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ON STABLE TRANSFORMATIONS[1]

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SUMMARY. Let T be a measure preserving transformation of a probability space $(\Omega, \mathcal{A}, P)$ into itself. We shall say that T is a *stable* transformation if for every $A, B \in \mathcal{A}$, $\lim\limits_{n\to\infty} P(T^{-n}A \cap B)$ exists. Stable transformations are investigated in this article with the aid of Rényi's results on stable sequences of events. The concept of a stable transformation generalises that of a mixing transformation.

1. Introduction

Let $(\Omega, \mathcal{A}, P)$ be a probability space. Let T be a measurable transformation (not necessarily one to one) of Ω into itself. Assume further that T is measure preserving, that is $P(T^{-1}A) = P(A)$ for every $A \in \mathcal{A}$. Following Rényi (1963), we shall say that T is *stable* if for every $A \in \mathcal{A}$, $\{T^{-n} A, n = 1, 2, \ldots\}$ is a stable sequence of events, that is, if for every $A, B \in \mathcal{A}$, $\lim\limits_{n\to\infty} P(T^{-n}A \cap B)$ exists. The purpose of this article is to study such transformations.

The concept of stability generalises that of mixing. A mixing transformation is, of course, always stable. It will be shown that a stable transformation T is mixing if and only if the σ-field of invariant sets is trivial (a measurable set A is said to be invariant if $T^{-1} A = A$).

As the present investigation relies heavily on the results proved in Rényi (1963), we shall for the sake of completeness give a résumé of these in Section 2. In Section 3 the analogues of results for stable sequences of events will be proved for stable transformations. Examples of stable transformations, including a counter-example to disprove a reasonable conjecture, will be given in Section 4.

2. Résumé of results on stable sequences of events

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\{A_n, n = 1, 2, \ldots\}$ be a sequence of events. We shall say that $\{A_n\}$ is a *stable* sequence of events if for every $B \in \mathcal{A}$

$$\lim_{n\to\infty} P(A_n \cap B) = Q(B)$$

exists.

Theorem 2.1 : *If $\{A_n\}$ is a stable sequence of events and Q is as above, then Q is a measure on $(\Omega, \mathcal{A})$ and is absolutely continuous with respect to P.*

Denote by α the Radon-Nikodym derivative of Q with respect to P. α is said to be the *local density* of the stable sequence of events $\{A_n\}$.

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A sequence of events $\{A_n, n = 1, 2, \ldots\}$ is said to be *mixing* if there exists β, $0 \leqslant \beta \leqslant 1$, such that for every $B \in \mathcal{A}$

$$\lim_{n \to \infty} P(A_n \cap B) = \beta P(B).$$

β is called the density of the mixing sequence$\{A_n\}$.

Corollary 2.1 : *If $\{A_n\}$ is a stable sequence of events with local density α, then $\{A_n\}$ is mixing if and only if α is a constant almost surely.*

Theorem 2.2 : *The sequence of events $\{A_n, n = 1, 2, \ldots\}$ is stable if and only if*

$$\lim_{n \to \infty} P(A_k \cap A_n) = Q_k, \quad k = 1, 2, \ldots$$

exists. If, in addition, $P(A_k) > 0$, $k = 1, 2, \ldots$, set $q_k = Q_k/P(A_k)$, $k = 1, 2, \ldots$, and $q_0 = \lim_{n \to \infty} P(A_n)$. Then $\{A_n\}$ is mixing if and only if the q_k's $(k = 0, 1, 2, \ldots)$ are all equal..

The property of stability is preserved if the underlying probability measure P is replaced by a probability measure absolutely continuous with repect to it. More explicitly we have the following theorom.

Theorem 2.3 : *Let $\{A_n, n = 1, 2, \ldots, \}$ be a stable sequence of events with local density α on the probability space $(\Omega, \mathcal{A}, P)$. Let P^* be a probability measure on $(\Omega, \mathcal{A})$, absolutely continuous with respect to P. Then $\{A_n\}$ is stable on $(\Omega, \mathcal{A}, P^*)$ with local density α.*

3. Some general theorems on stable transformations

We shall now prove some theorems about stable transformations.

Theorem 3.1: *Let T be a stable measure preserving transformation on $(\Omega, \mathcal{A}, P)$. Then*

$$\lim_{n \to \infty} P(T^{-n}A \cap B) = \int_B P(A/\mathcal{J})dP$$

for every $A, B \in \mathcal{A}$. Here $\mathcal{J}$ is the invariant σ-field and $P(A/\mathcal{J})$ is the conditional probability of A given $\mathcal{J}$.

