EXACT POWER OF SOME RANK TESTS

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1. Summary.

The objects of this investigation are :

1/ To derive the exact power functions of Wilcoxon's, van der Waerden's and Terry's two-sample tests for small sample values against parametric shift alternatives of exponential and rectangular populations.

2/ To compare the power functions of these tests.

This is done for sample sizes of $m = n = 3, 4, 5, 6$ and various significance levels.

2. Introduction and literature.

Assume there are given two independent random samples

$$\mathit{x}_1,\ \mathit{x}_2, \ldots,\ \mathit{x}_m \quad \text{and} \quad \mathit{y}_1,\ \mathit{y}_2, \ldots,\ \mathit{y}_n{}^{(2)}$$

from two populations with continuous cumulative distribution functions $F(x) = P[\mathit{x} < x]$ and $G(y) = P[\mathit{y} < y]$ respectively. We wish to test the null-hypothesis :

$$H_0 : F(x) \equiv G(x) \tag{1}$$

against the alternative hypothesis :

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(1) Report S 331 of the Statistical Department of the Mathematical Centre, Amsterdam.
Paper presented at the I. M. S. conference at Berne (14 th - 18 th September 1964).

(2) Random variables will-be distinguished from numbers (e. g. from values they assume in an experiment) by italicizing their symbols.

$$H_1 : F(x) \not\equiv G(x) \tag{2}$$

For this two-sample problem various distribution-free tests are proposed. In this paper we shall consider three of these, namely :

1/ Two-sample test of Wilcoxon (Mann-Whitney),

2/ " " " " van der Waerden (X-test),

3 " " " " Terry (Fisher-Yates, Hoeffding).

We shall only be interested in the power of each of these three tests. For shifted Exponential and Rectangular alternatives we shall evaluate their power functions :

$$\beta_W(\lambda ; \alpha), \quad \beta_X(\lambda ; \alpha) \quad \text{and} \quad \beta_T(\lambda ; \alpha) \tag{3}$$

respectively, with shift λ and significance level α, for small sample sizes $(m = n = 3, 4, 5$ and $6)$.

In Savage [18] it is indicated that for the two-sample problem with such alternatives as slippage, there do not exist optimum non-parametric tests. Dwass [6] has studied the locally best rank order statistic for testing $H_0 : \theta = 0$ against $H_1 : \theta$ "close" to zero, if the samples are drawn from populations with density functions $f_1(x, \theta)$ and $f_2(y, \theta)$ (if $\theta = 0$ both density functions are identical). Under certain regularity conditions it is possible to determine the large sample power.

Different investigations of the efficiencies and the powers of the three tests, mentioned above, and comparisons with Student's two-sample test have already been made. To get a rough idea of the relative powers we shall give a brief survey of the literature on this subject. We are mainly interested in the powers for small samples, so our references, especially those to asymptotic results, will be far from complete.

Pitman has proved in [17] that the Pitman asymptotic relative efficiency of Wilcoxon's test against Student's test for the shifted Normal distribution is equal to $\frac{3}{\pi}$. In [10], Hodges and Lehmann proved that for all distributions the efficiency is greater than or equal to 0.864 for shift alternatives. Regarding the power efficiencies of Wilcoxon's test for discrete populations relative to the most powerful tests available for such distributions there is the article of Chanda [1]. In [34], Witting derived a generalized Pitman efficiency (correction terms for finite sample sizes) for the Wilcoxon test relative to the Standard Normal and t-test for nearby alternatives. Efficiency values are given in the case of Normal and Rectangular distributions for various sample sizes.

Van der Vaart [24] found in 1950 that under Student's conditions for one-and two-sided testing with $m + n \leq 5$ and $m + n \leq 6$, respectively, the difference in power of Wilcoxon's and Student's tests is rather small in the neighbourhood of the null-hypothesis and an indication that for large sample sizes the difference is not great either : for two-sided testing he shows that $\beta_W''(o)/\beta_{St}''(o) \approx \frac{3}{\pi}$, where β_{St}'' is the second derivative of the power function of Student's test. In 1951 van Dantzig [3] obtained a lower bound for the power of Wilcoxon's test against alternatives of the form $P[x > y] >$ or $< \frac{1}{2}$. Using Pitman's formula for the variance of a distribution - free test of two-samples, due to Wilcoxon, Sundrum [20] determined the power of the test against Normal and Rectangular shift alternatives for large samples and compared this test with certain distributional tests.

Numerical calculations with regard to the power of Wilcoxon's test for Normal alternatives and for $m = n = 3, 4, 5$ and significance levels $\frac{1}{10}, \frac{1}{35}, \frac{1}{126}$ were made by Dixon [4] in 1954.

Experimental comparisons between Student's and Wilcoxon's tests are made by Dixon and Teichroew [5] (see also [21]) and Hemelrijk [9]. They investigated for $m=n=10$ the powers of both tests for Normal alternatives. The results in [5] also cover other distribution - free tests and different sample sizes. Hemelrijk made similar investigations for the Exponential distribution. The alternative hypotheses were not chosen in the form of shift but of multiplication factors $\left(G(x) \equiv F\left(\frac{x}{k}\right)\right)$.

Thompson [23] has constructed 90 % confidence limits of the power of the Wilcoxon test for Normall shift alternatives with the aid of Monte Carlo methods. This is done for $m = n = 5$ and 10 and rejection limits 35.5 and 127.5 respectively.

In 1953 Lehmann [12] has proposed to use the so-called Lehmann alternatives of the form $G(x) \equiv F^k(x)$. In [12] one can find some numerical results for the power of Wilcoxon's test for equal sample sizes of 4 and 6 and significance level 0.1 ($k = 2$ and 3). Eisenberg [7] has computed exact power values of Wilcoxon's test against Lehmann's alternatives $\left(k = \frac{1}{2}, \ldots, 4\right)$ for sample sizes $2 \leq m \leq n \leq 10$ and several significance levels.

The tests of van der Waerden and Terry are under Student's conditions asymptotically equivalent to the test of Student.

