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On stable transformations

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ON STABLE TRANSFORMATIONS

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Summary. Let T be a measure preserving transformation of a probability space $(\Omega, \mathcal{A}, P)$ into itself.
We will say that T is a stable transformation if for every $A, B \in \mathcal{A}$, $\lim_{n\to\infty} P(T^{-n} A \cap B)$ exists.
Stable transformations are investigated in this article with the aid of Rényi's results on stable sequences of events. The concept of a stable transformation generalises that of a mixing transformation.

1. Introduction

Let $(\Omega, \mathcal{A}, P)$ be a probability space.
Let T be a measurable transformation (not necessarily one to one) of Ω into itself. Assume further that T is measure preserving, that is, $P(T^{-1} A) = P(A)$ for every $A \in \mathcal{A}$. Following Rényi [5], we will say that T is stable if for every $A \in \mathcal{A}$, $\{T^{-n} A, n = 1,2,\ldots\}$ is a stable sequence of sets, that is, for every $A, B \in \mathcal{A}$, $\lim_{n\to\infty} P(T^{-n} A \cap B)$ exists.
The purpose of this article is to study such transformations.

The concept of stability generalises that of mixing. It will be shown that a stable transformation T is mixing if and only if the σ-field of invariant sets is trivial. [A measurable set A is said to be invariant if $T^{-1} A = A$].

As the present investigation relies heavily on the results proved in [5], we will for the sake of completeness give a resumé of these in section 2. In section 3 the analogues of results for stable sequences of sets will be proved for stable transformations. Examples of stable transformations will be given in section 4.

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2. Resumé of results on stable sequences of events

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\{A_n, n = 1,2,\ldots\}$ be a sequence of events. We will say that $\{A_n\}$ is a stable sequence of events if for every $B \in \mathcal{A}$

$$\lim_{n\to\infty} P(A_n \cap B) = Q(B)$$

exists.

Theorem 2.1. If $\{A_n\}$ is a stable sequence of events and Q is as above, then Q is a measure on $(\Omega, \mathcal{A})$ and is absolutely continuous with respect to P.

Denote by α the Radon-Nikodym derivative of Q with respect to P. α is said to be the local density of the stable sequence of sets $\{A_n\}$.

A sequence of events $\{A_n, n = 1,2,\ldots\}$ is said to be mixing if there exists β, $0 \leq \beta \leq 1$ such that for every $B \in \mathcal{A}$

$$\lim_{n\to\infty} P(A_n \cap B) = \beta\, P(B).$$

β is called the density of the mixing sequence $\{A_n\}$.

Corollary 2.1. If $\{A_n\}$ is stable with local density α, then $\{A_n\}$ is mixing if and only if α is a constant almost surely.

Theorem 2.2. The sequence of events $\{A_n, n = 1,2,\ldots\}$ is stable if and only if

$$\lim_{n\to\infty} P(A_k \cap A_n) = Q_k, \quad k = 1,2,\ldots$$

exists. If, in addition, $P(A_k) > 0$, $k = 1,2,\ldots$, set $q_k = \frac{Q_k}{P(A_k)}$, $k = 1,2,\ldots$, and $q_0 = \lim_{n\to\infty} P(A_n)$. Then $\{A_n\}$ is mixing if and only if the q_k's $(k = 0,1,2,\ldots)$ are all equal.

The property of stability is preserved if the underlying probability measure P is replaced by a probability measure absolutely continuous with respect to P. More explicitly, we have

Theorem 2.3. Let $\{A_n, n = 1,2,\ldots\}$ be a stable sequence of events with local density α on the probability space $(\Omega, \mathcal{A}, P)$. Let P^* be a probability measure on $(\Omega, \mathcal{A})$, absolutely continuous with respect to P. Then $\{A_n\}$ is stable on $(\Omega, \mathcal{A}, P^*)$ with local density α.

3. Some general theorems on stable transformations

Let T be a stable transformation on $(\Omega,\mathcal{A},P)$, that is, T is measure preserving and $\lim_{n\to\infty} P(T^{-n} A\cap B)$ exists for every A, $B\in\mathcal{A}$. The limit is easy to find.