Proof : By definition, the sequence $\{T^{-n} A, n = 1, 2, \ldots\}$, where $A \in \mathcal{A}$, is stable. Hence $\lim_{n \to \infty} P(T^{-n}A \cap B)$ exists for every $B \in \mathcal{A}$. But by the Individual Ergodic Theorem, we have : $\frac{1}{n} \sum_{k=1}^{n-1} I_{T^{-k}A}$ converges almost surely to $P(A/\mathcal{J})$, where I_C is the indicator of the set C. Hence, if $B \in A$, $\frac{1}{n} \sum_{k=0}^{n-1} I_{T^{-k}A} . I_B$ converges almost surely to $P(A/\mathcal{J})\, I_B$. Apply the Dominated Convergence Theorem. We get

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} P(T^{-k}A \cap B) = \int_B P(A/\mathcal{J})dP$$

that is, the sequence $\{P(T^{-n}A \cap B)\}$ is Ĉesaro-summable to $\int_B P(A/\mathcal{J})\, dP$. The result now follows from the remark made at the beginning of the proof.

Remark : Denote by α_A the local density of the stable sequence $\{T^{-n}A\}$, $A\epsilon\mathscr{A}$. What we have proved then is that $\int_B \alpha_A dP = \int_B P(A/\mathscr{I})\ dP$ for every $B\epsilon\mathscr{A}$. But α_A and $P(A/\mathscr{I})$ are $\mathscr{A}$-measurable functions. Hence $\alpha_A = P(A/\mathscr{I})$ almost surely. Therefore the local density of $\{T^{-n}\ A\}$ is simply $P(A/\mathscr{I})$.

In order to check if a measure preserving transformation T is stable, it is in fact sufficient to verify that $\lim_{n\to\infty} P(T^{-n}A \cap B)$ exists for $A = B\epsilon\mathscr{A}$.

Theorem 3.2 : *A measure preserving transformation T is stable if and only if $\lim_{n\to\infty} P(T^{-n}A \cap A)$ exists for every $A\epsilon\mathscr{A}$.*

Proof : The "only if" part is trivial. Consider now the sequence $\{T^{-n}A, n=1, 2, \ldots\}$, $A\epsilon\mathscr{A}$. We want to show that $\{T^{-n}\ A\}$ is stable. Note that since T is measure preserving, $P(T^{-k}A \cap T^{-n}A) = P(T^{-k}\ (T^{-(n-k)}A \cap A)) = P(T^{-(n-k)}\ A \cap A)$, where $n > k$. But by hypothesis, $\lim_{n\to\infty} P(T^{-(n-k)}A \cap A)$ exists and so $\lim_{n\to\infty} P(T^{-k}A \cap T^{-n}A)$ exists, $k = 1, 2, \ldots$. Hence, by Theorem 2.2, $\{T^{-n}A\}$ is stable. This completes the "if" part of the proof.

A measure preserving transformation T is *mixing* if for every $A\epsilon\mathscr{A}$, the sequence of events $\{T^{-n}A,\ n = 1, 2, \ldots\}$ is mixing with density $P(A)$, that is, if for every A, $B\epsilon\mathscr{A}$

$$\lim_{n\to\infty} P(T^{-n}A\cap B) = P(A).\ P(B).$$

Clearly a mixing transformation is stable. When is the converse true ?

Corollary 3.1: *In order that a stable transformation T be mixing, it is necessary and sufficient that $\mathscr{I}$, the σ- field of invariant sets, be trivial under P.*

Proof : Suppose that $\mathscr{I}$ is trivial under P, that is, if $A\epsilon\mathscr{I}$, then $P(A) = 0$ or 1. By Theorem 3.1, since T is stable, we have

$$\lim_{n\to\infty} P(T^{-n}A\cap B) = \int_B P(A/\mathscr{I})dP$$

for every A, $B\ \epsilon\mathscr{A}$. But as $\mathscr{I}$ is trivial, $P(A/\mathscr{I}) = P(A)$ almost surely for every $A\epsilon\mathscr{A}$. Hence $\lim_{n\to\infty} P(T^{-n}A\cap B) = P(A).\ P(B)$ for every A, $B\ \epsilon\mathscr{A}$, so that T is mixing. Conversely assume that T is mixing. Let $A\ \epsilon\ \mathscr{I}$. Then $T^{-n}\ A = A$ for $n = 1, 2, \ldots$. But $\{T^{-n}A,\ n = 1, 2, \ldots\}$ is mixing. Hence for every $B\ \epsilon\ \mathscr{I}$, $P(A\cap B) = P(A).\ P(B)$, that is $P(A) = 0$ or 1. Therefore, $\mathscr{I}$ is trivial, which concludes the proof.