Some numerical calculations concerning van der Waerden's test are made by van der Waerden [26] for $m = n = 3$ and a Normal alternative. For $m = 2$ and $n \longrightarrow \infty$ he makes a comparison between Student's and van der Waerden's tests for a shifted Normal population. Van der Waerden [27] and [28] investigated the powers of Wilcoxon's and van der Waerden's tests when two samples with sizes 4 and 6 are drawn from populations with Rectangular distribution functions with ranges $(0, 1)$ and $(0, 1 + \mu)$ respectively $(\mu \geq 0)$. The powers of both tests tend to 1 if μ tends to infinity. He proved that the power of Student's test does not tend to 1 if μ tends to infinity. In a seperate paper [29] he gives an example where the power of Student's test can be made small, whereas the power of van der Waerden's test does not change.

Noether [16] has given some examples where van der Waerden's and Terry's tests do not always lead to the same decision. However, when H_0 is true, he proved that the correlation-coefficient between the two test statistics tends to 1 as $m+n$ increases, and the two tests are equivalent. The asymptotic equivalence of van der Waerden's and Terry's tests was established for a wide class of cases by Chernoff and Savage [2]. They proved that the test for translation based on Terry's statistic is at least as efficient as the t-test, and Pitman's efficiency is 1 only if the underlying distribution is Normal.

In 1963 Jean D. Gibbons [8] investigated the powers of Wilcoxon's and Terry's test for sample sizes $m, n \leq 6$ and alternatives of the form : $F(x) = 1 - (1 - H(x))^k$ and $G(x) = H(x)^k$ $(k = 2, 3, 4)$ where $H(x)$ is a continuous, but arbitrary, distribution function, and significance levels 0.01, 0.05 and 0.10. She also gives some numerical results against Normal alternatives differing only in location.

Hodges and Lehmann [11] give the Pitman efficiencies of Wilcoxon's test against the Normal Scores test (The two asymptotically equivalent versions of this test are van der Waerden's and Terry's test) for various underlying distributions against shift alternatives. In particular they found for the Exponential and Rectangular distributions that the efficiency is zero. They stated that for bell-shaped densities the efficiency of Wilcoxon's test relative to the Normal Scores test seems to increase as the tails of the underlying distribution grow heavier. This result was made precise by a theorem due to van Zwet [35].

3. Determination of the power in general.

Let $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_n$ be two independent random samples from populations with continuous cumulative distribution functions $F(x) = P[x < x]$ and $G(y) = P[y < y]$ respectively. Null and alternative hypotheses will be of the form :

$$H_0 : F(x) \equiv G(x) \qquad \text{and} \qquad H_1 : F(x) \equiv G(x-\lambda) \tag{4}$$

respectively, where λ is the shift of the distribution function of x.

Let $x_{[i]}$ denote the i th order statistic of the x-sample and $y_{[j]}$ be the j th order statistic of the y-sample (i = 1, 2, ..., m; j = 1, 2, ..., n). Furthermore let $z = (z_1, z_2, \ldots, z_{m+n})$ denote the order statistics of the combined sample $(x_1, x_2, \ldots, y_n)$. We define the random vector $t = (t_1, t_2, \ldots, t_{m+n})$ where

$$t_i = \begin{cases} 0 & \text{if } z_i \text{ belongs to the x-sample} \\ 1 & \text{if } z_i \text{ " " " y- "} \end{cases}$$

Under the null-hypothesis every value of the vector t is equally likely, under the alternative hypothesis they are not.

In order to compute the power functions of the tests of Wilcoxon, van der Waerden and Terry for shift alternatives, one has to compute the probabilities under H_1 of orderings z lying in the critical regions of each test. It is therefore sufficient to compute integrals of the form :

$$m!\; n! \underset{-\infty<z_1<z_2<\ldots<z_{m+n}<\infty}{\iint \ldots \int} \prod_{i=1}^{m+n} d\, H(z_i\; ;\; t_i) \tag{5}$$

where

$$H(z_i\; ;\; t_i) = \begin{cases} F(z_i) & \text{if } t_i = 1 \\ G(z_i) & \text{if } t_i = 0 \end{cases} \tag{6}$$

The sums of the probabilities under H_1 of those orderings which lie in each test's rejection region are the power functions of the three tests. If the underlying distribution function is exponential or homogeneous, it is possible to evaluate the integrals of form (5) and hence, also the power functions.

4. Some remarks on the probabilities of orderings.

a. EXPONENTIAL DISTRIBUTION -

Let :

$$F(x) = 1 - e^{-(x-\lambda)} \qquad \lambda \leq x < \infty \qquad (0 \leq \lambda < \infty) \tag{7}$$
$$= 0 \qquad x < \lambda$$

and

$$G(y) = 1 - e^{-y} \qquad 0 \leq y < \infty \tag{8}$$
$$= 0 \qquad y < 0$$

If x is a exponentially distributed random variable then we know that

$$P[x < x + c \mid x \geq c] = P[x < x] \tag{9}$$

where c is a positive constant. Hence it is easy to see that we can draw the following conclusion :

CONCLUSION 1 : Random vectors $t = (t_1, t_2, \ldots, t_{m+n})$ *with* $t_1 = t_2 = \ldots = t_k = 0$ *and* $t_{k+1} = 1$ *have equal probabilities* $(m > 1 ; k = 0, 1, \ldots, n-1)$.

Proof : $$P[t = t] = m!\ n! \int\int_{-\infty<z_1<z_2<\ldots<z_{m+n}<\infty} \ldots \int \prod_{i=1}^{m+n} dH(z_i ; t_i) =$$

$$= m!\ n! \int\int_{\substack{0<y_{[1]}<\ldots<y_{[k]}<x_{[1]}<z_{k+2}<\ldots<z_{m+n}<\infty \\ \lambda<x_{[1]}}} \ldots \int \prod_{i=1}^{k} dG(y_{[i]})\, dF(x_{[1]}) \prod_{j=k+2}^{m+n} dH(z_j ; t_j) =$$

$$m!\, n!\, e^{(m-1)\lambda} \int\int_{\substack{0<y_{[1]}<\ldots<y_{[k]}<x_{[1]}<z_{k+2}<\ldots<z_{m+n}<\infty \\ -\lambda<x_{[1]}}} \ldots \int \prod_{i=1}^{k} dG(y_{[i]})\, dF(x_{[1]})\, e^{-(\sum_2^m x_{[i]} + \sum_{k+1}^n y_{[j]})} \prod_2^m dx_{[i]} \prod_{k+1}^n dy_{[j]}$$

