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A probabilistic interpretation of some formulae in queueing theory

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Introduction

In this paper some known formulae, which are of importance for the theory of queueing with one server, are derived by means of a probabilistic interpretation of generating and moment generating functions, according to a method introduced in [1] Van Dantzig (1947, 1948) and applied to some problems in these and later publications (Van Dantzig (1955, 1957), Van Dantzig and Scheffer (1954), Van Dantzig and Zoutendijk (1958)), and to queueing problems in Kesten and Runnenburg (1957). In particular the present paper contains the answers to questions recently put in the Royal Statistical Society by D.R. Cox, D.G. Kendall and F.G. Foster, concerning the possibility of giving a probabilistic interpretation to some formulae occuring in queueing theory.

In the three applications we treat here, the following situation is considered (described for the non-equilibrium case).

Customers are served at a counter in the order in which they arrive from time t=0 onwards, $\underline{t}_r$ is the time of arrival of the r^{th} customer, r=1,2,... and $\underline{s}_r$ his servicetime [2]. If

(1) $$\underline{y}_r \stackrel{\text{def}}{=\!=} \underline{t}_r - \underline{t}_{r-1} \quad \text{for} \quad r=1,2,\ldots \text{ (with } \underline{t}_0=0) ,$$

then the $\underline{y}_r$ and $\underline{s}_r$ are taken to be nonnegative independent random variables, with all $\underline{y}_r$ having the same distributionfunction

(2) $$A(y) \stackrel{\text{def}}{=\!=} \begin{cases} 1 - e^{-\lambda y} & \text{if} \quad y \geq 0 \\ 0 & \text{if} \quad y \leq 0 , \end{cases}$$

where λ is a positive constant, and all $\underline{s}_r$ having the same known distributionfunction B(s), with B(0-)=0. By choosing an appropriate unit of time we assume without restriction $\lambda=1$. We further assume, that $\mathcal{E}\,\underline{s}_1$ exists and define [3]

(3) $$\rho \stackrel{\text{def}}{=\!=} \mathcal{E}\,\underline{s}_1 .$$

Let $\underline{w}_r$ denote the waitingtime of the r^{th} customer. Define [3]

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1) See the list of references at the end of this paper

2) Random variables are distinguished from numbers (e.g. from the values they take in an experiment) by underlining their symbols

3) $\mathcal{E}\,\underline{x}$ denotes the mathematical expectation of a stochastic variable $\underline{x}$, P{A} is written for the probability of event A.

(4) $$C_r(w) \stackrel{\text{def}}{=\!=} P\{\underline{w}_r \leq w\} .$$

Following Tákacs we introduce a function $\underline{w}(t)$, denoting the time needed to complete the service of all those present at time t.

Further (either with or without a suffix on both sides)

(5) $$\beta(\xi) \stackrel{\text{def}}{=\!=} \int_{0-}^{\infty} e^{-\xi s}\, d\, B(s) \qquad (\mathrm{Re}\,\xi \geq 0) ,$$

(6) $$\gamma_r(\xi) \stackrel{\text{def}}{=\!=} \int_{0-}^{\infty} e^{-\xi w}\, d\, C_r(w) \qquad (\mathrm{Re}\,\xi \geq 0) .$$

A Tákacs formula

In Tákacs (1955), a theorem is proved (theorem 2), which we shall prove here in a slightly less general form. (From (2) we have that the probability that a customer arrives in the interval dt is $\lambda\, dt + O(dt)$, where λ is a constant; Tákacs assumed that λ is a function of t). The theorem as we prove it, reads

The Laplace-Stieltjes transform

(7) $$\Phi(t,\xi) \stackrel{\text{def}}{=\!=} \int_{0-}^{\infty} e^{-\xi w}\, d\, F(t,w)$$

of the function

(8) $$F(t,w) \stackrel{\text{def}}{=\!=} P\{\underline{w}(t) \leq w\}$$

may be written in the form

(9) $$\Phi(t,\xi) = e^{\xi t - \{1-\beta(\xi)\}t} \left\{ 1 - \xi \int_0^t e^{-\xi u + \{1-\beta(\xi)\}u}\, F(u,0)\, du \right\} ,$$

where $F(u,0)$ denotes the probability, that at time u the counter is free.