Let us now turn to the functional form of stability. Let $\mathscr{L}_2(\Omega,\mathscr{A},P)$ be the class of complex-valued random variables f on $(\Omega, \mathscr{A}, P)$ such that $\int|f|^2dP < \infty$. Identify all functions $\mathscr{L}_2$ which differ on a set of measure zero. Then $\mathscr{L}_2$ is a Hilbert space over the field of complex numbers with inner product $(f, g) = \int f\,\bar{g}dP$ (here $\bar{x}$ is the complex conjugate of x) and norm $\|f\| = (\int|f|^2\ dP)^{\frac{1}{2}}$. If T is a measure preserving transformation of Ω into itself, we can define a transformation U of $\mathscr{L}_2$ into itself as follows : $Uf = f \circ T$, $f \in \mathscr{L}_2$. Then U is an isometry, that is, U is a bounded linear transformation such that $\|Uf\| = \|f\|$ for every $f \in \mathscr{L}_2$ (see Halmos, 1956, p. 14). Denote by U^n the n-th iterate of U.

Call a function $f \in \mathscr{L}_2$ *invariant* if $Uf = f$. Denote by E_0 the projection on the closed subspace of invariant functions in $\mathscr{L}_2$. We can now characterise stability of T as follows.

Theorem 3.3 : *A measure preserving transformation T is stable if and only if $\lim_{n\to\infty} (U^n f, g) = (E_0 f, g)$ for every $f, g \in \mathscr{L}_2$, that is, U^n converges to E_0 in the weak operator topology.*

Proof : Straightforward.

Remark : Let $\{f_j, j \in J\}$ be a complete orthonormal set for $\mathscr{L}_2$. Then a measure preserving transformation T is stable if and only if $\lim_{n\to\infty} (U^n f_i, f_j) = (E_0 f_i, f_j)$ for all $i, j \in J$. This follows directly from the linearity and continuity of U.

In the case of mixing, $\mathscr{J}$ is trivial so that all invariant functions in $\mathscr{L}_2$ are constants. Hence $E_0 f = (f, 1)1$ for every $f \in \mathscr{L}_2$, where 1 stands for the function which is equal to one everywhere.

Corollary 3.2 : *A measure preserving transformation T is mixing if and only if $\lim_{n\to\infty} (U^n f, g) = ((f, 1)\, 1, g) = (f, 1)(1, g)$ for every $f, g \in \mathscr{L}_2$.*

We may add here that if T is a stable measure preserving transformation, then U^n converges to E_0 in the strong operator topology only in a rather trivial and uninteresting case. In fact, U^n converges to E_0 if and only if U is the identity. To prove this statement, note that since U^n converges weakly to E_0, U^n will converge strongly to E_0 if and only if $\lim_{n\to\infty} \|U^n f\| = \|E_0 f\|$ for each $f \in \mathscr{L}_2$. But $\|U^n f\| = \|f\|$ for $n = 1, 2, \ldots$. Note also that for any $f \in L_2$, $\|f\|^2 = \|E_0 f\|^2 + \|f - E_0 f\|^2$ by the Decomposition Theorem. Hence $\|f\| = \|E_0 f\|$ if and only if $E_0 f = f$. It follows that U^n converges strongly to E_0 if and only if $U f = f$ for each $f \in \mathscr{L}_2$.

The property of stability is preserved if the underlying measure is replaced by a measure absolutely continuous with respect to it. More explicitly, we have the following theorem.