The form $e^{-(\sum_2^m x_{[i]} + \sum_{k+1}^n y_{[j]})}$ is a symmetric function of $x_{[i]}$ and $y_{[j]}$ $(i = 2, 3, \ldots, m;\ j = k+1,\ k+2, \ldots, n)$, so the order of the $x_{[i]}$'s and $y_{[j]}$'s in the sequence $z_{k+2} < z_{k+3} < \ldots < z_{m+n}$ does not matter, which proves the assertion. It is clear that this conclusion gives a substantial reduction of the computations.

b. HOMOGENEOUS DISTRIBUTION -

Let

$$\begin{aligned} F(x) &= 0 & x &< \lambda & (0 \le \lambda \le 1) \qquad (10)\\ &= x - \lambda & \lambda \le x &< \lambda + 1 & \\ &= 1 & \lambda + 1 &\le x & \end{aligned}$$

and

$$\begin{aligned} G(y) &= 0 & y &< 0 & \\ &= y & 0 \le y &< 1 & \qquad (11)\\ &= 1 & 1 &\le y & \end{aligned}$$

From the fact that the density functions are constants, it is easy to see that we can draw the following two conclusions :

CONCLUSION 2 : Random vectors $t = (t_1, t_2, \ldots, t_{m+n})$ *with* $t_1 = t_2 = \ldots = t_k = 0$ *and* $t_{k+1} = 1$,

$$t_{m+n-l} = 0 \quad \text{and} \quad t_{m+n-l+1} = t_{m+n-l+2} = \ldots = t_{m+n} = 1$$

have equal probabilities

$$(k = 0, 1, \ldots, n-2\ ;\ l = 0, 1, \ldots, m-2).$$

Proof :

$$P[t = t] = m!\, n! \underset{-\infty < z_1 < z_2 < \ldots < z_{m+n} < \infty}{\iint \cdots \int} \prod_{i=1}^{m+n} dH(z_i\ ;\ t_i) = m!\, n! \iint_G \cdots \int \prod_{i=1}^{m} dx_{[i]} \prod_{j=1}^{n} dy_{[j]}$$

where the domain of integration G is defined by :

$$G = \{ 0 < y_{[1]} < \ldots < y_{[k]} < x_{[1]} < z_{k+2} < \ldots < z_{m+n-l-1} < y_{[n]} < x_{[m-l+1]} < \ldots < x_{[m]} < \lambda + 1\ ;\ \lambda < x_{[1]},\ y_{[n]} < 1 \}.$$

The integrand is 1, so the order of the $x_{[i]}$'s and $y_{[j]}$'s $(i = 2, 3, \ldots,$

$m - l$; $j = k+1, k+2, \ldots, n-1$) in the sequence $z_{k+2} < z_{k+3} < \ldots < z_{m+n-l-1}$ does not matter, which proves the assertion.

For reasons of symmetry the next conclusion is clear.

CONCLUSION 3 : If two random vectors $t_i = (t_{i1}, t_{i2}, \ldots, t_{i,m+n})$ $(i=1,2)$ *are known to satisfy*

$$t_{i1} = t_{i2} = \ldots = t_{i,r(i)} = 0 \quad \text{and} \quad t_{i,r(i)+1} = 1$$

$$t_{i,m+n-s(i)} = 0 \quad \textit{and} \quad t_{i,m+n-s(i)+1} = t_{i,m+n-s(i)+2} = \ldots = t_{i,m+n} = 1$$

where $r(1) = s(2) = k$ *and* $r(2) = s(1) = l$, *then they are equally likely.*

(If $m = n$: $k = 0, 1, \ldots, m-2$; $l = 0, 1, \ldots, n-2$
" $m > n$: $k = 0, 1, \ldots, n-1$; $l = 0, 1, \ldots, n-2$
" $m < n$: $k = 0, 1, \ldots, m-2$; $l = 0, 1, \ldots, m-1$).

5. Results.

We have determined the power functions for one and two sided testing with exact significance levels α in the neigbourhood of 0.01 and 0.05 and sample sizes of $m = n = 3, 4, 5$ and 6. It turned out that only for $m = n = 6$ in the case of one-sided testing with significance level in the neighbourhood of 0.05 the power function of Wilcoxon differs from that of van der Waerden and Terry, whose power functions are the same in this case too. To make a comparison between the tests possible we determined the power function of Wilcoxon at exact significance level of $\alpha \doteq \frac{43}{924}$ (≈ 0.0465) and the power function of van der Waerden (and Terry) at exact significance levels of $\alpha = \frac{42}{924}$ (≈ 0.0455) and of $\alpha = \frac{48}{924}$ (≈ 0.0519), while randomization was used to get the same significance level as for Wilcoxon's test. Sometimes we denote the tests of Wilcoxon, van der Waerden and Terry by W, X and T respectively. The common power function of van der Waerden's and Terry's test are denoted by $\beta_{X,T}$.

a. EXPONENTIAL DISTRIBUTION -

The power functions are given as functions of λ, the shift parameter. $(0 \leq \lambda < \infty)$.

Table 1

Power functions ; m = n = 3.

	α	$\beta_W(\lambda;\alpha) = \beta_{X,T}(\lambda;\alpha)$
One-sided	$\frac{1}{100}$	0
	$\frac{1}{20}$	$(1-e^{-\lambda})^3 + \frac{3}{4}(1-e^{-\lambda})^2 e^{-\lambda} + \frac{3}{10}(1-e^{-\lambda})e^{-2\lambda} + \frac{1}{20}e^{-3\lambda}$
Two-sided	$\frac{1}{100}$	0
	$\frac{1}{20}$	0

Table 2

Power functions ; m = n = 4.

	α	$\beta_W(\lambda;\alpha) = \beta_{X,T}(\lambda;\alpha)$
One-sided	$\frac{1}{100}$	0
	$\frac{1}{35}$	$(1-e^{-\lambda})^4 + \frac{8}{5}(1-e^{-\lambda})^3 e^{-\lambda} + \frac{4}{5}(1-e^{-\lambda})^2 e^{-2\lambda} + \frac{8}{35}(1-e^{-\lambda})e^{-3\lambda} + \frac{1}{35}e^{-4\lambda}$
Two-sided	$\frac{1}{100}$	0
	$\frac{1}{35}$	$(1-e^{-\lambda})^4 + \frac{4}{5}(1-e^{-\lambda})^3 e^{-\lambda} + \frac{2}{5}(1-e^{-\lambda})^2 e^{-2\lambda} + \frac{4}{35}(1-e^{-\lambda})e^{-3\lambda} + \frac{1}{35}e^{-4\lambda}$

Table 3

Power functions ; m = n = 5.

