WA

STICHTING

MATHEMATISCH CENTRUM

2e BOERHAAVESTRAAT 49

AMSTERDAM

STATISTISCHE AFDELING

Report S 283 [VP 16 ; SP 76]

Renewal theory for Markov-dependent random variables

by

Dr. J. Th. Runnenburg

May 1961

Summary

Given a sequence of positive integer-valued Markov-dependent random variables $\underline{y}_0, \underline{y}_1, \underline{y}_2, \ldots$, one may ask whether the well-known renewal-theorems still apply. Two theorems are proved, which very closely resemble the classical results for independent random variables.

Résumé

Pour une chaîne de Markoff $\underline{y}_0, \underline{y}_1, \underline{y}_2, \ldots$ avec variables aléatoires positives et entières, on prouve deux théorèmes qui ressemblent parfaitement aux résultats classiques de la théorie de renouvellement pour variables aléatoires independantes.

Conventions

1. Random variables are underlined.

2. The set of all integers is denoted by I, the set of all integers $\geqslant 0$ by N, the set of all integers $\geqslant 1$ by N', the set of all integers $\geqslant 2$ by N" and the set of all integers $\geqslant 3$ by N'''. If at the end of a numbered line a symbol like $i \in N''$ is added, this means that the preceding equation holds for all $i \in N''$.

1. Introduction, basic assumptions

Let $\underline{t}_0, \underline{t}_1, \underline{t}_2, \ldots$ be a sequence of random integers, also described as points on a time-axis, which are called moments of renewal, with

(1.1) $$\underline{t}_0 \leqslant 0 < \underline{t}_1 < \underline{t}_2 < \ldots \quad .$$

The renewal-intervals are denoted by $\underline{y}_n$, where

(1.2) $$\underline{y}_n \overset{\text{def}}{=} \underline{t}_{n+1} - \underline{t}_n \qquad n \in N.$$

For any integers $r,n \in N$ we define

(1.3) $$\underline{n}_r \overset{\text{def}}{=} n \qquad \text{if } \underline{t}_n \leqslant r < \underline{t}_{n+1},$$

(1.4) $$\underline{u}_r \overset{\text{def}}{=} r - \underline{t}_n \qquad \text{if } \underline{n}_r = n,$$

(1.5) $$\underline{v}_r \overset{\text{def}}{=} \underline{t}_{n+1} - r \quad \text{if } \underline{n}_r = n,$$

hence we have in particular

(1.6) $$\underline{u}_0 = -\underline{t}_0 \ ,$$

(1.7) $$\underline{v}_0 = \underline{t}_1 \ ,$$

(1.8) $$\underline{y}_0 = \underline{u}_0 + \underline{v}_0 .$$

These quantities have a physical interpretation: $\underline{n}_r$ denotes the number of renewals in the (left open, right closed) interval $(0,r]$, $\underline{u}_r$ is the distance between r and the last moment of renewal before r, and $\underline{v}_r$ is the distance between r and the first moment of renewal after r.

We say that the interval $[a,b)$ covers time c, if $a \leqslant c < b$. Hence $[\underline{t}_0, \underline{t}_1)$ covers time 0, because $\underline{t}_0 \leqslant 0 < \underline{t}_1$. It is often convenient to renumber the intervals $[\underline{t}_0, \underline{t}_1), [\underline{t}_1, \underline{t}_2), \ldots$. We may start by giving the number 0 to the interval covering time r, where r is any nonnegative integer. Accordingly we define

(1.9) $$\underline{y}_{k,r} \overset{\text{def}}{=} \begin{cases} \underline{y}_{k+n} & \text{if } \underline{n}_r = n \text{ and } k+n \in N, \\ 0 & \text{otherwise} \end{cases}$$

The $\underline{y}_0, \underline{y}_1, \underline{y}_2, \ldots$ are (simple) Markov-dependent random variables, i.e. there is given a set $Y = \{y_{(1)}, y_{(2)}, \ldots\}$ of states, which are here integers, with

(1.10) $$1 \leqslant y_{(1)} < y_{(2)} < \ldots \ ,$$

an initial probability distribution

$$(1.11) \qquad q_y^{(0)} \stackrel{\text{def}}{=} P\{\underline{y}_0 = y\} \qquad y \in Y,$$

and a <u>set of transition probabilities</u> (independent of n)

$$(1.12) \qquad p_{y,z} \stackrel{\text{def}}{=} P\{\underline{y}_{n+1} = z \mid \underline{y}_n = y\} \qquad y \in Y,\ z \in Y.$$

In this paper a <u>probability distribution</u> is any sequence of real numbers $p_k \geqslant 0$ with $k \in N'$, such that $\sum_{k=1}^{\infty} p_k = 1$. A <u>set of transition probabilities</u> is any set of real numbers $p_{i,j} \geqslant 0$ with $i,j \in N'$ and $\sum_{j=1}^{\infty} p_{i,j} = 1$ for all i. The probabilities may be indexed in a different manner.