Theorem 3.1. Let T be a stable transformation. Then

$$\lim_{n\to\infty} P(T^{-n} A\cap B) = \int_B P(A/\mathcal{J})\, dP$$

for every A, $B\in\mathcal{A}$. Here $\mathcal{J}$ is the invariant σ-field and $P(A/\mathcal{J})$ is the conditional probability of A given $\mathcal{J}$.

Proof. By definition, the sequence $\{T^{-n} A,\ n = 1,2,\dots\}$, where $A\in\mathcal{A}$ is stable. Hence $\lim_{n\to\infty} P(T^{-n} A\cap B)$ exists for every $B\in\mathcal{A}$. But by the Individual Ergodic Theorem, we have:

$\frac{1}{n}\sum_{k=0}^{n-1} I_{T^{-k}A}$ converges almost surely to $P(A/\mathcal{J})$, where I_C is the indicator of the set C. Hence if $B\in\mathcal{A}$, $\frac{1}{n}\sum_{k=0}^{n-1} I_{T^{-k}A}\cdot I_B$ converges almost surely to $P(A/\mathcal{J})\cdot I_B$. Apply the Dominated Convergence Theorem. We get:

$\lim_n \frac{1}{n}\sum_{k=0}^{n-1} P(T^{-k} A\cap B) = \int_B P(A/\mathcal{J})\, dP$, that is, the sequence $\{P(T^{-n} A\cap B)\}$ is Cesaro-summable to $\int_B P(A/\mathcal{J})\, dP$. The result now follows from the remark made at the beginning of the proof.

Remark. Denote by α_A the local density of the stable sequence $\{T^{-n} A\}$, $A\in\mathcal{A}$. What we have proved then is that $\int_B \alpha_A\, dP = \int_B P(A/\mathcal{J})\, dP$ for every $B\in\mathcal{A}$. But α_A and $P(A/\mathcal{J})$ are $\mathcal{A}$-measurbale functions. Hence $\alpha_A = P(A/\mathcal{J})$ almost surely. Therefore the local density of $\{T^{-n} A\}$ is simply $P(A/\mathcal{J})$.

In order to check if a mesure preserving transformation T is stable, it is in fact sufficient to verify that $\lim_{n\to\infty} P(T^{-n} A\cap B)$ exists for $A = B\in\mathcal{A}$.

Theorem 3.2. A measure preserving transformation T is stable if and only if $\lim_{n\to\infty} P(T^{-n} A\cap A)$ exists for every $A\in\mathcal{A}$.

Proof. The "only if" part is trivial. Consider now the sequence $\{T^{-n} A,\ n = 1,2,\dots\}$, $A\in\mathcal{A}$. We want to show that $\{T^{-n} A\}$ is stable. Note that since T is measure preserving, $P(T^{-k} A\cap T^{-n} A) = P(T^{-k}(T^{-(n-k)}A\cap A)) =$

$= P(T^{-(n-k)} A \cap A)$, where $n > k$. But by the hypothesis, $\lim\limits_{n\to\infty} P(T^{-(n-k)} A \cap A)$ exists and so $\lim\limits_{n\to\infty} P(T^{-k} A \cap T^{-n} A)$ exists, $k = 1,2,\ldots$. Hence, by Theorem 2.2., $\{T^{-n} A\}$ is stable. This completes the "if" part of the proof.

A measure preserving transformation T is <u>mixing</u> if for every $A \in \mathcal{A}$ the sequence of events $\{T^{-n} A,\ n = 1,2,\ldots\}$ is mixing with density P(A), that is, if for every $A, B \in \mathcal{A}$

$$\lim_{n\to\infty} P(T^{-n} A \cap B) = P(A) \cdot P(B).$$

Clearly a mixing transformation is stable. When is the converse true?