	α	$\beta_W(\lambda;\alpha) = \beta_{X,T}(\lambda;\alpha)$
One-sided	$\frac{1}{126}$	$(1-e^{-\lambda})^5 + \frac{5}{3}(1-e^{-\lambda})^4 e^{-\lambda} + \frac{20}{21}(1-e^{-\lambda})^3 e^{-2\lambda} + \frac{5}{14}(1-e^{-\lambda})^2 e^{-3\lambda} + \frac{5}{63}(1-e^{-\lambda}) e^{-4\lambda} + \frac{1}{126} e^{-5\lambda}$
	$\frac{1}{21}$	$(1-e^{-\lambda})^5 + \frac{25}{6}(1-e^{-\lambda})^4 e^{-\lambda} + \frac{30}{7}(1-e^{-\lambda})^3 e^{-2\lambda} + \frac{55}{28}(1-e^{-\lambda})^2 e^{-3\lambda} + \frac{10}{21}(1-e^{-\lambda}) e^{-4\lambda} + \frac{1}{21} e^{-5\lambda}$
Two-sided	$\frac{1}{126}$	$(1-e^{-\lambda})^5 + \frac{5}{6}(1-e^{-\lambda})^4 e^{-\lambda} + \frac{10}{21}(1-e^{-\lambda})^3 e^{-2\lambda} + \frac{5}{28}(1-e^{-\lambda})^2 e^{-3\lambda} + \frac{5}{126}(1-e^{-\lambda}) e^{-4\lambda} + \frac{1}{126} e^{-5\lambda}$
	$\frac{2}{63}$	$(1-e^{-\lambda})^5 + \frac{5}{2}(1-e^{-\lambda})^4 e^{-\lambda} + \frac{40}{21}(1-e^{-\lambda})^3 e^{-2\lambda} + \frac{5}{7}(1-e^{-\lambda})^2 e^{-3\lambda} + \frac{10}{63}(1-e^{-\lambda}) e^{-4\lambda} + \frac{2}{63} e^{-5\lambda}$

Table 4

Power functions ; m = n = 6.

	α	$\beta_W(\lambda;\alpha) = \beta_{X,T}(\lambda;\alpha)$
One-sided	$\frac{1}{132}$	$(1-e^{-\lambda})^6 + \frac{24}{7}(1-e^{-\lambda})^5 e^{-\lambda} + \frac{45}{14}(1-e^{-\lambda})^4 e^{-2\lambda} + \frac{5}{3}(1-e^{-\lambda})^3 e^{-3\lambda} + \frac{1}{2}(1-e^{-\lambda})^2 e^{-4\lambda} + \frac{1}{11}(1-e^{-\lambda}) e^{-5\lambda} + \frac{1}{132} e^{-6\lambda}$
	$\frac{43}{924}$	See Table 5.
Two-sided	$\frac{2}{231}$	$(1-e^{-\lambda})^6 + \frac{18}{7}(1-e^{-\lambda})^5 e^{-\lambda} + \frac{15}{7}(1-e^{-\lambda})^4 e^{-2\lambda} + \frac{20}{21}(1-e^{-\lambda})^3 e^{-3\lambda} + \frac{2}{7}(1-e^{-\lambda})^2 e^{-4\lambda} + \frac{4}{77}(1-e^{-\lambda}) e^{-5\lambda} + \frac{2}{231} e^{-6\lambda}$
	$\frac{19}{462}$	$(1-e^{-\lambda})^6 + \frac{36}{7}(1-e^{-\lambda})^5 e^{-\lambda} + \frac{45}{7}(1-e^{-\lambda})^4 e^{-2\lambda} + \frac{80}{21}(1-e^{-\lambda})^3 e^{-3\lambda} + \frac{9}{7}(1-e^{-\lambda})^2 e^{-4\lambda} + \frac{19}{77}(1-e^{-\lambda}) e^{-5\lambda} + \frac{19}{462} e^{-6\lambda}$

Table 5

Power functions ; m = n = 6, one sided testing.

Test	α	Power function
W	$\frac{43}{924}$	$(1-e^{-\lambda})^6+6(1-e^{-\lambda})^5e^{-\lambda}+\frac{285}{28}(1-e^{-\lambda})^4e^{-2\lambda}+\frac{50}{7}(1-e^{-\lambda})^3e^{-3\lambda}+$ $+\frac{37}{14}(1-e^{-\lambda})^2e^{-4\lambda}+\frac{41}{77}(1-e^{-\lambda})e^{-5\lambda}+\frac{43}{924}e^{-6\lambda}$
X and T	$\frac{42}{924}$	$(1-e^{-\lambda})^6+6(1-e^{-\lambda})^5e^{-\lambda}+\frac{135}{14}(1-e^{-\lambda})^4e^{-2\lambda}+\frac{50}{7}(1-e^{-\lambda})^3e^{-3\lambda}+$ $+\frac{37}{14}(1-e^{-\lambda})^2e^{-4\lambda}+\frac{41}{77}(1-e^{-\lambda})e^{-5\lambda}+\frac{1}{22}e^{-6\lambda}$
"	$\frac{48}{924}$	$(1-e^{-\lambda})^6+6(1-e^{-\lambda})^5e^{-\lambda}+\frac{285}{28}(1-e^{-\lambda})^4e^{-2\lambda}+\frac{55}{7}(1-e^{-\lambda})^3e^{-3\lambda}+$ $+\frac{43}{14}(1-e^{-\lambda})^2e^{-4\lambda}+\frac{47}{77}(1-e^{-\lambda})e^{-5\lambda}+\frac{4}{77}e^{-6\lambda}$
"	rand. $\frac{43}{924}$	$(1-e^{-\lambda})^6+6(1-e^{-\lambda})^5e^{-\lambda}+\frac{545}{56}(1-e^{-\lambda})^4e^{-2\lambda}+\frac{305}{42}(1-e^{-\lambda})^3e^{-3\lambda}+$ $+\frac{19}{7}(1-e^{-\lambda})^2e^{-4\lambda}+\frac{6}{11}(1-e^{-\lambda})e^{-5\lambda}+\frac{43}{924}e^{-6\lambda}$

The following table gives some values of the power functions of Wilcoxon's, van der Waerden's and Terry's test, if we use randomization to get the same significance level.