We may now introduce

$$(1.13) \qquad \underline{n}_r(i,j) \stackrel{\text{def}}{=} n \qquad \text{if} \quad \underline{n}_r = n,\ \underline{u}_0 = i \text{ and } \underline{v}_0 = j.$$

Clearly $\underline{n}_r(i,j)$ denotes the number of renewals in $(0,r]$ under the condition $\underline{u}_0=i$, $\underline{v}_0=j$.

It is also assumed, that there exists an <u>invariant distribution</u> to the sequence of random variables $\underline{y}_0, \underline{y}_1, \underline{y}_2, \ldots$ (which is further called the Markov chain M_1). This means, that there exists a probability distribution π_y with $y \in Y$, such that

$$(1.14) \qquad \sum_{y \in Y} \pi_y = 1, \quad \sum_{y \in Y} \pi_y p_{y,z} = \pi_z \qquad z \in Y,$$

with

$$(1.15) \qquad \mu_1 \stackrel{\text{def}}{=} \sum_{y \in Y} y\, \pi_y < \infty .$$

Sequences $\underline{t}_0, \underline{t}_1, \underline{t}_2, \ldots$ of the foregoing description have been used extensively in the author's thesis to describe the moments of arrival of customers at a counter (cf. Runnenburg (1960)). The assumption of Markov dependence rather than independence was introduced as a useful generalization and a certainly more realistic description.

The material given here is extracted from the author's thesis. The restrictions imposed (integer valued random variables!) have been chosen in order to make the results easily presentable and accessible.

2. Stationary situation

In this section we show, that a proper choice of the distribution of $\underline{u}_0$ and $\underline{v}_0$ leads to a stationary covering of the time-axis with renewal-intervals.

Theorem 2.1. If the Markov chain M_1, i.e. the sequence $\underline{y}_0=\underline{u}_0+\underline{v}_0$, $\underline{y}_1,\underline{y}_2,\dots$ is given the initial distribution

$$(2.1)\qquad P\{\underline{u}_0=i,\ \underline{v}_0=j\} \overset{\text{def}}{=} \frac{1}{\mu_1}\pi_{i+j} \qquad i\in N,\ j\in N',i+j\in Y,$$

then for any $r\in N$ the sequence $\underline{y}_{0,r}=\underline{u}_r+\underline{v}_r$, $\underline{y}_{1,r},\underline{y}_{2,r},\dots$ is also a Markov chain with transition probabilities $p_{y,z}$ and invariant distribution π_y, while

$$(2.2)\qquad P\{\underline{u}_r=i,\underline{v}_r=j\} = \frac{1}{\mu_1}\pi_{i+j} \qquad i\in N,\ j\in N',\ i+j\in Y.$$

Also for any $i<j$

$$(2.3)\qquad \sum_{n=1}^{\infty} P\{\underline{v}_r+\sum_{k=1}^{n-1}\underline{y}_{k,r}=i,\underline{v}_r+\sum_{k=1}^{n}\underline{y}_{k,r}=j\} = \frac{1}{\mu_1}\pi_{j-i}$$
$$r\in N,\ i\in N',\ j\in N'',\ j-i\in Y,$$

hence in particular (sum in (2.3) over $j\geqslant i+1$ for fixed i)

$$(2.4)\qquad \sum_{n=1}^{\infty} P\{\underline{v}_r+\sum_{k=1}^{n-1}\underline{y}_{k,r}=i\} = \frac{1}{\mu_1} \qquad r\in N,\ i\in N'.$$

Remark 2.1. From (2.4) we conclude, that if the points $\underline{t}_{n+1}=\underline{v}_0+\sum_{k=1}^{n}\underline{y}_k$ with $n\in N$ are moments of renewal, then the expected number of renewals in $(0,r]$ is $\frac{r}{\mu_1}$, provided $\underline{v}_0,\underline{y}_1,\underline{y}_2,\dots$ satisfy the conditions of theorem 2.1.