Theorem 3.4 : *Let T be a stable measure preserving transformation on $(\Omega, \mathscr{A}, P)$. Let Q be a probability measure on $(\Omega, \mathscr{A})$ absolutely continuous with respect to P on $\mathscr{J}$. Assume further that Q is preserved by T. Then T is stable on $(\Omega, \mathscr{A}, Q)$ and for every $A \in \mathscr{A}$, $P(A/\mathscr{J}) = Q(A/\mathscr{J})$ almost surely $[Q]$.*

Proof : (1) First we prove that Q is absolutely continuous with respect to P on $\mathscr{A}$. Let $A \in \mathscr{A}$ and $P(A) = 0$. Since T preserves P, $P(\limsup T^{-n}A) = 0$. But $\limsup T^{-n}A \in \mathscr{J}$. Hence $Q(\limsup T^{-n}A) = 0$. It now follows from the fact that Q is preserved by T and the Recurrence Theorem (Halmos, 1956, p. 10) that $Q(A) = 0$.

(2) Now consider the sequence of events $\{T^{-n}A, n = 1, 2, \ldots\}$, $A \in \mathscr{A}$. Since Q is absolutely continuous with respect to P on $\mathscr{A}$, by Theorem 2.3, $\{T^{-n} A\}$ is stable with respect to Q. Hence T is stable on $(\Omega, \mathscr{A}, Q)$. Furthermore, by Theorem 2.3, $\lim_{n\to\infty} Q(T^{-n}A \cap B) = \int_B P(A/\mathscr{J})\, dQ$ for every A, $B \in \mathscr{A}$. Hence, by Theorem 3.1., we have $\int_B Q(A/\mathscr{J})dQ = \int_B P(A/\mathscr{J})dQ$ for every A, $B \in \mathscr{A}$. This proves the second assertion of the theorem.

Corollary 3.3 : *Let P and Q be probability measures on $(\Omega, \mathscr{A})$. Assume that T is stable and measure preserving with respect to both P and Q. Then, if $P = Q$ on $\mathscr{J}$, $P = Q$ on $\mathscr{A}$.*

Proof: Let $\mu(A) = \frac{1}{2}P(A)+\frac{1}{2}Q(A)$, $A \in \mathscr{A}$. It is easy to verify that T is stable and measure preserving with respect to μ. Note that P, Q are absolutely continuous with respect to μ. Furthermore, $\mu = P = Q$ on $\mathscr{J}$. By Theorem 3.4, $\mu(A/\mathscr{J}) = P(A/\mathscr{J})$ almost surely $[P]$ for every $A \in \mathscr{A}$. Note that the exceptional set above is $\mathscr{J}$-measurable and so must have μ-measure zero as well. Again, as $P(A/\mathscr{J})$ and $\mu(A/\mathscr{J})$ are $\mathscr{J}$-measureble functions, we have

$$\mu(A) = \int \mu(A/\mathscr{J})d\mu^{\mathscr{J}} = \int P(A/\mathscr{J})dP^{\mathscr{J}} = P(A)$$

for every $A \in \mathscr{A}$. Here $\mu^{\mathscr{J}}$, $P^{\mathscr{J}}$ denote the restriction of μ, P, respectively to $\mathscr{J}$. This proves the corollary.

Corollary 3.4 : *Let T be a measure preserving mixing transformation on $(\Omega, \mathscr{A}, P)$. Let Q be a probability measure on $(\Omega, \mathscr{A})$. Assume that Q is absolutely continuous with respect to P on $\mathscr{J}$ and that it is preserved by T. Then $P = Q$.*

Proof: Follows directly from Theorem 3.4.

Corollary 3.5 : *Let T be measure preserving and mixing with respect to probability measures P and Q on $(\Omega, \mathscr{A})$. Then either $P = Q$ or P and Q are mutually singular.*

Proof: Suppose $P \neq Q$. Then, by Corollary 3.3., there exists a set $A \in \mathscr{J}$ such that $P(A) \neq Q(A)$. But since T is mixing for both P and Q, either $P(A) = 1$ and $Q(A) = 0$ or $P(A) = 0$ and $Q(A) = 1$. In either case, P and Q are mutually singular.

In the rest of this section, we shall investigate stable transformations which are not necessarily measure preserving. As before, we shall say that a measurable transformation T on $(\Omega, \mathscr{A}, P)$ is *stable* if $\lim_{n\to\infty} P(T^{-n}A \cap B)$ exists for every A, $B \in \mathscr{A}$. Under certain additional assumptions, we shall prove that stability of a transformation makes it potentially measure preserving. Before making this last statement precise, we need a couple of definitions.

