0.010 3 0.010
4	— —	4 0.037	4 0.016	4 0.008 4 0.008	4 0.005 4 0.005	4 0.003 4 0.003	4 0.002 4 0.002	3 0.045 4 0.001	3 0.037 4 0.001
5	— —	5 0.012	5 0.004 5 0.004	4 0.034 5 0.002	4 0.020 5 0.001	4 0.013 5 0.000	4 0.009 4 0.009	4 0.006 4 0.006	4 0.005 4 0.005
6	6 0.031 — —	6 0.004 6 0.004	5 0.019 6 0.001	5 0.008 5 0.008	5 0.004 5 0.004	4 0.035 5 0.002	4 0.024 5 0.001	4 0.017 5 0.001	4 0.013 5 0.001
7	7 0.016 — —	6 0.021 7 0.001	6 0.005 6 0.005	5 0.023 6 0.002	5 0.012 6 0.001	5 0.007 5 0.007	5 0.004 5 0.004	4 0.037 5 0.003	4 0.027 5 0.002
8	8 0.008 8 0.008	7 0.008 7 0.008	6 0.017 7 0.002	6 0.006 6 0.006	5 0.028 6 0.003	5 0.016 6 0.001	5 0.010 5 0.010	5 0.006 5 0.006	5 0.004 5 0.004
9	8 0.039 9 0.004	7 0.025 8 0.003	6 0.040 7 0.005	6 0.015 7 0.002	6 0.007 6 0.007	5 0.032 6 0.003	5 0.020 6 0.002	5 0.013 6 0.001	5 0.009 5 0.009
10	9 0.021 10 0.002	8 0.010 9 0.001	7 0.014 8 0.002	6 0.032 7 0.004	6 0.015 7 0.002	6 0.008 6 0.008	5 0.036 6 0.004	5 0.024 6 0.002	5 0.016 6 0.001
11	10 0.012 11 0.001	8 0.027 9 0.004	7 0.030 8 0.005	7 0.010 7 0.010	6 0.028 7 0.004	6 0.015 7 0.002	6 0.008 6 0.008	5 0.040 6 0.005	5 0.028 6 0.003
12	10 0.039 11 0.006	9 0.012 10 0.002	8 0.011 9 0.002	7 0.020 8 0.003	6 0.048 7 0.008	6 0.026 7 0.004	6 0.015 7 0.002	6 0.009 6 0.009	5 0.043 6 0.005
13	11 0.022 12 0.003	9 0.027 10 0.005	8 0.023 9 0.004	7 0.035 8 0.006	7 0.015 8 0.002	6 0.042 7 0.007	6 0.024 7 0.003	6 0.015 7 0.002	6 0.009 6 0.009
14	12 0.013 13 0.002	10 0.012 11 0.002	8 0.041 9 0.009	8 0.012 9 0.002	7 0.025 8 0.004	7 0.012 8 0.002	6 0.038 7 0.006	6 0.023 7 0.003	6 0.015 7 0.002
15	12 0.035 13 0.007	10 0.026 11 0.005	9 0.017 10 0.003	8 0.021 9 0.004	7 0.040 8 0.008	7 0.019 8 0.003	7 0.010 8 0.001	6 0.035 7 0.005	6 0.022 7 0.003
16	13 0.021 14 0.004	10 0.048 12 0.002	9 0.030 10 0.007	8 0.035 9 0.007	8 0.013 9 0.002	7 0.030 8 0.005	7 0.016 8 0.002	7 0.009 7 0.009	6 0.033 7 0.005
17	13 0.049 15 0.002	11 0.024 12 0.006	9 0.050 11 0.002	9 0.013 10 0.002	8 0.021 9 0.004	7 0.045 8 0.009	7 0.024 8 0.004	7 0.013 8 0.002	6 0.047 7 0.008
18	14 0.031 15 0.008	11 0.044 13 0.003	10 0.022 11 0.005	9 0.021 10 0.005	8 0.032 9 0.007	8 0.014 9 0.003	7 0.035 8 0.007	7 0.020 8 0.003	7 0.012 8 0.002
19	15 0.019 16 0.004	12 0.022 13 0.006	10 0.036 11 0.009	9 0.033 10 0.008	8 0.048 10 0.002	8 0.021 9 0.004	7 0.050 9 0.002	7 0.028 8 0.005	7 0.017 8 0.003
20	15 0.041 17 0.003	12 0.039 14 0.003	11 0.016 12 0.004	9 0.050 11 0.003	9 0.017 10 0.004	8 0.031 9 0.007	8 0.015 9 0.003	7 0.040 8 0.008	7 0.024 8 0.004
21	16 0.027 17 0.007	13 0.021 14 0.006	11 0.026 12 0.007	10 0.020 11 0.005	9 0.026 10 0.006	8 0.044 10 0.002	8 0.022 9 0.004	8 0.011 9 0.002	7 0.033 8 0.006
22	17 0.017 18 0.004	13 0.035 15 0.003	11 0.040 13 0.003	10 0.031 11 0.008	9 0.037 10 0.009	9 0.015 10 0.003	8 0.031 9 0.007	8 0.016 9 0.003	7 0.044 8 0.009
23	17 0.035 19 0.003	14 0.019 15 0.005	12 0.019 13 0.005	10 0.045 12 0.003	10 0.014 11 0.003	9 0.022 10 0.005	8 0.042 9 0.010	8 0.022 9 0.004	8 0.012 9 0.002
24	18 0.023 19 0.007	14 0.031 15 0.010	12 0.029 13 0.008	11 0.019 12 0.005	10 0.020 11 0.005	9 0.030 10 0.007	9 0.014 10 0.003	8 0.030 9 0.006	8 0.017 9 0.003
25	18 0.043 20 0.004	14 0.049 16 0.005	12 0.043 14 0.004	11 0.028 12 0.008	10 0.029 11 0.008	9 0.041 11 0.002	9 0.019 10 0.004	8 0.040 9 0.009	8 0.023 9 0.005