<u>Corollary 3.1. In order that a stable transformation</u> T <u>be mixing, it is necessary and sufficient that</u> $\mathcal{J}$, <u>the</u> σ-<u>field of invariant sets, be trivial under</u> P.

<u>Proof</u>. Suppose that $\mathcal{J}$ is trivial under P, that is, if $A \in \mathcal{J}$ then $P(A) = 0$ or 1. By Theorem 3.1., since T is stable, we have

$$\lim_{n\to\infty} P(T^{-n} A \cap B) = \int_B P(A/\mathcal{J})\, dP$$

for every $A, B \in \mathcal{A}$. But as $\mathcal{J}$ is trivial, $P(A/\mathcal{J}) = P(A)$ almost surely for every $A \in \mathcal{A}$. Hence $\lim\limits_{n\to\infty} P(T^{-n} A \cap B) = P(A).P(B)$ for every $A, B \in \mathcal{A}$, so that T is mixing. Conversely, assume that T is mixing. Let $A \in \mathcal{J}$. Then $T^{-n} A = A$ for $n = 1,2,\ldots$. But $\{T^{-n} A,\ n = 1,2,\ldots\}$ is mixing. Hence for every $B \in \mathcal{A}$, $P(A \cap B) = P(A) \cdot P(B)$, that is, $P(A) = 0$ or 1. Therefore, $\mathcal{J}$ is trivial, which concludes the proof.

Let us now turn to the functional form of stability. Let $\mathcal{L}_2(\Omega, \mathcal{A}, P)$ be the class of complex-valued random variables f on $(\Omega, \mathcal{A}, P)$ such that $\int |f|^2\, dP < \infty$. Identify all functions in $\mathcal{L}_2$ which differ on a set of measure zero. Then $\mathcal{L}_2$ is a Hilbert space over the field of complex numbers with inner-product $(f,g) = \int f\, \overline{g}\, dP$ (here $\overline{x}$ is the complex-conjugate of x) and norm $||f|| = (\int |f|^2\, dP)^{\frac{1}{2}}$. If T is a measure preserving transformation of Ω into itself we can define a transformation $\mathcal{U}$ of $\mathcal{L}_2$ into itself as follows: $\mathcal{U}f = f \circ T$, $f \in \mathcal{L}_2$. Then $\mathcal{U}$ is an isometry, that is, $\mathcal{U}$ is a bounded linear transformation such that $||\mathcal{U}f|| = ||f||$ for every $f \in \mathcal{L}_2$ (see [2], page 14). Denote by $\mathcal{U}^n$ the n-th iterate of $\mathcal{U}$.

Call a function $f \in \mathcal{L}_2$ <u>invariant</u> if $\mathcal{U}f = f$. Denote by E_0 the projection on the closed subspace of invariant functions in $\mathcal{L}_2$. We can now characteris stability of T as follows.

Theorem 3.3. A measure preserving transformation T is stable if and only if $\lim_{n\to\infty} (U^n f, g) = (E_0 f, g)$ for every $f, g \in \mathcal{L}_2$ that is, U^n converges to E_0 in the weak operator topology.

Proof. The proof depends on the remark that the conditional expectation of f given $\mathcal{J}$ is almost surely equal to $E_0 f$. If f and g are indicators of sets F and G respectively, then the functional form simply reduces to the set-theoretic definition of stability. To go the other way, use a double approximation process as follows: let g be a fixed indicator in $\mathcal{L}_2$. The result holds for simple functions $f \in \mathcal{L}_2$ and so by $\mathcal{L}_2$- approximation holds for functions $f \in \mathcal{L}_2$. Now let f be a fixed function in $\mathcal{L}_2$ and a similar argument about g yields the result.

In the case of mixing, $\mathcal{J}$ is trivial so that all invariant functions in $\mathcal{L}_2$ are constants. Hence $E_0 f = (f,1)1$ for every $f \in \mathcal{L}_2$, where 1 stands for the function which is equal to one everywhere.

Corollary 3.2. A measure preserving transformation T is mixing if and only if $\lim_{n\to\infty} (U^n f, g) = ((f,1)1,g) = (f,1)(1,g)$ for every $f, g \in \mathcal{L}_2$.