Remark 2.2. Instead of (2.1) we could have used (equivalently)

$$(2.5)\qquad \begin{cases} P\{\underline{y}_0=y\} = \frac{y}{\mu_1}\pi_y & y\in Y,\\ P\{\underline{u}_0=i\mid \underline{y}_0=y\} = \frac{1}{y} & i\in N,\ y\in Y,\ 0\leqq i<y,\end{cases}$$

as can easily be verified.

Proof of theorem 2.1. First note that for any $i\in N$, $r\in N$ and $j\in N'$

$$(2.6)\quad P\{\underline{u}_r=i,\underline{v}_r=j\} = \begin{cases} P\{\underline{u}_0=i-r,\ \underline{v}_0=j+r\} & \text{if } i\geqslant r,\\ \sum_{n=1}^{\infty} P\{\underline{v}_0+\sum_{k=1}^{n-1}\underline{y}_k=r-i,\ \underline{v}_0+\sum_{k=1}^{n}\underline{y}_k=r+j\} & \text{if } i<r.\end{cases}$$

The sequence $\underline{y}_{0,r},\underline{y}_{1,r},\underline{y}_{2,r},\dots$ has the described properties, if (2.2) holds. From (2.6) we find, that (2.2) is trivially true for $i\geqslant r$. For $i<r$ there remains to prove

$$(2.7)\qquad \sum_{n=1}^{\infty} P\{\underline{v}_0+\sum_{k=1}^{n-1}\underline{y}_k=r-i,\ \underline{v}_0+\sum_{k=1}^{n}\underline{y}_k=r+j\}=\frac{1}{\mu_1}\pi_{i+j}$$

$$r-i\in N',r+j\in N'',\ i+j\in Y,$$

i.e. equation (2.3) for $r=0$.

Now for any $n\in N''$, $h\in N''$, $l\in N'''$ with $h<l$ (and $\pi_i \overset{\text{def}}{=} 0$ for $i\notin Y$)

$$(2.8)\qquad P\{\underline{v}_0+\sum_{k=1}^{n-1}\underline{y}_k=h,\underline{v}_0+\sum_{k=1}^{n}\underline{y}_k=l\} =$$

$$=\frac{1}{\mu_1}\sum_{i=0}^{\infty}\sum_{j=1}^{h-1} P\{\sum_{k=1}^{n-1}\underline{y}_k=j,\ \sum_{k=1}^{n}\underline{y}_k=l-h+j\,|\,\underline{u}_0=i,\ \underline{v}_0=h-j\}\ \pi_{i+h-j} =$$

$$=\frac{1}{\mu_1}\sum_{j=1}^{h-1}\sum_{i=0}^{\infty} P\{\sum_{k=1}^{n-1}\underline{y}_k=j,\ \sum_{k=1}^{n}\underline{y}_k=l-h+j\,|\,\underline{y}_0=i+h-j\}\ \pi_{i+h-j} =$$

$$=\frac{1}{\mu_1}\sum_{j=1}^{h-1} P^{st}\{\sum_{k=1}^{n-1}\underline{y}_k=j,\ \sum_{k=1}^{n}\underline{y}_k=l-h+j\} +$$

$$-\frac{1}{\mu_1}\sum_{j=1}^{h-1}\sum_{i=-j+1}^{-1} P^{st}\{\sum_{k=1}^{n-1}\underline{y}_k=j,\ \sum_{k=1}^{n}\underline{y}_k=l-h+j,\ \underline{y}_0=i+h-j\},$$

where we have written P^{st} instead of P to indicate that the probability between brackets must be evaluated using the invariant probability distribution π_y. Write

$$(2.9)\qquad \frac{1}{\mu_1}\sum_{j=1}^{h-1}\sum_{i=-j+1}^{-1} P^{st}\{\sum_{k=1}^{n-1}\underline{y}_k=j,\ \sum_{k=1}^{n}\underline{y}_k=l-h+j,\ \underline{y}_0=i+h-j\} =$$

$$=\frac{1}{\mu_1}\sum_{j=1}^{h-1}\sum_{i=-j+1}^{-1} P^{st}\{\sum_{k=0}^{n-1}\underline{y}_k=i+h,\ \sum_{k=0}^{n}\underline{y}_k=i+l,\ \underline{y}_0=i+h-j\}.$$