TABLE 11.2

Critical values  $s_\alpha$  of the test statistic  $\min s_i$  for the slippage test to the left for the method of  $m$  rankings. Level of significance  $\alpha$ , number of rankings  $m$ , number of ranked objects  $k$ . The approximated true levels of significance are written behind the corresponding critical values

$k$	$m \backslash \alpha$	3	4	5	6	7	8	9
2	0.05	— —	— —	— —	6 0.031	7 0.016	8 0.008	10 0.039
	0.025	— —	— —	— —	— —	7 0.016	8 0.008	9 0.004
	0.01	— —	— —	— —	— —	— —	8 0.008	9 0.004
3	0.05	— —	4 0.037	5 0.012	7 0.029	9 0.049	10 0.021	12 0.032
	0.025	— —	— —	5 0.012	6 0.004	8 0.011	10 0.021	11 0.008
	0.01	— —	— —	— —	6 0.004	7 0.001	9 0.004	11 0.008
4	0.05	— —	4 0.016	6 0.023	8 0.027	10 0.029	12 0.030	14 0.029
	0.025	— —	4 0.016	6 0.023	7 0.007	9 0.009	11 0.010	13 0.011
	0.01	— —	— —	5 0.004	7 0.007	9 0.009	10 0.003	12 0.003
5	0.05	3 0.040	5 0.040	7 0.034	9 0.027	11 0.021	14 0.038	16 0.028
	0.025	— —	4 0.008	6 0.010	8 0.009	11 0.021	13 0.016	15 0.013
	0.01	— —	4 0.008	6 0.010	8 0.009	10 0.008	12 0.006	14 0.005
6	0.05	3 0.028	5 0.023	8 0.043	10 0.027	13 0.037	16 0.045	18 0.028
	0.025	— —	5 0.023	7 0.016	9 0.011	12 0.017	15 0.023	17 0.014
	0.01	— —	4 0.005	6 0.005	8 0.004	11 0.007	13 0.005	16 0.007
7	0.05	3 0.020	6 0.044	8 0.023	11 0.027	14 0.029	17 0.029	21 0.048
	0.025	3 0.020	5 0.014	8 0.023	10 0.012	13 0.015	16 0.016	19 0.016
	0.01	— —	4 0.003	7 0.009	9 0.005	12 0.007	15 0.008	18 0.008
8	0.05	3 0.016	6 0.029	9 0.031	12 0.028	16 0.043	19 0.035	23 0.046
	0.025	3 0.016	5 0.010	8 0.014	11 0.014	15 0.025	18 0.021	21 0.017
	0.01	— —	5 0.010	7 0.005	10 0.006	13 0.007	16 0.006	20 0.010
9	0.05	4 0.049	7 0.048	10 0.038	13 0.029	17 0.036	21 0.042	25 0.045
	0.025	3 0.012	6 0.021	9 0.019	12 0.016	16 0.022	19 0.016	23 0.019
	0.01	— —	5 0.007	8 0.009	11 0.008	14 0.006	18 0.009	21 0.007
10	0.05	4 0.040	7 0.035	11 0.046	14 0.030	18 0.032	23 0.048	27 0.045
	0.025	3 0.010	6 0.015	9 0.013	13 0.017	17 0.019	21 0.020	25 0.020
	0.01	3 0.010	5 0.005	8 0.006	12 0.009	15 0.006	19 0.008	23 0.008