We may add here that if T is stable, then U^n converges to E_0 in the strong operator topology only in a rather trivial and uninteresting case. In fact, U^n converges to E_0 in the strong operator topology if and only if every function in $\mathcal{L}_2$ is invariant. To prove this statement, note that since U^n converges weakly to E_0, U^n will converge strongly to E_0 if and only if $\lim_{n\to\infty} ||U^n f|| = ||E_0 f||$ for each $f \in \mathcal{L}_2$. But $||U^n f|| = ||f||$. Note also that for any $f \in \mathcal{L}_2$, $||f||^2 = ||E_0 f||^2 + ||f - E_0 f||^2$ by the Decomposition Theorem. Hence $||E_0 f|| = ||f||$ for each $f \in \mathcal{L}_2$ if and only if $E_0 f = f$ for each $f \in \mathcal{L}_2$. This completes the proof.

The property of stability is preserved if the underlying measure is replaced by a measure absolutely continuous with respect to it. Explicitly we have:

Theorem 3.4. Let T be a stable transformation on $(\Omega, \mathcal{A}, P)$. Let Q be a probability measure on $(\Omega, \mathcal{A})$ such that Q is absolutely continuous with respect to P. Assume further that Q is preserved by T. Then T is stable on $(\Omega, \mathcal{A}, Q)$ and for every $A \in \mathcal{A}$, $P(A/\mathcal{J}) = Q(A/\mathcal{J})$ almost surely $[Q]$.

Proof. Consider the sequence of sets $\{T^{-n} A, n = 1, 2, \ldots\}$, $A \in \mathcal{A}$. Since Q is absolutely continuous with respect to P, by Theorem 2.3., $\{T^{-n} A\}$ is stable with respect to Q. Hence T is stable on $(\Omega, \mathcal{A}, Q)$. Furthermore, by Theorem 2.3., $\lim_{n\to\infty} Q(T^{-n} A \cap B) = \int_B P(A/\mathcal{J})\, dQ$ for every A, $B \in \mathcal{A}$. Hence by Theorem 3.1 we have: $\int_B Q(A/\mathcal{J})\, dQ = \int_B P(A/\mathcal{J})\, dQ$ for every A, $B \in \mathcal{A}$. This proves the second assertion of the theorem.

Corollary 3.3. Let P and Q be probability measures on $(\Omega, \mathcal{A})$. Assume that T is stable for both P and Q. Then, if P = Q on $\mathcal{J}$, P = Q on $\mathcal{A}$.

Proof. Let $\mu(A) = \frac{1}{2} P(A) + \frac{1}{2} Q(A)$, $A \in \mathcal{A}$. It is easy to verify that T is stable for μ. Note that P, Q are absolutely continuous with respect to μ. Furthermore, $\mu = P = Q$ on $\mathcal{J}$. By Theorem 3.4, $\mu(A/\mathcal{J}) = P(A/\mathcal{J})$ almost surely $[P]$ for every $A \in \mathcal{A}$. Note that the exceptional set above is $\mathcal{J}$-measurable and so must have μ-measure zero as well. Again, as $P(A/\mathcal{J})$, $\mu(A/\mathcal{J})$ are $\mathcal{J}$-measurable functions, we have:

$$\mu(A) = \int \mu(A/\mathcal{J})\, d\mu^{\mathcal{J}} = \int P(A/\mathcal{J})\, dP^{\mathcal{J}} = P(A)$$

for every $A \in \mathcal{A}$. Here $\mu^{\mathcal{J}}$, $P^{\mathcal{J}}$ denote the restrictions of μ, P, respectively to $\mathcal{J}$. This proves the corollary.

Corollary 3.4. Let T be a mixing transformation on $(\Omega, \mathcal{A}, P)$. Let Q be a probability measure on $(\Omega, \mathcal{A})$. Assume that Q is absolutely continuous with respect to P and that it is preserved by T. Then P = Q.