We shall say that a measurable transformation T on $(\Omega, \mathscr{A}, P)$ is *non-singular* if $P(A) = 0$ implies $P(T^{-1} A) = 0$. We shall call T *conservative* if $A, T^{-1} A, T^{-2}A, \ldots$, $(A \in \mathscr{A})$, mutually disjoint implies $P(A) = 0$.

We are now in a position to state our theorem.

Theorem 3.5 : *Let T be a stable, non-singular, conservative transformation on $(\Omega, \mathcal{A}, P)$. Then there exists a probability measure Q on $(\Omega, \mathcal{A})$ with the following properties :*

(i) *P and Q agree on $\mathcal{J}$,*

(ii) *T is a stable, measure preserving transformation on $(\Omega, \mathcal{A}, Q)$,*

(iii) *P and Q are equivalent, i.e. they vanish on the same sets,*

(iv) $\lim\limits_{n\to\infty} P(T^{-n}A \cap B) = \int_B Q(A/\mathcal{J})\, dP$ *for every* $A, B \in \mathcal{A}$.

Proof: Define $Q(A) = \lim\limits_{n\to\infty} P(T^{-n}A)$, $A \in \mathcal{A}$. The existence of the limit is guaranteed by the stability of T. It follows from the Vitali-Hahn-Saks Theorem (Halmos, 1950, p. 170) that Q is a probability measure. (i) is obvious. Clearly, $Q(A) = Q(T^{-1}A)$ for every $A \in \mathcal{A}$. Furthermore, non-singularity of T (with respect to P) implies that Q is absolutely continuous with respect to P. Now we can use Theorem 2.3. to conclude that T is stable with ıespect to Q. Thus (ii).

Now let $Q(A) = 0$. Since Q is preserved by T, $Q(\limsup T^{-n}A) = 0$. But $\limsup T^{-n}A \in \mathcal{J}$, so that $P(\limsup T^{-n}A) = 0$ by (i). Since T is conservative we can invoke the Recurrence Theorem for conservative transformations (Sucheston, 1957, p. 445) and conclude that $P(A) = 0$. We have already shown that $P(A) = 0$ implies $Q(A) = 0$. Hence (iii).

(iv) now follows from (iii), Theorem 2.3. and the remark following Theorem 3.1. This completes the proof of Theorem 3.5.

Remark : Conservativeness of T was used to prove that P is absolutely continuous with respect to Q. If T is invertible and both ways measurable, then the assumption of conservativeness can be dropped from the preceeding theorem. For now $\bigcup\limits_{n=\infty}^{\infty} T^n A$ plays the role of $\limsup T^{-n}A$.

4. Examples of stable transformations

Example 1 : Let T be the identity transfo rmation on a probability space $(\Omega, \mathcal{A}, P)$. Then T is a stable measure preserving transformation. If $\mathcal{A}$ is non-trivial, we get an example of a stable transformation that is not mixing.

Example 2 : Let $(\Omega_0, \mathcal{A}_0)$ be a measurable space and let $(\Omega_n, \mathcal{A}_n) = (\Omega_0, \mathcal{A}_0)$, $n = 1, 2, \ldots$. Let $(\Omega, \mathcal{A}) = \prod\limits_{n=1}^{\infty} (\Omega_n, \mathcal{A}_n)$. Denote by ω_n $(n = 1, 2, \ldots)$ the n-th coordinate of a point ω in Ω. We shall use the following notation for finite dimensional rectangles : $C\left(\underset{1}{E}^{(i_1)}, \ldots, \underset{n}{E}^{(i_n)}\right)$, where $i_1 << i_2 < \ldots < i_n$, is the set of all ω such that $\omega_{i_k} \in E_k$, $k = 1, \ldots, n$. If $i_k = k$, $k = 1, \ldots, n$, we shall write

$C(E_1, \dots, E_n)$. Let T be the shift operator on Ω, that is, $T\omega = \omega^1$, where $\omega^1_n = \omega_{n+1}$, $n = 1, 2, \dots$. Consider a symmetric probability measure P on $(\Omega, \mathscr{A})$, that is, P satisfies the following condition :

$$P\Big(C\Big(E_1^{(i_1)}, \dots, E_n^{(i_n)}\Big)\Big) = P\Big(C\Big(E_1^{(j_1)}, \dots, E_n^{(j_n)}\Big)\Big)$$

for all $n = 1, 2, \dots$, all $E_1, \dots, E_n \in \mathscr{A}_0$ and all sequences of positive integers $i_1, \dots, i_n$ and $j_1, \dots, j_n$ (i's all distinct and j's all distinct).