On applying the transformation

$$(2.10)\qquad i'=i+h,\quad j'=i+h-j,$$

the right-hand side of (2.9) changes to

$$(2.11)\qquad \frac{1}{\mu_1}\sum_{i'=1}^{h-1}\sum_{j'=1}^{i'-1} P^{st}\{\sum_{k=0}^{n-1}\underline{y}_k=i',\ \sum_{k=0}^{n}\underline{y}_k=l-h+i',\ \underline{y}_0=j'\} =$$

$$=\frac{1}{\mu_1}\sum_{i'=1}^{h-1} P^{st}\{\sum_{k=0}^{n-1}\underline{y}_k=i',\ \sum_{k=0}^{n}\underline{y}_k=l-h+i'\}.$$

For the stationary chain the vector $\left(\sum_{k=0}^{n-1}\underline{y}_k,\ \sum_{k=0}^{n}\underline{y}_k\right)$ has the same distribution as the vector $\left(\sum_{k=1}^{n}\underline{y}_k,\ \sum_{k=1}^{n+1}\underline{y}_k\right)$. Hence we find, combining (2.8), (2.9), (2.10) and (2.11)

$$(2.12)\qquad P\left\{\underline{v}_0+\sum_{k=1}^{n-1}\underline{y}_k=h,\ \underline{v}_0+\sum_{k=1}^{n}\underline{y}_k=l\right\}=$$
$$=\frac{1}{\mu_1}\sum_{j=1}^{h-1}P^{st}\left\{\sum_{k=1}^{n-1}\underline{y}_k=j,\ \sum_{k=1}^{n}\underline{y}_k=l-h+j\right\}+$$
$$-\frac{1}{\mu_1}\sum_{j=1}^{h-1}P^{st}\left\{\sum_{k=1}^{n}\underline{y}_k=j,\ \sum_{k=1}^{n+1}\underline{y}_k=l-h+j\right\},$$

which holds for $n\in N''$, $h\in N''$ and $l\in N'''$ with $h<l$. For $n=1$, $h\in N'$ and $l\in N''$ with $h<l$ we find in a similar way

$$(2.13)\qquad P\left\{\underline{v}_0=h,\ \underline{v}_0+\underline{y}_1=l\right\}=\frac{1}{\mu_1}P^{st}\left\{\underline{y}_1=l-h\right\}-\frac{1}{\mu_1}\sum_{j=1}^{h-1}P^{st}\left\{\underline{y}_1=j,\underline{y}_1+\underline{y}_2=l-h+j\right\}.$$

From (2.12) and (2.13) we have for $m\in N'$

$$(2.14)\qquad \sum_{n=1}^{m}P\left\{\underline{v}_0+\sum_{k=1}^{n-1}\underline{y}_k=h,\ \underline{v}_0+\sum_{k=1}^{n}\underline{y}_k=l\right\}=$$
$$=\frac{1}{\mu_1}\pi_{l-h}-\frac{1}{\mu_1}\sum_{j=1}^{h-1}P^{st}\left\{\sum_{k=1}^{m}\underline{y}_k=j,\ \sum_{k=1}^{m+1}\underline{y}_k=l-h+j\right\},$$

the last sum being equal to 0 for $m\geqslant h$. Hence we may take $m\to\infty$ or (2.3) has been proved for $r=0$.

In proving (2.2) we have shown (2.3) to be true for $r=0$. But then (2.3) holds for any $r\in N$ as the distributions of the variables involved do not depend on r, if both (2.1) and (2.2) hold.

This completes the proof of theorem 2.1.

3. Renewal theorem for Markov-dependent renewal-intervals

It will be convenient to make use of the notion of a returnpath. For the Markov chain M_1 with transition probabilities $p_{y,z}$ and states $y\in Y$, $z\in Y$, a path from state y to state z of order n with length λ is by definition a sequence of $n+1$ states $y_0,y_1,\ldots,y_n$ with $y_0=y$ and $y_n=z$, where $n\in N'$, such that $p_{y_{\nu-1},y_\nu}>0$ for $\nu\in\{1,2,\ldots,n\}$ and $\sum_{\nu=1}^{n}y_\nu=\lambda$. A returnpath from state y to state y is a path from state y to state y of arbitrary order and of arbitrary length, such that none of the intermediate states (i.e. $y_1,y_2,\ldots,y_{n-1}$ if the order is n) is the state y.