Tákacs first derived an integro-differential equation for $F(t,w)$ and then passed to the Laplace-Stieltjes transform $\Phi(t,\xi)$. We obtain his theorem with the help of a probabilistic interpretation, which might equally well have been used to derive his more general result. To do this we write (9) in the equivalent form

(10) $$e^{-t\{1-\beta(\xi)\}} = e^{-\xi t}\, \Phi(t,\xi) + \int_0^t e^{-(t-u)\{1-\beta(\xi)\}}\, F(u,0)\, \xi e^{-\xi u}\, du.$$

Let $\underline{t}_1'$, $\underline{t}_2'$,... be moments at which catastrophe E_ξ occurs [1],

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1) This is an example of the kind mentioned in Cox (1957).

these catastrophes being in no way connected to the problem under discussion, with

(11) $$\underline{y}'_r = \underline{t}'_r - \underline{t}'_{r-1} \quad \text{for} \quad r=1,2,\ldots \quad (\text{with } \underline{t}'_o = 0)\ ,$$

all $\underline{y}'_r$ being independent random variables, drawn from the distribution

(12) $$P\{\underline{y}'_r \leqslant y\} = \begin{cases} 1-e^{-\xi y} & \text{if} \quad y \geqslant 0 \\ 0 & \text{if} \quad y \leqslant 0\ , \end{cases}$$

where ξ is a positive constant.

We now introduce the three events

(13) $\mathcal{A} \stackrel{\text{def}}{=}$ no E_ξ occurs during the time the counter is occupied by customers, who arrive before t,

(14) $\mathcal{B} \stackrel{\text{def}}{=}$ E_ξ occurs for the first time after all customers arriving before t have been served,

(15) $\mathcal{C} \stackrel{\text{def}}{=}$ E_ξ occurs for the first time before t at a moment u at which the counter is free and after that no E_ξ occurs during the remaining servicetime of the customers who arrive before t, $0 \leqslant u \leqslant t$.

If exactly n customers arrive before t (an event with probability $e^{-t}.(t^n)/n!$), the probability, that no E_ξ occurs during the servicetime of anyone of these customers is equal to $\{\beta(\xi)\}^n$, as these servicetimes are mutually exclusive and stochastically independent. Therefore

(16) $$P\{\mathcal{A}\} = \sum_{n=0}^{\infty} e^{-t}\,\frac{t^n}{n!}\,\{\beta(\xi)\}^n = e^{-t\{1-\beta(\xi)\}}\ .$$

For event $\mathcal{B}$ we have

(17) $$P\{\mathcal{B}\} = P\{\underline{y}'_1 > t+\underline{w}(t)\} =$$
$$= \int_{0-}^{\infty} e^{-\xi(t+w)}\, dP\{\underline{w}(t) \leqslant w\} = e^{-\xi t}\,\Phi(t,\xi).$$

From (16) we have for the probability, that no E_ξ occurs during the time the counter is occupied by customers, who arrive in the interval $[u,t)$

(18) $$e^{-(t-u)\{1-\beta(\xi)\}}$$

while F(u,0) is the probability, that at time u the counter is free. Therefore

(19) $$P\{\mathcal{C}\} = \int_0^t e^{-(t-u)\{1-\beta(\xi)\}} F(u,0)\, \xi e^{-\xi u}\, du \ .$$

As event $\mathcal{A}$ is clearly the conjunction of the disjoint events $\mathcal{B}$ and $\mathcal{C}$, we have

(20) $$P\{\mathcal{A}\} = P\{\mathcal{B}\} + P\{\mathcal{C}\} \ ,$$

which combined with (16), (17) and (19) leads to (10).

Therefore Tákacs' result has now been derived by a probabilistic interpretation, for the relation (10) holds for all ξ with $\mathrm{Re}\,\xi \geqslant 0$ by analytic continuation.

B Pollaczek's formula [1)]

Let E be an incident (catastrophe), which happens with probability 1-X to a customer, these events being independent for the different customers and from each other. Consider the events

(21) $\mathcal{A}_r \overset{\text{def}}{=}$ E does not happen with respect to any of the customers arriving in $\underline{w}_r + \underline{s}_r$,

(22) $\mathcal{B}_r \overset{\text{def}}{=}$ E happens with respect to customer r+1 and does not happen with respect to any of the customers arriving in $\underline{w}_r + \underline{s}_r$ (or equivalently) = E happens with respect to customer r+1 and $\underline{w}_{r+1} = 0$,

(23) $\mathcal{C}_r \overset{\text{def}}{=}$ E does not happen with respect to customer r+1 and does not happen with respect to any of the customers arriving in $\underline{w}_{r+1}$ (where either $\underline{w}_{r+1} = 0$ or $\underline{w}_{r+1} > 0$) .