Then T is a stable, measure preserving transformation on $(\Omega, \mathscr{A}, P)$. Clearly T is measure preserving. Let B be a measurable $\{1, \dots, m\}$-cylinder, that is, $B = F \times \Omega_{m+1} \times \Omega_{m+2} \times \dots$, where F is a measurable subset of $\prod_{k=1}^{m} \Omega_k$. Let $B_k = T^{-k}B$, $k = 1, 2, \dots$. It is clear that $B_k = \Omega_1 \times \dots \times \Omega_k \times F \times \Omega_{k+m+1} \times \Omega_{k+m+2} \times \dots$, that is, B_k is a $\{k+1, \dots, k+m\}$-cylinder with base F. Hence, as P is a symmetric measure, for all large n and fixed k, $P(B_k \cap B_n) = P(D)$, where D is the $\{1, \dots, 2m\}$-cylinder, $F \times F \times \Omega_{2m+1} \times \Omega_{2m+2} \times \dots$. Therefore, $\lim_{n\to\infty} P(B_k \cap B_n)$ exists for every $k = 1, 2, \dots$. Consequently, the sequence of events $\{T^{-k}B, k = 1, 2, \dots\}$ is stable by virtue of Theorem 2.2. Now any set $A \in \mathscr{A}$ can be approximated arbitrarily closely by a measurable $\{1, \dots, m\}$-cylinder B (for some m), from which it follows that $\{T^{-n}A, n = 1, 2, \dots\}$ is a stable sequence of events for every $A \in \mathscr{A}$. This proves that T is a stable transformation.

In particular, let P be a product measure with identical components. The arguments of the last paragraph show that T is mixing. Conversely, assume that T is mixing for a symmetric measure P. Let $A = C(E_1, \dots, E_m)$ be a measurable finite dimensional rectangle. It is easy to see that

$$\lim_{n\to\infty} P(T^{-k}A \cap T^{-n}A) = P(C(E_1, \dots, E_m, E_1, \dots, E_m)), \quad k = 1, 2, \dots .$$

The limit is independent of k. But the sequence $\{T^{-n}A\}$ is mixing. Hence, by Theorem 2.2. we must have

$$P(C(E_1, \dots, E_m, E_1, \dots, E_m)) = P^2(C(E_1, \dots, E_m)).$$

As T is mixing, this last relation holds for all measurable finite-dimensional rectangles. Hence, by Theorems 5.2. and 5.3 in Hewitt and Savage (1955, pp. 477-78), P must be a product measure with identical components. We have thus proved :

Theorem 4.1 : *Let P be a symmetric probability on $(\Omega, \mathscr{A})$. Then T is a stable measure preserving transformation on $(\Omega, \mathscr{A}, P)$ and T is mixing if and only if P is a product measure with identical components.*

Example 3 : Let $\{x_n, n = 0, 1, \dots\}$ be a stationary, aperiodic Markov chain with countable state space I. Elements of I will be denoted by i with or without subscripts. Assume that the Markov chain is defined on the appropriate (unilateral) sequence space $(\Omega, \mathscr{A})$ and let T be the shift operator on $(\Omega, \mathscr{A})$. If P is the relevant probability measure on $(\Omega, \mathscr{A})$, T is a stable measure preserving transformation on $(\Omega, \mathscr{A}, P)$.

To see that T is stable, let us note that it is sufficient to demonstrate stability of sequences of events $\{T^{-n}A,\ n = 1, 2, \ldots\}$, where A is a finite-dimensional rectangle of the form $(x_0 = i_0, \ldots, x_m = i_m)$, the i's being ergodic states belonging to the same class. We have for fixed k and large n

$$P(T^{-k}A \cap T^{-n}A) = p_{i_0} p_{i_0 i_1} \cdots p_{i_{m-1} i_m} p^{(n-m-k)}_{i_m i_0} p_{i_0 i_1} \cdots p_{i_{m-1} i_m},$$

where p_i denotes the stationary distribution, p_{ij} the one-step transition probability and $p^{(n)}_{ij}$ the n-step transition probability.