Theorem 3.1. If the Markov chain M_1, i.e. the sequence $\underline{y}_0=\underline{u}_0+\underline{v}_0$, $\underline{y}_1,\underline{y}_2,\ldots$, has the following properties

(a) the chain M_1 is irreducible,

(b) the greatest common divisor of the lengths of the returnpaths of at least one state y of the chain M_1 is equal to 1, then

$$(3.1)\quad U_r(i,j) \overset{\text{def}}{=} \sum_{n=1}^{\infty} P\left\{\underline{v}_0 + \sum_{k=1}^{n-1} \underline{y}_k = r \,\middle|\, \underline{u}_0 = i,\ \underline{v}_0 = j\right\} \qquad i+j\in Y, i\in N, j\in N', r\in N,$$

satisfies

$$(3.2)\quad \lim_{r\to\infty} U_r(i,j) = \frac{1}{\mu_1} \qquad i+j\in Y, i\in N, j\in N',$$

while for all $z_\nu \in Y$ with $-\alpha\leq\nu\leq\beta$, where $\alpha, \beta, \nu \in N$, $z_0=i+j$, $i,i'\in N$, $j,j'\in N'$ and $i'+j'\in Y$

$$(3.3)\quad \lim_{r\to\infty} P\left\{\underline{y}_{\nu,r} = z_\nu \text{ for } -\alpha\leq\nu\leq\beta,\ \underline{u}_r=i, \underline{v}_r=j \,\middle|\, \underline{u}_0=i', \underline{v}_0=j'\right\} = \frac{1}{\mu_1}\pi_{z_{-\alpha}} \prod_{\nu=-\alpha}^{\beta-1} p_{z_\nu, z_{\nu+1}} .$$

In particular we have (for $\alpha=\beta=0$)

$$(3.4)\quad \lim_{r\to\infty} P\left\{\underline{u}_r=i,\ \underline{v}_r=j \,\middle|\, \underline{u}_0=i',\ \underline{v}_0=j'\right\} = \frac{1}{\mu_1}\pi_{z_0} \qquad z_0\in Y, i'+j\in Y.$$

<u>Remark 3.1.</u> For the inverse chain to the chain M_1 (with the invariant distribution as initial distribution) the transition probabilities $\bar{p}_{y,z}$ are given by [1)]

$$(3.5)\quad \bar{p}_{y,z} \overset{\text{def}}{=} \frac{\pi_z p_{z,y}}{\pi_y} \qquad y,z\in Y.$$

The limit in (3.3) can be obtained from a stationary process, defined in the following way. For the sequence of random variables $\ldots, \underline{\bar{y}}_{-1}, \underline{\bar{y}}_0, \underline{\bar{y}}_1, \ldots$ taking only positive integer values, we define

$$(3.6)\quad \underline{\bar{y}}_0 = \underline{\bar{u}}_0 + \underline{\bar{v}}_0 ,$$

$$(3.7)\quad P\left\{\underline{\bar{u}}_0 = i,\ \underline{\bar{v}}_0 = j\right\} = \frac{1}{\mu_1}\pi_{i+j} \qquad i+j\in Y,$$

$$(3.8)\quad P\left\{\underline{\bar{y}}_{-n-1} = z \,\middle|\, \underline{\bar{y}}_{-n} = y\right\} = \bar{p}_{y,z} \qquad n\in N, y\in Y, z\in Y,$$

$$(3.9)\quad P\left\{\underline{\bar{y}}_{n+1} = z \,\middle|\, \underline{\bar{y}}_n = y\right\} = p_{y,z} \qquad n\in N, y\in Y, z\in Y,$$

and we assume that under the condition $\underline{\bar{y}}_0 = y\in Y$ the sequences $\underline{\bar{y}}_0, \underline{\bar{y}}_{-1}, \underline{\bar{y}}_{-2}, \ldots$ and $\underline{\bar{y}}_0, \underline{\bar{y}}_1, \underline{\bar{y}}_2, \ldots$ are independent (simple) Markov chains for each $y\in Y$.