Because $\mathcal{A}_r$ is the conjunction of the disjoint events $\mathcal{B}_r$ and $\mathcal{C}_r$ we have

(24) $$P\{\mathcal{A}_r\} = P\{\mathcal{B}_r\} + P\{\mathcal{C}_r\} \ .$$

If t is the length of an interval, the probability of no customer arriving in that interval with respect to whom E happens, is given by (see (16) and its derivation)

(25) $$\sum_{n=0}^{\infty} e^{-t} \frac{t^n X^n}{n!} = e^{-t(1-X)} \ ,$$

so

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1) The results under B and C were obtained in collaboration with Prof. Dr D. van Dantzig.

(26) $$P\{\mathcal{A}_r\} = \mathcal{E}\, e^{-(\underline{w}_r+\underline{s}_r)(1-X)} = \gamma_r(1-X)\ \beta_r(1-X)\ ,$$

because of the independence of $\underline{w}_r$ and $\underline{s}_r$. Further

(27) $$P\{\mathcal{B}_r\} = (1-X)\ P\{\underline{w}_{r+1} = 0\}$$

and

(28) $$P\{\mathcal{C}_r\} = X\mathcal{E}\, e^{-\underline{w}_{r+1}(1-X)} = X\,\gamma_{r+1}\,(1-X)\ .$$

If we write

$$\xi = 1-X\ ,$$

then we have by (24), (26), (27) and (28)

(29) $$\gamma_r(\xi)\,\beta_r(\xi) = \xi\, P\{\underline{w}_{r+1} = 0\} + (1-\xi)\,\gamma_{r+1}(\xi)\ .$$

If we consider the stationary situation connected with the process described on page 1, we may drop [1] the suffixes r and r+1 from (29) to obtain

(30) $$\gamma(\xi)\,\beta(\xi) = \xi\, P\{\underline{w} = 0\} + (1-\xi)\,\gamma(\xi)\ .$$

This identity holds not only for $0 \leqslant X \leqslant 1$ (or $0 \leqslant \xi \leqslant 1$), but for all ξ with $\mathrm{Re}\,\xi \geqslant 0$. From (27) we find by differentiation with respect to ξ, upon taking $\xi = 0$

(31) $$P\{\underline{w} = 0\} = 1 - \mathcal{E}\,\underline{s} = 1 - \rho\ ,$$

from which we see, that $\rho \leq 1$ is necessary for stationarity. As is well known $\rho < 1$ is the necessary and sufficient condition (see e.g. Kendall (1951)) for a stationary system.

From the relation (24) we have thus derived the well known Pollaczek-formula (30) [2]. An equivalent form of (30) is

(32) $$\gamma(\xi) = 1-\rho + \frac{\rho\,\gamma(\xi)\{1-\beta(\xi)\}}{\xi\,\mathcal{E}\,\underline{s}}\ .$$

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1) In Kesten and Runnenburg (1957) the details of this procedure are given. By specialization of the derivations given there to the case of one priority, a slightly less direct proof of (29) is obtained by the same method as is used here.

2) This formula was given in Pollaczek (1930) for the first time, see footnote on page 105 in Pollaczek (1957). For another probabilistic interpretation, see Foster's comment in Kendall (1957), page 213.

C Kendall's decomposition

If we consider the incident E in B as a mark, which a customer may have, where again the probability of a customer having that mark is 1-X, we can infer a "principle" from equation (32), which can be used to give a probabilistic interpretation to the decomposition in components, as indicated in Kendall (1957) (see first footnote on page 208 and the corresponding passage in the text).

We suppose the system to be in statistical equilibrium. Arriving customers take a seat in a waitingroom, in which they stay during their waitingtime, i.e. from the moment they arrive until the counter can attend to them. Call a customer having mark E an E-customer. The "principle" can now be stated: the probability, that during the waitingtime of a customer, K_0 say, no E-customer enters the waitingroom equals the probability, that no E-customer leaves that room during that time. As "statistical equilibrium" may be regarded as "statistical equilibrium in the waitingroom", this principle seems quite natural. One can prove that it is true by making use of the truth of (32).

For the event

(33) $\mathcal{A}_0 \overset{\text{def}}{=\!=}$ during the waitingtime of K_0 no E-customer enters the waitingroom

clearly

(34) $$P\{\mathcal{A}_0\} = \gamma(\xi)$$

holds.

We further consider the events

(35) $\mathcal{A}_0' =$ during K_0's waitingtime no E-customer leaves the waitingroom,

(36) $\mathcal{B}_0' =$ customer K_0 finds an empty counter on arrival (in which case during his waitingtime certainly no customer, be it an E-customer or otherwise, leaves the waitingroom),

(37) $\mathcal{C}_0' =$ customer K_0 finds the counter occupied by a customer K_{-1}, and no E-customer is present in the waitingroom (or equivalently) = customer K_0 finds the counter occupied by a customer K_{-1}, and no E-customer arrived during K_{-1}'s waitingtime nor during that part of

K_{-1}'s servicetime which lies before K_o's arrival.