Remembering that $\lim\limits_{n\to\infty} p^{(n)}_{ij} = \pi_{ij}$ for j ergodic, we obtain

$$\lim_{n\to\infty} P(T^{-k}A \cap T^{-n}A) = p_{i_0} p_{i_0 i_1} \cdots p_{i_{m-1} i_m} \pi_{i_m i_0} p_{i_0 i_1} \cdots p_{i_{m-1} i_m}, \quad k = 1, 2, \ldots$$

Hence, by Theorem 2.2, $T^{-n}A$ is stable. This proves the assertion.

Example 4 : Let Ω be a compact Abelian group, $\mathcal{A}$ the σ-field of Borel subsets of Ω and P normalised Haar measure on $(\Omega, \mathcal{A})$. Let T be a continuous automorphism of Ω. Then T is measure preserving with respect to P (Halmos, 1956, p. 7).

Let C be the character group of Ω, that is, C is the set of all continuous homomorphisms of Ω into the circle group. Denote by U the unitary operator on $\mathcal{L}_2(\Omega, \mathcal{A}, P)$ induced by T. U restricted to C is an automorphism of the group C. If $f \epsilon C$, by the orbit of f under U, we shall mean the set $\{U^n f,\ n = 0, \pm 1, \pm 2, \ldots\}$. If the orbit is finite, the least positive integer m such that $U^m f = f$ will be called the order of the orbit. The order of the orbit of an invariant character f(i.e. $f = Uf$) under U is clearly 1. We remark for later use that C forms a complete orthonormal set in $\mathcal{L}_2(\Omega, \mathcal{A}, P)$. (These facts may be found in Halmos (1956, p. 53)).

We want to characterise continuous automorphisms of Ω which are stable.

Theorem 4.2 : *A continuous automorphism T of a compact Abelian group Ω is stable if and only if the induced automorphism U on the character group C has no finite orbits of order $m > 1$.*

Proof : Assume that T is stable and that there is a $f \epsilon C$ such that the orbit of f under U is finite and of order $m > 1$. Then, it is clear that $\lim\limits_{n\to\infty} \sup\, (U_n f, f) = 1$ and $\lim\limits_{n\to\infty} \inf\, (U^n f, f) = 0$, so that $\lim\limits_{n\to\infty} (U^n f, f)$ does not exist. We have thus arrived at a contradiction.

Conversely, suppose that U has only finite orbits of order 1 or infinite orbits. If $f \epsilon C$ is such that $Uf = f$, then it is easy to see that for every $g \epsilon C$, $\lim\limits_{n\to\infty} (U^n f, g) = (f, g)$ $= 0$ or 1 according as $g \neq f$ or $g = f$. If the orbit $f \epsilon C$ under U is infinite, then clearly $\lim_{n \to \infty} (U^n f, g) = 0$ for every $g \epsilon C$. Hence, in either case, $\lim_{n \to \infty} (U^n f, g) = (E_0 f, g)$ for every $f, g \epsilon C$, where E_0 is the projection on the closed subspace of invariant functions in $\mathcal{L}_2(\Omega, \mathcal{A}, P)$. It now follows from the fact that C forms a complete orthonormal set and the remark made after Theorem 3.3 that T is stable. This completes the proof of Theorem 4.2.

Since a stable transformation T is mixing if and only if every invariant function in $\mathcal{L}_2$ is a constant, we can now characterise continuous automorphisms which are mixing as follows :

Corollary 4.1 : *A continuous automorphism T of a compact Abelian group Ω is mixing if and only if the induced automorphism U on the character group C has only infinite orbits, other than the trivial orbit $\{1\}$ (here 1 stands for the function where value is one everywhere on Ω).*

Example 5 : It is known that, under suitable assumptions on the measure space, a measure preserving transformation can be expressed as a direct sum (direct integral) of ergodic transformations (see, for instance, Halmos (1941)).

The question then naturally arises whether a stable measure preserving transformation is always a direct sum of mixing transformations. We give an example below which answers the question in the negative. (The reader is referred to Halmos (1941) for a precise definition of the concept of direct sum).