Table 6

Some power values.
One-sided testing, m = n = 6.
(randomization is used)

	λ									
	0.0	0.1	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5
$\beta_W(\lambda;\frac{43}{924})$	.0465	.0776	.120	.234	.368	.500	.618	.825	.927	.971
$\beta_{X,T}(\lambda;\frac{43}{924})$	.0465	.0788	.122	.235	.365	.495	.610	.817	.922	.969

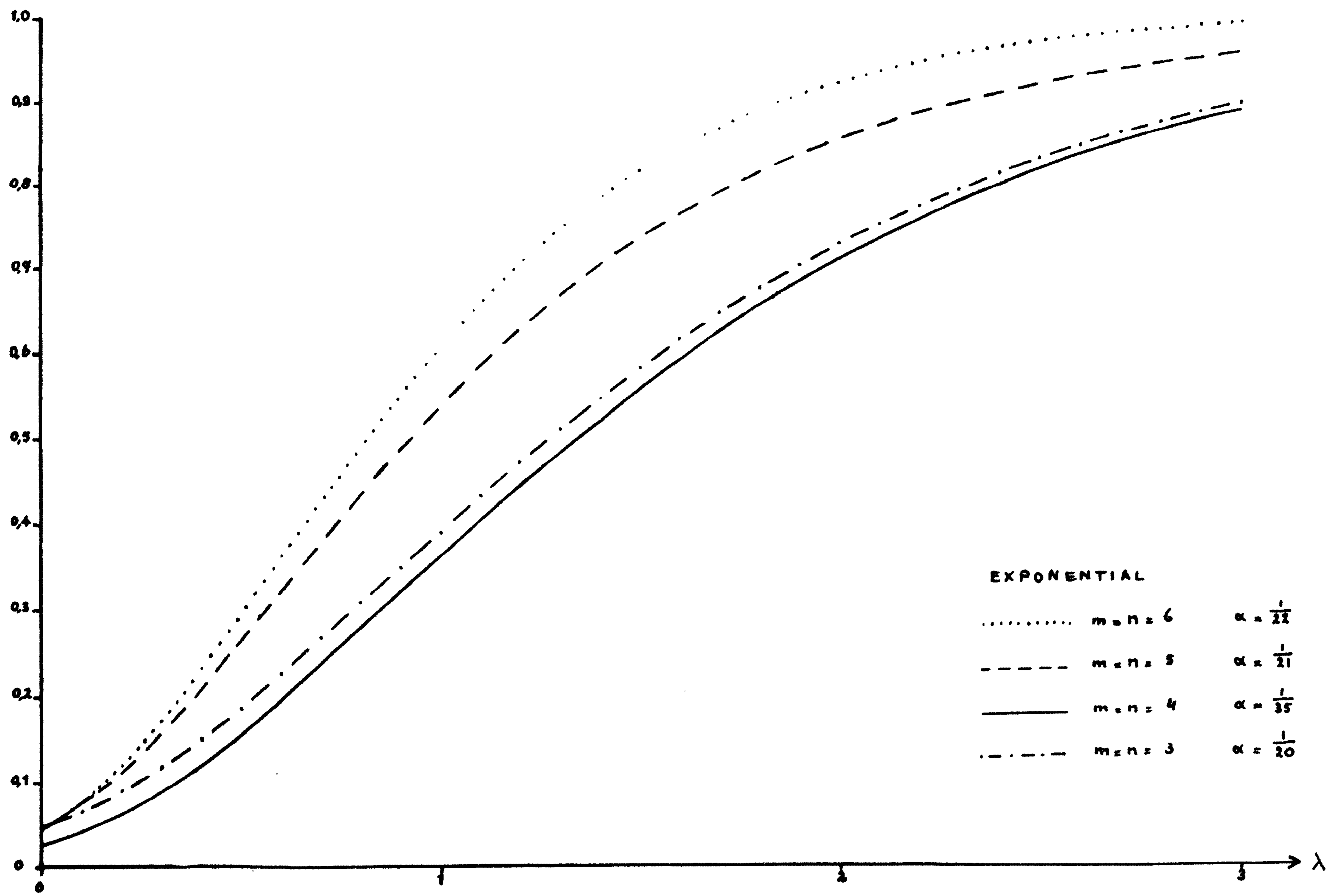

Figure 1. Power functions of van der WAERDEN's (and TERRY's) test. One-sided testing.