We may now calculate

---

1) It will be seen that $\pi_y > 0$ for $y\in Y$.

$$(3.10)\quad P\left\{\bar{\underline{y}}_\nu = z_\nu \text{ for } -\alpha\leq\nu\leq\beta\ ,\ \bar{\underline{u}}_0 = i,\ \bar{\underline{v}}_0 = j\right\} =$$

$$= P\left\{\bar{\underline{y}}_\nu = z_\nu \text{ for } -\alpha\leq\nu\leq\beta \,\middle|\, \bar{\underline{u}}_0 = i,\ \bar{\underline{v}}_0 = j\right\} P\left\{\bar{\underline{u}}_0=i,\ \bar{\underline{v}}_0=j\right\} =$$

$$= \prod_{\nu=-(\alpha-1)}^{0} \bar{p}_{z_\nu, z_{\nu-1}} \prod_{\nu=0}^{\beta-1} p_{z_\nu, z_{\nu+1}} \frac{1}{\mu_1} \pi_{z_0} =$$

$$= \frac{1}{\mu_1} \pi_{z_0} \prod_{\nu=-(\alpha-1)}^{0} \pi_{z_{\nu-1}} p_{z_{\nu-1}, z_\nu} \frac{1}{\pi_{z_\nu}} \prod_{\nu=0}^{\beta-1} p_{z_\nu, z_{\nu+1}} =$$

$$= \frac{1}{\mu_1} \pi_{z_{-\alpha}} \prod_{\nu=-\alpha}^{\beta-1} p_{z_\nu, z_{\nu+1}},$$

which is equal to the limit in (3.3). This result is by no means trivial, because $\bar{\underline{y}}_0$ does <u>not</u> have the invariant probability distribution (cf. (2.5) for $\underline{y}_0$), i.e. $P\left\{\bar{\underline{y}}_0=y\right\} \neq \pi_y$.

<u>Remark 3.2.</u> From theorem 3.1 the weaker

$$(3.11)\quad \lim_{r\to\infty} \frac{1}{r} \sum_{n=1}^{\infty} P\left\{\underline{v}_0 + \sum_{k=1}^{n-1} \underline{y}_k \leq r \,\middle|\, \underline{u}_0=i,\ \underline{v}_0=j\right\} = \frac{1}{\mu_1} \qquad i+j\in Y,$$

is a trivial consequence. It may be formulated in terms of $\underline{n}_r(i,j)$. The equivalent of (3.11) is

$$(3.12)\qquad \lim_{r\to\infty} \frac{1}{r}\mathcal{E}\, \underline{n}_r(i,j) = \frac{1}{\mu_1} \qquad i+j\in Y,$$

because

$$(3.13)\quad \mathcal{E}\,\underline{n}_r(i,j) = \sum_{n=1}^{\infty} nP\left\{\underline{n}_r(i,j)=n\right\} = \sum_{n=1}^{\infty} P\left\{\underline{n}_r(i,j)\geq n\right\} =$$

$$= \sum_{n=1}^{\infty} P\left\{\underline{n}_r\geq n \,\middle|\, \underline{u}_0=i,\ \underline{v}_0=j\right\} = \sum_{n=1}^{\infty} P\left\{\underline{v}_0 + \sum_{k=1}^{n-1} \underline{y}_k \leq r \,\middle|\, \underline{u}_0=i, \underline{v}_0=j\right\}.$$

Instead of (3.2) we may write

$$(3.14)\qquad \lim_{r\to\infty}\left\{\mathcal{E}\,\underline{n}_r(i,j) - \mathcal{E}\,\underline{n}_{r-1}(i,j)\right\} = \frac{1}{\mu_1} \qquad i+j\in Y.$$

We have here extended two well-known theorems from renewal theory to what can be called renewal theory for Markov-dependent renewal-intervals. If

$$(3.15)\qquad p_{y,z} = \pi_z \qquad y\in Y,\ z\in Y,$$

then the $\underline{y}_0, \underline{y}_1, \underline{y}_2, \ldots$ are independent random variables, all having the same distribution function. For these variables (3.12) and (3.14) hold if the conditions of theorem 3.1 apply. Condition (a) is trivially satisfied, condition (b) can be simplified to Feller's: the greatest common divisor of the $y_{(1)}, y_{(2)}, \ldots$ is equal to 1. Cf.