If K_o finds the counter occupied on arrival, we call the customer who is served at that moment customer K_{-1}. Customer K_{-1} may be called the "ancestor" of customer K_o, in distinction to the "predecessor" of customer K_o, who is the last one arriving before K_o. If $\underline{w}_{-1}$ is the waitingtime of K_{-1} and $\underline{x}_{-1}$ the time between the start of K_{-1}'s service and K_o's arrival, then $\underline{w}_{-1}$ and $\underline{x}_{-1}$ are independent random variables. The probability, that K_o finds the counter occupied and that no E-customer leaves the waitingroom during K_o's waitingtime is trivially equal to the probability, that neither during K_{-1}'s waitingtime $\underline{w}_{-1}$ nor during the time $\underline{x}_{-1}$ spend by K_{-1} at the counter before K_o's arrival an E-customer enters the waitingroom. The probability, that no E-customer enters during a given interval of length t is $e^{-t(1-X)}$ (see (25)).

Take the moment of K_{-1}'s arrival as the initial point of this interval. The probability, that a customer enters during an interval dt is dt+O(dt). Hence the probability that K_o enters during K_{-1}'s servicetime $\underline{s}_{-1}$ and that no E-customer has entered after K_{-1}'s and before K_o's arrival is given by

$$(38)\quad P\{\mathcal{C}'_o\} = \mathcal{E} \int_{\underline{w}_{-1}}^{\underline{w}_{-1}+\underline{s}_{-1}} e^{-t(1-X)}\,dt =$$

$$= \mathcal{E}\; e^{-\underline{w}_{-1}(1-X)}\,(1-e^{-\underline{s}_{-1}(1-X)})(1-X)^{-1} =$$

$$= \gamma(1-X)\{1-\beta(1-X)\}(1-X)^{-1} = \frac{\rho\gamma(\xi)\{1-\beta(\xi)\}}{\xi\,\mathcal{E}\,\underline{s}},$$

because $\underline{w}_{-1}$ and $\underline{s}_{-1}$ are independent.

For $\mathcal{B}'_o$ we have (see (31))

$$(39)\qquad P\{\mathcal{B}'_o\} = 1-\rho\;.$$

Again $\mathcal{A}'_o$ is the conjunction of the disjoint events $\mathcal{B}'_o$ and $\mathcal{C}'_o$, so

$$(40)\qquad P\{\mathcal{A}'_o\} = P\{\mathcal{B}'_o\} + P\{\mathcal{C}'_o\}\;.$$

Because of (38), (39) and (40)

(41) $$P\{\mathcal{A}'_o\} = 1 - \rho + \frac{\rho\gamma(\xi)\{1-\beta(\xi)\}}{\xi\,\mathcal{E}\,\underline{s}},$$

so we have proved with the help of (32)

(42) $$P\{\mathcal{A}_o\} = P\{\mathcal{A}'_o\} \quad ,$$

which is just the "principle" stated earlier.

If we substitute for $\gamma(\xi)$ on the right hand side in (32) the whole right hand side of that equation and iterate this procedure, we obtain Kendall's decomposition of (32)

(43) $$\gamma(\xi) = \sum_{n=o}^{\infty} (1-\rho)\rho^n \left\{\frac{1-\beta(\xi)}{\xi\,\mathcal{E}\,\underline{s}}\right\}^n \quad .$$

This relation shows, that the waitingtime $\underline{w}$ of any customer may be written (with $\underline{w}=0$ if $\underline{n}=0$)

(44) $$\underline{w} = \sum_{i=1}^{\underline{n}} \underline{z}_i \quad ,$$

where the $\underline{z}_i$ are independent random variables, all having the same distributionfunction, the Laplace-Stieltjes transform of which is

(45) $$\frac{1-\beta(\xi)}{\xi\,\mathcal{E}\,\underline{s}}$$

and $\underline{n}$ has a Pascaldistribution, with

(46) $$P\{\underline{n}=n\} = (1-\rho)\rho^n \quad (n=0,1,\dots) \quad .$$

So far we considered only customers K_o and K_{-1}, K_{-1} being the ancestor of K_o, if such an ancestor existed. Let K_{-i} be the ancestor of K_{-i+1} if K_{-i+1} has an ancestor, i.e. if the counter is occupied upon K_{-i+1}'s arrival, we call the customer who is served at that moment K_{-i}. Then $\underline{n}$ is defined to be the number of ancestors of customer K_o. K_{-n} is thus the first customer (going back from K_o to K_{-1} etc.), who found an empty counter on arrival. Now

(47) $$P\{\underline{n}=n\} = (1-\rho)\rho^n \quad (n=0,1,\dots)$$

because whether K_{-i+1} finds the counter occupied or not does not depend on what happens in his servicetime, so K_{-i+1} finds with probability ρ that customer K_{-i} is being served and with probability $1-\rho$ an empty counter, whence (47) holds.