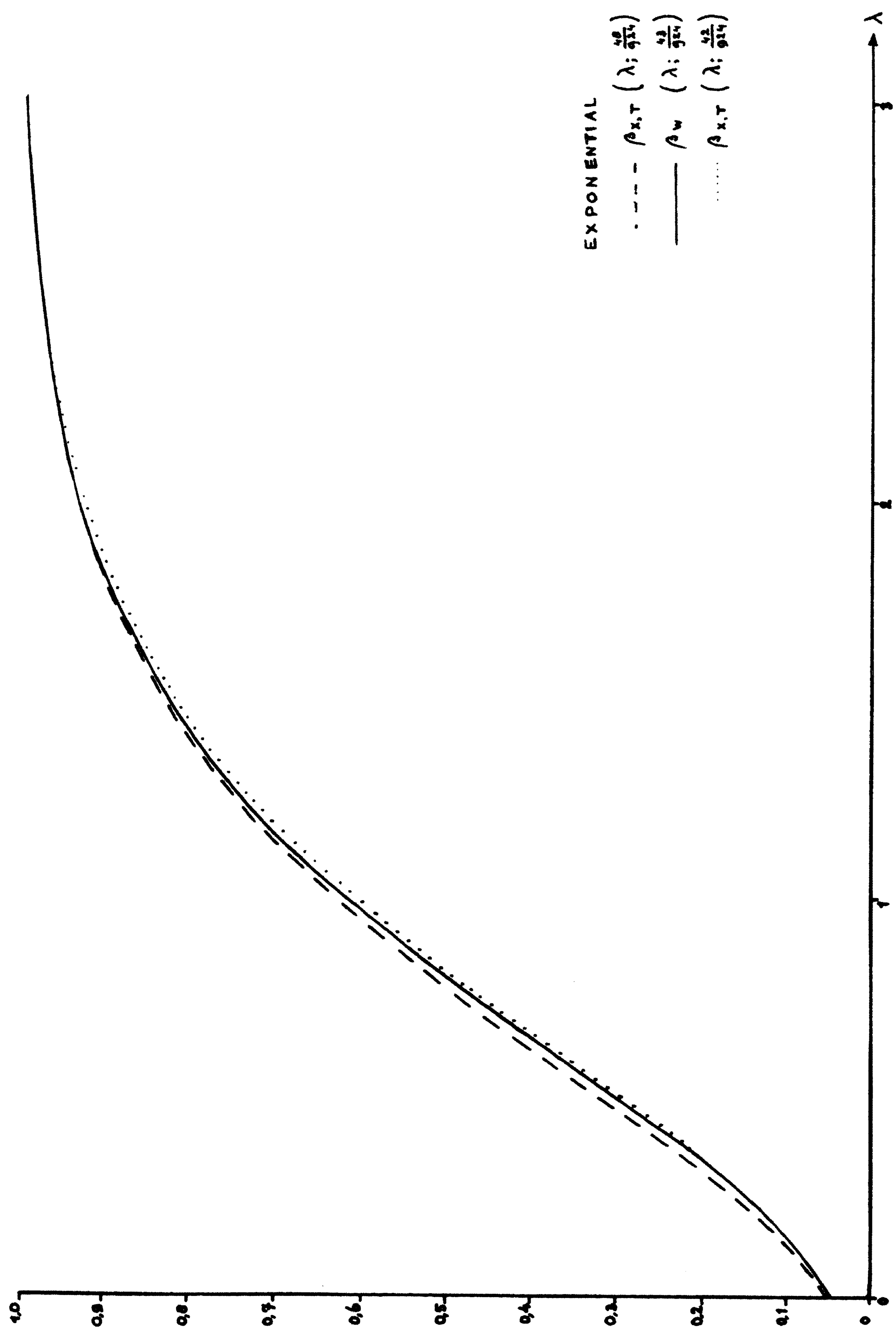

Figure 2. Power functions. One-sided testing, m=n=6.

If we fix the significance level at e.g. 0.05 (two-sided) we get different power functions because of the discrete character of the distribution functions of the test statistics (m = n = 5) :

$$\beta_W(\lambda;.05) = \beta_W\left(\lambda;\frac{2}{63}\right).$$

$$\beta_{X,T}(\lambda;.05) = \beta_{X,T}\left(\lambda;\frac{5}{126}\right) = (1-e^{-\lambda})^5 + \frac{5}{2}(1-e^{-\lambda})^4 e^{-\lambda} + \frac{50}{21}(1-e^{-\lambda})^3 e^{-2\lambda} +$$
$$+ \frac{25}{28}(1-e^{-\lambda})^2 e^{-3\lambda} + \frac{25}{126}(1-e^{-\lambda}) e^{-4\lambda} + \frac{5}{126} e^{-5\lambda}.$$

Some numerical values may be found in the next table.

Table 7

Some power values by two-sided testing with fixed significance level of .05 ; m = n = 5.

	λ									
	0.0	0.1	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5
$\beta_W\left(\lambda;\frac{2}{63}\right)$	.0317	.0357	.0475	.0926	.160	.242	.329	.536	.696	.808
$\beta_{X,T}\left(\lambda;\frac{5}{126}\right)$	.0397	.0446	.0588	.110	.182	.264	.349	.548	.702	.810

b. HOMOGENEOUS DISTRIBUTION -

The power functions are given as functions of λ, the shift parameter $(0 \le \lambda \le 1)$.

Table 8

Power functions ; m = n = 3.

	α	$\beta_W(\lambda;\alpha) = \beta_{X,T}(\lambda;\alpha)$
One-sided	$\frac{1}{100}$	0
	$\frac{1}{20}$	$\frac{1}{20} + \frac{3}{10}\lambda + \frac{3}{4}\lambda^2 + \lambda^3 - \frac{3}{4}\lambda^4 - \frac{3}{10}\lambda^5 - \frac{1}{20}\lambda^6$
Two-sided	$\frac{1}{100}$	0
	$\frac{1}{20}$	0

Table 9

Power functions ; m = n = 4.

	α	$\beta_W(\lambda;\alpha) = \beta_{X,T}(\lambda;\alpha)$
One-sided	$\frac{1}{100}$	0
	$\frac{1}{35}$	$\frac{1}{35} + \frac{8}{35}\lambda + \frac{4}{5}\lambda^2 + \frac{8}{5}\lambda^3 - \frac{8}{5}\lambda^5 - \frac{4}{5}\lambda^6 - \frac{8}{35}\lambda^7 + \frac{34}{35}\lambda^8$
Two-sided	$\frac{1}{100}$	0
	$\frac{1}{35}$	$\frac{1}{35} + \frac{4}{5}\lambda^2 + 2\lambda^4 - \frac{8}{5}\lambda^5 - \frac{8}{35}\lambda^7$

Table 10

Power functions ; m = n = 5.

	α	$\beta_W(\lambda;\alpha) = \beta_{X,T}(\lambda;\alpha)$
One-sided	$\frac{1}{126}$	$\frac{1}{126} + \frac{5}{63}\lambda + \frac{5}{14}\lambda^2 + \frac{20}{21}\lambda^3 + \frac{5}{3}\lambda^4 - \frac{5}{3}\lambda^6 - \frac{20}{21}\lambda^7 -$ $- \frac{5}{14}\lambda^8 - \frac{5}{63}\lambda^9 + \frac{125}{126}\lambda^{10}$
	$\frac{1}{21}$	$\frac{1}{21} + \frac{10}{21}\lambda + \frac{25}{14}\lambda^2 + \frac{20}{7}\lambda^3 - \frac{5}{3}\lambda^4 - 10\lambda^5 - 10\lambda^6 +$ $+ \frac{1000}{21}\lambda^7 - \frac{295}{7}\lambda^8 + \frac{260}{21}\lambda^9 - \frac{5}{14}\lambda^{10}$
Two-sided	$\frac{1}{126}$	$\frac{1}{126} + \frac{5}{14}\lambda^2 + \frac{5}{3}\lambda^4 - \frac{20}{21}\lambda^7 - \frac{5}{63}\lambda^9$
	$\frac{2}{63}$	$\frac{2}{63} + \frac{10}{7}\lambda^2 + 5\lambda^4 - 6\lambda^5 - \frac{80}{21}\lambda^7 + \frac{610}{63}\lambda^9 - \frac{16}{3}\lambda^{10}$

Table 11

Power functions ; m = n = 6.