Feller (1950), theorem 3 on page 244 and problem 7 on page 262. Two further results are known for independent $\underline{y}_0, \underline{y}_1, \ldots$, they also appear as problems in Feller (1950). Cf. problems 10 and 11 on page 263. In the present notation they read

$$(3.16)\qquad \lim_{r \to \infty} \left\{ \mathcal{E}\, \underline{n}_r(i,j) - \frac{r}{\mu_1} \right\} = \frac{\mu_2 + \mu_1}{2\mu_1^2} - \frac{j}{\mu_1} \qquad i+j \in Y,$$

and

$$(3.17)\qquad \lim_{r \to \infty} \frac{1}{r} \operatorname{var} \underline{n}_r(i,j) = \frac{\mu_2 - \mu_1^2}{\mu_1^3} \qquad i+j \in Y,$$

where

$$(3.18)\qquad \mu_2 \overset{\text{def}}{=} \sum_{y \in Y} y^2 \pi_y .$$

Similar results hold for Markov-dependent renewal-intervals if a further restriction to finitely many states is made. They can be proved with the technique of the next proof (cf. also Runnenburg (1961)).

Proof of theorem 3.1. First we summarize part of the relevant theory from Feller (1950), Chapter 15, on Markov chains.

Consider an irreducible Markov chain with states $y \in Y$, initial probability distribution $q_y^{(0)}$ and transition probabilities $p_{y,z}$. The chain is called irreducible, because it is possible to reach every state from any state in a finite number of steps with positive probability. The absolute probabilities $q_y^{(r)}$ of reaching state y at the r^{th} step are then such, that for each $y \in Y$

$$(3.19)\qquad \pi_y \overset{\text{def}}{=} \lim_{r \to \infty} \frac{1}{r} \sum_{k=1}^{r} q_y^{(k)}$$

exists and is independent of the initial probabilities $q_z^{(0)}$ for all $z \in Y$. If the chain is aperiodic, the Cesàro limit in (3.19) may be replaced by an ordinary limit. Furthermore, for all $y \in Y$

$$(3.20)\qquad \begin{cases} \pi_y > 0, \ \sum_{z \in Y} \pi_z p_{z,y} = \pi_y \text{ and } \sum_{z \in Y} \pi_z = 1 & \text{if the chain is ergodic,} \\ \pi_y = 0 & \text{if the chain is not ergodic.} \end{cases}$$

If the chain is ergodic, then to any solution x_y with $y \in Y$ of

$$(3.21)\qquad \sum_{y \in Y} x_y p_{y,z} = x_z, \quad \sum_{y \in Y} |x_y| < \infty ,$$

there exists a constant c with $x_z = c\,\pi_z$ for all z. Hence the π_z are the unique solution of (3.21), for which $\sum_{y \in Y} \pi_y = 1$.

Theorem 1 in Foster (1953) may be reformulated so it holds for periodic Markov chains too. Here we only need: An irreducible Markov chain is ergodic if there exists a nonnegative nonnull solution x_y with $y \in Y$ of

$$(3.22) \qquad \sum_{y \in Y} x_y p_{y,z} = x_z, \quad \sum_{y \in Y} x_y < \infty .$$

Regarding the foregoing as known, consider the Markov chain M_1. We assumed in section 1 the existence of an invariant distribution π_y. These π_y are unique and equal to the π_y defined in (3.19) because of the quoted theory and assumption (a) of theorem 3.1. Moreover, the chain is ergodic, because the invariant distribution of section 1 provides a nonnull solution of (3.22). Hence $\pi_y > 0$ for $y \in Y$.

From the chain M_1 we construct a Markov chain M_2 with states (y,i), where $y \in Y$, $i \in N'$ and $i \leq y$. The chain M_2 is the chain of random vectors $(\underline{y}_{0,r}, \underline{v}_r)$. Hence chain M_2 is in state (y,i) at the r^{th} step if $\underline{y}_{0,r} = y$ and $\underline{v}_r = i$. It is easily verified, that chain M_2 has transition probabilities (defined only for $y \in Y$, $z \in Y$ and $i \in N'$, $j \in N'$)

$$(3.23) \qquad p_{y,i;z,j} \overset{\text{def}}{=} \begin{cases} 1 & \text{if } y=z \text{ and } j=i-1, \\ p_{y,z} & \text{if } i=1 \text{ and } j=z, \\ 0 & \text{otherwise.} \end{cases}$$