Let $X = Y =$ circumference of the unit circle, $\mathcal{A}_1 = \mathcal{A}_2 = \sigma$-field of Borel subsets of $X = Y$, and $P_1 = P_2 =$ normalised Lebesgue measure on $\mathcal{A}_1 = \mathcal{A}_2$. Let $(\Omega, \mathcal{A}, P) = (X, \mathcal{A}_1, P_1) \times (Y, \mathcal{A}_2, P_2)$. Ω is then a compact Abelian group, the group operation being coordinatervise multiplication, $\mathcal{A}$ is the σ-field of Borel subsets of Ω and P is normalised Haar measure. We shall denote points of Ω by ordered pairs (x, y), where $x \epsilon X$, $y \epsilon Y$. We now define a transformation T of Ω onto Ω as follows : $T(x, y) = (x, xy) \epsilon \Omega$. In fact, T is a continuous automorphism of Ω and is, consequently, measure preserving with respect to P. Now the character group C of Ω is easily seen to be the set of functions $f_{m,n} (m, n = 0, \pm 1, \pm 2, \ldots)$, where $f_{m,n}(x, y) = x^m y^n$, $(x, y) \epsilon \Omega$. It follows from a straightforward application of Theorem 4.2 that T is stable. Thus, we have proved that T is a stable measure preserving transformation.

We assert that T is a direct sum of transformations, none of which is mixing. First note that the invariant σ-field $\mathcal{J}$ of T is the σ-field of sets of the form $A \times Y$, $A \epsilon \mathcal{A}_1$. The atoms of $\mathcal{J}$ are of the form $\{x\} \times Y$, $x \epsilon X$. We shall denote atoms of $\mathcal{J}$ by Y_x. Now, each Y_x being invariant, T induces a transformation, say T_x, on each Y_x. In fact, $T_x y = xy$ for $(x, y) \epsilon Y_x$. It is easy to see that T is a direct sum of these transformations T_x, $x \epsilon X$. Now T_x is a rotation on the circle group for every $x \epsilon X$. Consequently, for each $x \epsilon X$, T_x is measure preserving with respect to Lebesgue measure (Halmos, 1956, p. 7); furthermore, for all x, except for the countable number of x's such that $x^n = 1$ for some natural number n, T_x is ergodic (Halmos, 1956, p. 26).

But for no $x \epsilon X$ is T_x mixing (Halmos, 1956, p. 37). Thus we have shown that T is a direct sum of ergodic measure preserving transformations, none of which is mixing. It follows now, since the transformation T_x were defined on the atoms of $\mathcal{J}$, that T cannot be expressed as a direct sum of mixing transformations.

Example 6 : We conclude with an example of a stable, non-singular transformation which is not measure preserving.

Let $\Omega = [0, 1]$, $\mathcal{A}$ the σ-field of Borel subsets of Ω and P Lebesgue measure on $\mathcal{A}$. Define a transformation T of Ω onto itself as follows :

$$Tx = \begin{cases} 2x & \text{if } x \epsilon [0, \frac{1}{2}) \\ x & \text{if } x \epsilon [\frac{1}{2}, 1] \end{cases}$$

T is clearly measurable.

Since for any set $A \epsilon \mathcal{A}$, $P(T^{-1})A) \leqslant 2P(A)$, T is non-singular with respect to P. For $A \epsilon \mathcal{A}$ and $A \subset [0, \frac{1}{2})$, it is clear that $\lim_{n \to \infty} P(T^{-n}A) = 0$, so that $\lim_{n \to \infty} P(T^{-n}A \cap B) = 0$ for every $B \epsilon \mathcal{A}$. Hence $\{T^{-n}A,\ n = 1, 2, \ldots\}$ is a stable sequence of events. If $A \epsilon \mathcal{A}$ and $A \subset [\frac{1}{2}, 1)$, then $T^{-n}A$ is a non-decreasing sequence of sets. Hence $\lim_{n \to \infty} P(T^{-n}A \cap B) = P\left(\bigcup_{n \to 0}^{\infty} T^{-n}A \cap B\right)$ for every $B \epsilon \mathcal{A}$. Therefore $\{T^{-n}A, n = 1, 2, \ldots\}$ is stable. It now follows that $\lim_{n \to \infty} P(T^{-n}A \cap B)$ exists for every $A, B \epsilon \mathcal{A}$. Thus T is a stable transformation.

But T is not measure preserving with respect to P; indeed, T is not measure preserving with respect to any finite measure equivalent to P. To prove this, it suffices to show that T is not conservative (Halmos, 1956, p. 84). Consider $B = [\frac{1}{4}, \frac{1}{2})$. Then $B, T^{-1}\ B, T^{-2}B, \ldots$ are mutually disjoint and $P(B) = \frac{1}{4}$. Hence T is not conservative.

This example shows that the assumption of conservativeness cannot be dropped from Theorem 3.5, if T is not invertible.

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