	α	$\beta_W(\lambda;\alpha) = \beta_{X,T}(\lambda;\alpha)$
One-sided	$\frac{1}{132}$	$\frac{1}{132}+\frac{1}{11}\lambda+\frac{1}{2}\lambda^2+\frac{5}{3}\lambda^3+\frac{75}{28}\lambda^4+\frac{6}{7}\lambda^5-5\lambda^6-6\lambda^7-\frac{15}{4}\lambda^8-\frac{5}{3}\lambda^9+\frac{95}{2}\lambda^{10}-\frac{3967}{77}\lambda^{11}+\frac{14447}{924}\lambda^{12}$
	$\frac{43}{924}$	See Table 12
Two-sided	$\frac{2}{231}$	$\frac{2}{231}+\frac{4}{7}\lambda^2+\frac{30}{7}\lambda^4-\frac{12}{7}\lambda^5+2\lambda^6-\frac{48}{7}\lambda^7-\frac{40}{21}\lambda^9+\frac{916}{77}\lambda^{11}-\frac{51}{7}\lambda^{12}$
	$\frac{19}{462}$	$\frac{19}{462}+\frac{18}{7}\lambda^2-\frac{10}{7}\lambda^3+\frac{90}{7}\lambda^4-\frac{156}{7}\lambda^5+2\lambda^6-\frac{228}{7}\lambda^7+\frac{255}{2}\lambda^8-\frac{2980}{21}\lambda^9+\frac{444}{7}\lambda^{10}-\frac{654}{77}\lambda^{11}-\frac{5}{7}\lambda^{12}$

Table 12

Power functions ; m = n = 6, one-sided testing.

Test	α	Power function
W	$\frac{43}{924}$	$\frac{43}{924}+\frac{39}{77}\lambda+\frac{31}{14}\lambda^2+\frac{85}{21}\lambda^3-\frac{75}{28}\lambda^4-\frac{174}{7}\lambda^5+21\lambda^6-\frac{6}{7}\lambda^7+\frac{2145}{28}\lambda^8-\frac{3725}{21}\lambda^9+\frac{2139}{14}\lambda^{10}-\frac{4531}{77}\lambda^{11}+\frac{7771}{924}\lambda^{12}$
X and T	$\frac{42}{924}$	$\frac{1}{22}+\frac{40}{77}\lambda+\frac{16}{7}\lambda^2+\frac{30}{7}\lambda^3-\frac{45}{14}\lambda^4-24\lambda^5+10\lambda^6+24\lambda^7+\frac{1065}{14}\lambda^8-\frac{1640}{7}\lambda^9+\frac{1594}{7}\lambda^{10}-\frac{7610}{77}\lambda^{11}+\frac{2527}{154}\lambda^{12}$
"	$\frac{48}{924}$	$\frac{4}{77}+\frac{46}{77}\lambda+\frac{19}{7}\lambda^2+\frac{30}{7}\lambda^3-\frac{75}{14}\lambda^4-\frac{204}{7}\lambda^5+4\lambda^6+\frac{972}{7}\lambda^7-\frac{1380}{7}\lambda^8+\frac{450}{7}\lambda^9+\frac{397}{7}\lambda^{10}-\frac{3854}{77}\lambda^{11}+\frac{1723}{154}\lambda^{12}$
"	rand. $\frac{43}{924}$	$\frac{43}{924}+\frac{41}{77}\lambda+\frac{33}{14}\lambda^2+\frac{30}{7}\lambda^3-\frac{25}{7}\lambda^4-\frac{174}{7}\lambda^5+9\lambda^6+\frac{302}{7}\lambda^7+\frac{855}{28}\lambda^8-\frac{3875}{21}\lambda^9+\frac{2789}{14}\lambda^{10}-\frac{6984}{77}\lambda^{11}+\frac{2393}{154}\lambda^{12}$