Because M_1 is irreducible, the same holds for M_2. Now consider the equations equivalent to (3.22)

$$(3.24) \qquad \sum_{y \in Y} \sum_{i=1}^{y} x_{y,i}\, p_{y,i;z,j} = x_{z,j} ,$$

where $j \leq z$, $z \in Y$ and $j \in N'$. With (3.23) they can be replaced by

$$(3.25) \qquad \sum_{y \in Y} x_{y,1}\, p_{y,z} = x_{z,j} ,$$

where $j \leq z$, $z \in Y$ and $j \in N'$. Any solution of (3.25) must satisfy

$$(3.26) \qquad x_{z,j} = c\, \pi_z \qquad j \leq z,\ z \in Y,\ j \in N',$$

as the chain M_1 is ergodic. If we take $c > 0$, then by Foster's result the chain M_2 is ergodic. If we take $c = \frac{1}{\mu_1}$, then

$$(3.27) \qquad \pi_{z,j} \overset{\text{def}}{=} \frac{1}{\mu_1} \pi_z \qquad j \leq z,\ z \in Y,\ j \in N',$$

is indeed a probability distribution. Hence, similar to (3.19)

$$\pi_{z,j} = \lim_{r \to \infty} \frac{1}{r} \sum_{k=1}^{r} q_{z,j}^{(k)} , \tag{3.28}$$

for any initial probability distribution $q_{y,i}^{(0)}$.

The Markov chain M_2 is aperiodic, if at least one state is aperiodic, because the chain is irreducible. By assumption (b) of the theorem an aperiodic state of M_2 exists. Therefore M_2 is aperiodic (although M_1 may be periodic!) and (3.28) holds for the ordinary limit. In particular we have for the r-step transition probabilities $p_{y,i;z,j}^{(r)}$ of the chain M_2

$$\pi_{z,j} = \lim_{r \to \infty} p_{y,i;z,j}^{(r)} \tag{3.29}$$

for all states (z,j), independent of the initial state (y,i).

As

$$\liminf_{r \to \infty} \sum_{z \in Y} p_{y,i;z,z}^{(r)} \geq \sum_{z \in Y} \liminf_{r \to \infty} p_{y,i;z,z}^{(r)} = \tag{3.30}$$

$$= \frac{1}{\mu_1} \sum_{z \in Y} \pi_z = \frac{1}{\mu_1}$$

and

$$\limsup_{r \to \infty} \sum_{z \in Y} p_{y,i;z,z}^{(r)} = 1 - \liminf_{r \to \infty} \sum_{z \in Y} \sum_{j=1}^{z-1} p_{y,i;z,j}^{(r)} \leq \tag{3.31}$$

$$\leq 1 - \frac{1}{\mu_1} \sum_{z \in Y} \sum_{j=1}^{z-1} \pi_z = \frac{1}{\mu_1} ,$$

for all states (y,i) of the chain M_2 we have

$$\lim_{r \to \infty} \sum_{z \in Y} p_{y,i;z,z}^{(r)} = \frac{1}{\mu_1} . \tag{3.32}$$

If now we note that for $i+j \in Y$

$$U_r(i,j) = \sum_{z \in Y} p_{i+j,z,z}^{(r)} , \tag{3.33}$$

combination of (3.32) and (3.33) yields (3.2). Because

$$P\left\{ \underline{y}_{\nu,r} = z_\nu \text{ for } -\alpha \leq \nu \leq \beta ,\ \underline{u}_r = i, \underline{v}_r = j \,\middle|\, \underline{u}_0 = i', \underline{v}_0 = j' \right\} = \tag{3.34}$$

$$= p_{i'+j',j';z_{-\alpha},z_{-\alpha}}^{(s)} \prod_{\nu=-\alpha}^{\beta-1} p_{z_\nu,z_{\nu+1}} ,$$

where $s \overset{\text{def}}{=} r - \sum_{\nu=-\alpha}^{-1} z_\nu - i$, by (3.27) and (3.29) we have (3.3). Hence theorem 3.1 is proved.

References

Feller (1950), W., An introduction to probability theory and its applications, Wiley, New York, 1950.

Foster (1953), F.G., On the stochastic matrices associated with certain queueing processes, Ann.Math.Stat. 24 (1953), 355-360.

Runnenburg (1960), J.Th., On the use of Markov processes in one-server waiting-time problems and renewal theory, thesis, Poortpers, Amsterdam, 1960.

Runnenburg (1961), J.Th., An example illustrating the possibilities of renewal theory and waiting-time theory for Markov-dependent arrival-intervals, Report S 285, Mathematical Centre, Amsterdam, 1961.