Proof. Follows directly from Theorem 3.4.

Corollary 3.5. Let P and Q be probability measures on $(\Omega, \mathcal{A})$ for which T is a mixing transformation. Then either P = Q or P and Q are mutually singular.

Proof. Suppose $P \neq Q$. Then by Corollary 3.3, there exists a set $A \in \mathcal{J}$ such that $P(A) \neq Q(A)$. But, since T is mixing for both P and Q, either $P(A) = 1$ and $Q(A) = 0$ or $P(A) = 0$ and $Q(A) = 1$. In either case, P and Q are mutually singular.

4. Examples of stable transformations

A. Let T be the identity transformation on a probability space $(\Omega, \mathfrak{A}, P)$, that is, $T\omega = \omega$, $\omega \in \Omega$. Then $\mathcal{J} = \mathfrak{A}$ and T is stable. If $\mathfrak{A}$ is non-trivial, we get an example of a stable transformation that is not mixing.

B. Let $(\Omega, \mathfrak{A})$ be a countably infinite product of a measurable space $(\Omega_o, \mathfrak{A}_o)$. Denote by ω_n $(n = 1, 2, \ldots)$ the n-th coordinate of a point ω in Ω. We will use the following notation for finite dimensional rectangles: $C(E_1^{(i_1)}, \ldots, E_n^{(i_n)})$ is the set of all ω such that $\omega_{i_k} \in E_k$, $k = 1, \ldots, n$. If $i_k = k$, $k = 1, \ldots, n$, we will write $C(E_1, \ldots, E_n)$. Let T be the shift operation on Ω, that is, $T\omega = \omega^1$, where $\omega^1_n = \omega_{n+1}$, $n = 1, 2, \ldots$. Consider a symmetric probability measure P on $(\Omega, \mathfrak{A})$, that is

$$P(C(E_1^{(i_1)}, \ldots, E_n^{(i_n)})) = P(C(E_1^{(j_1)}, \ldots, E_n^{(j_n)}))$$

for all $n = 1, 2, \ldots$, all $E_1, E_2, \ldots, E_n \in \mathfrak{A}_o$ and all sequences of positive integers $i_1, \ldots, i_n$ and $j_1, \ldots, j_n$ (i's all distinct and j's all distinct).

Then T is a stable transformation on $(\Omega, \mathfrak{A}, P)$. To see this, first note that T is measure preserving. Now let B be a measurable $\{1, \ldots, m\}$-cylinder, that is $B = A \times \Omega_o \times \Omega_o \times \ldots$ where A is a measurable subset of $\Omega_o \times \Omega_o \times \ldots \times \Omega_o$ (m times). Consider the sequence of sets $\{B_k, k = 1, 2, \ldots\}$ where $B_k = T^{-k} B$, $k = 1, 2, \ldots$. It is clear that B_k is a $\{k+1, \ldots, k+m\}$-cylinder with base B. Hence, as P is a symmetric measure, for n large $P(B_k \cap B_n) = P(C)$, where C is the $\{1, \ldots, 2m\}$-cylinder $B \times B \times \Omega_o \times \Omega_o \times \ldots$ Hence $\lim_{n\to\infty} P(B_k \cap B_n)$ exists for $k = 1, 2, \ldots$. Therefore, by Theorem 2.2, $T^{-k} B$ is stable. But for every set $A \in \mathfrak{A}$ and $\varepsilon > 0$, there exists a $\{1, \ldots, m\}$ cylinder B (for some m) such that $P(A \Delta B) < \varepsilon$. It is easy to see that the stability of the sequence $\{T^{-n} A\}$ follows from that of $\{T^{-n} B\}$. This proves that T is a stable transformation.