Let $\underline{w}_{-i}$ be the waitingtime of customer K_{-i} and $\underline{x}_{-i}$ the time from the start of K_{-i}'s service until K_{-i+1}'s arrival, then one can proceed in the following manner, the details of which are omitted.

The "principle" can be generalised (for $n \geq 1$) to

(48) $$P\{\text{no E-customer leaves in } \underline{w}_0 \mid \underline{n} = n\} = \\ = P\{\text{no E-customer arrives in } \underline{w}_{-1} + \underline{x}_{-1} \mid \underline{n} = n\},$$

where $(\underline{w}_{-1} \mid \underline{n} = n)$ and $(\underline{x}_{-1} \mid \underline{n} = n)$ are still independent random variables, so (for $n \geq 1$)

(49) $$P\{\text{no E-customer leaves in } \underline{w}_0 \mid \underline{n} = n\} = \\ = P\{\text{no E-customer arrives in } \underline{w}_{-1} \mid \underline{n} = n\} . \\ . P\{\text{no E-customer arrives } \underline{x}_{-1} \mid \underline{n} = n\} .$$

For $n \geq 1$ we also have

(50) $$P\{\text{no E-customer arrives in } \underline{w}_{-1} \mid \underline{n} = n\} = \\ = P\{\text{no E-customer arrives in } \underline{w}_0 \mid \underline{n} = n-1\}$$

and because $\underline{w}_0 = 0$ if $\underline{n} = 0$

(51) $$P\{\text{no E-customer arrives in } \underline{w}_0 \mid \underline{n} = 0\} = 1 ,$$

while further for $n \geq 1$

(52) $$P\{\text{no E-customer arrives in } \underline{x}_{-1} \mid \underline{n} = n\} = \\ = P\{\text{no E-customer arrives in } \underline{x}_{-1}\} .$$

Therefore because of (48), (49), (50), (51) and (52)

(53) $$P\{\text{no E-customer leaves in } \underline{w}_0 \mid \underline{n} = n\} = \\ = \prod_{i=1}^{n} P\{\text{no E-customer arrives in } \underline{x}_{-1}\} = \\ = \left\{ \frac{1 - \beta(\xi)}{\xi \mathcal{E} \underline{s}} \right\}^n ,$$

which means that we may take

(54) $$\underline{z}_i \overset{\text{def}}{=\!=} \underline{x}_{-i}$$

and that we have found a probabilistic interpretation of (43). This formula may now be read: the probability, that during the waitingtime $\underline{w}_0$ of a customer K_0 no E-customer arrives is equal to the probability, that no E-customer arrives during the time

$\sum_{i=1}^{n} \underline{x}_{-i}$, where $\underline{n}$ is the number of ancestors of K_o and $\underline{x}_{-i}$ the time between the start of K_{-i}'s service and K_{-i+1}'s arrival.

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Résumé

Dans cet article quelques formules connues, importantes pour la théorie d'attente à un guichet, sont dérivées à l'aide d'une interprétation probabilistique des fonctions génératrices et des fonctions génératrices des moments, suivant une méthode introduite par Van Dantzig [1)] (1947, 1948) et appliquée à quelques problèmes dans ces publications et d'autres (Van Dantzig (1955, 1957), Van Dantzig and Scheffer (1954), Van Dantzig and Zoutendijk (1958) et à des problèmes d'attente dans Kesten and Runnenburg (1957). En particulier on a traité quelques questions posées par D.R. Cox, D.G. Kendall et F.G. Foster dans le "Journal of the Royal Statistical Society", concernant la possibilité de telles interprétations.

Dans le présent article on donne une interprétation probabiliste pour la formule (9), due à L. Tákacs (Tákacs (1955)), la formule (32), due à F. Pollaczek (Pollaczek (1930) et la décomposition de (32) comme donnée par (43), due à D.G. Kendall (Kendall (1957)).

1) Une liste bibliographique se trouve à la fin de l'article.