In particular, let P be a product measure with identical components. Then it is well known that T is mixing (see [6], page 110). Conversely, assume that T is mixing for a symmetric measure P. Let $A = C(E_1, \ldots, E_m)$ be a measurable finite-dimensional rectangle. It is easy to see that

$$\lim_{n\to\infty} P(T^{-k} A \cap T^{-n} A) = P(C(E_1,\ldots,E_m, E_1,\ldots,E_m)), \; k = 1,2,\ldots \; .$$

The limit is independent of k. But the sequence $\{T^{-n} A\}$ is mixing. Hence, by Theorem 2.2, we must have

$$P(C(E_1,\ldots,E_m, E_1,\ldots,E_m)) = P^2(C(E_1,\ldots,E_m)).$$

As T is mixing, this last relation is true for all measurable finite-dimensional rectangles. Hence, by Theorems 5.2 and 5.3 in [3] (see pages 477-478), P must be a product measure with identical components. Hence we have

<u>Theorem 4.1. Let</u> P <u>be a symmetric probability measure on</u> $(\Omega,\mathcal{A})$. <u>Then</u> T <u>is stable and</u> T <u>is mixing if and only if</u> P <u>is a product measure with identical components</u>.

C. Let $\{x_n, n=0,1,\ldots\}$ be a stationary aperiodic Markov chain with countable state space I. Elements of I will be denoted by i with or without subscripts. Assume that the Markov chain is defined on the appropriate (unilateral) sequence space $(\Omega,\mathcal{A})$ and let T be the shift operator on $(\Omega,\mathcal{A})$. If P is the relevant probability measure on $(\Omega,\mathcal{A})$, T is stable on $(\Omega,\mathcal{A},P)$.

To prove this, let us note that it is sufficient to demonstrate stability of sequences of events $\{T^{-n} A, n = 1,2,\ldots\}$, where A is a finite-dimensional rectangle of the form $(x_o = i_o,\ldots,x_m = i_m)$, the i's being ergodic states belonging to the same class. We have for large n

$$P(T^{-k} A \cap T^{-n} A) = p_{i_o} p_{i_o i_1} \cdots p_{i_{m-1} i_m} p_{i_m i_o}^{(n-m-k)} p_{i_o i_1} \cdots p_{i_{m-1} i_m}$$

where p_i denotes the initial distribution, p_{ij} the one-step transition probability and $p_{ij}^{(n)}$ the n-step transition probability.
Clearly since $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_{ij}$ for j ergodic,

$$\lim_{n\to\infty} P(T^{-k} A \cap T^{-n} A) = p_{i_o} p_{i_o i_1} \cdots p_{i_{m-1} i_m} \pi_{i_m i_o} p_{i_o i_1} \cdots p_{i_{m-1} i_m}, \quad k = 1,2,\ldots \; .$$

Hence by Theorem 2.2, $\{T^{-n} A\}$ is stable. This proves the assertion.

D. We conclude with an example of a measure preserving transformation which is not stable.

Let $\Omega = [0,1]$, $\mathfrak{A}$ the σ-field of Borel subsets of Ω, P Lebesgue measure on $\mathfrak{A}$. Let T be an invertible, both ways measurable, measure preserving transformation of Ω onto Ω, which has strict period m ($m > 1$) at almost all [P] points of Ω.

According to a result of Halmos (see [2], page 70), there exists a set $E \in \mathfrak{A}$ such that $P(E) = 1/m$ and $E, T^{-1}E, \ldots, T^{-(m-1)}E$ are pairwise disjoint. It follows that $\limsup_{n\to\infty} P(T^{-n} E \cap E) = 1/m$ and $\liminf_{n\to\infty} P(T^{-n} E \cap E) = 0$, so that the sequence $\{T^{-n} E, n = 1,2,\ldots\}$ is not stable. Hence T is not stable.

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Résumé. Soit T une transformation, conservant la mesure, d'un espace de probabilité $(\Omega,\mathfrak{A},P)$ dans lui-même. On dira que T est stable si, pour tout $A, B \in \mathfrak{A}$, il existe $\lim_{n\to\infty} P(T^{-n} A \cap B)$. L'investigation des transformations stables est fondée sur des résultats de Rényi concernant les suites stables d'événements. La notion de transformation stable est une généralisation de celle de transformation melangée.