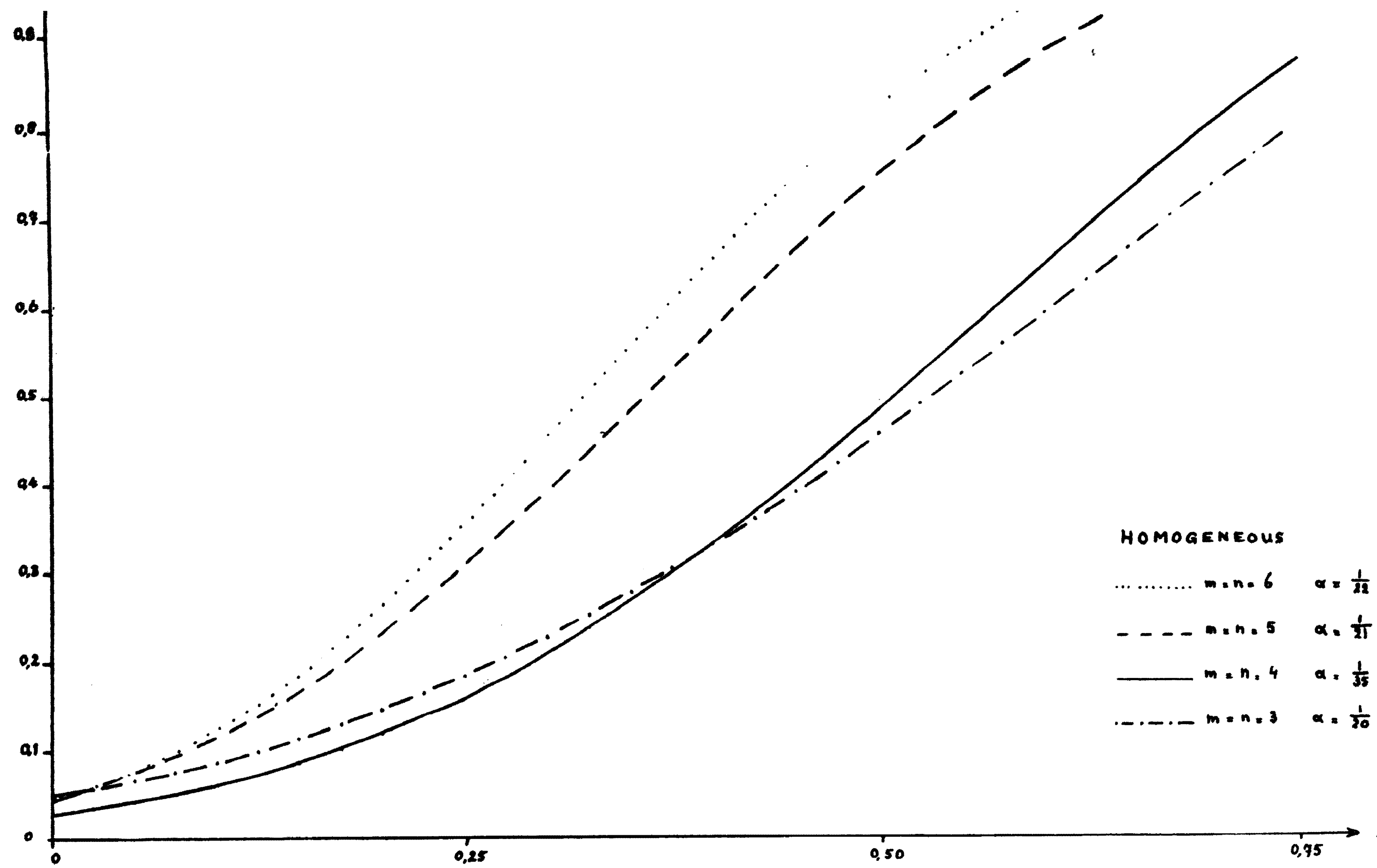

Figure 3. Power functions of van der WAERDEN's (and TERRY's) test. One-sided testing.

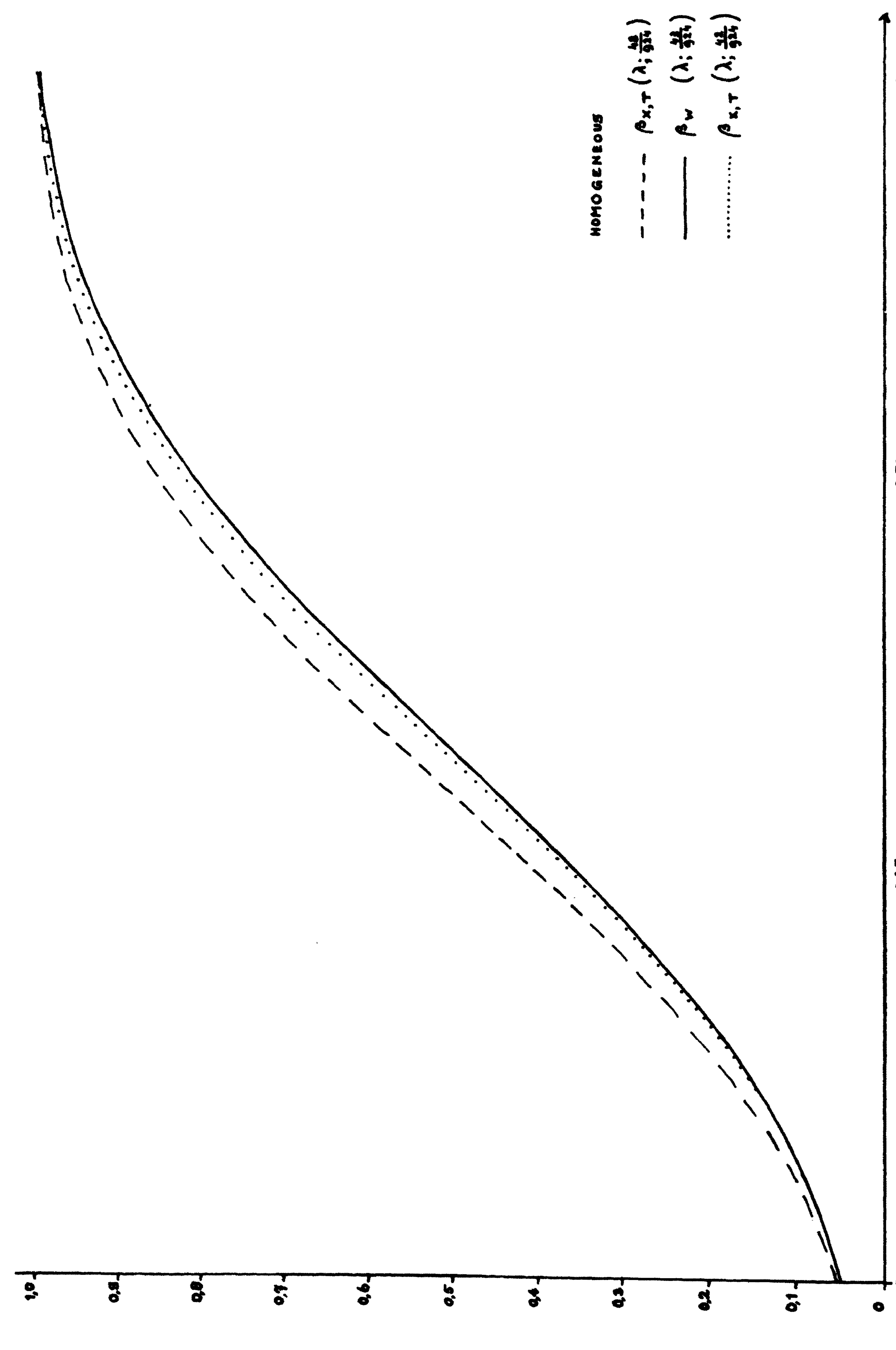

Figure 4 - Power functions. One-sided testing, m = n = 6.

The following table gives some values of the power functions of Wilcoxon's, van der Waerden's and Terry's test, if we use randomization to get the same level of significance.

Table 13

Some power values. One-sided testing. (randomization is used)

	λ									
	.00	.05	.10	.15	.20	.30	.40	.50	.60	.70
$\beta_W\left(\lambda;\ \frac{43}{924}\right)$	.0465	.0779	.123	.183	.258	.442	.641	.812	.925	.981
$\beta_{X,T}\left(\lambda;\ \frac{43}{924}\right)$	.0465	.0796	.127	.190	.269	.460	.662	.828	.935	.984

If we fix the significance level at e.g. 0.05 (two-sided) we get different power functions because of the discrete character of the distribution functions of the test statistics (m = n = 5) :

$$\beta_W(\lambda;\ .05) = \beta_W\left(\lambda;\ \frac{2}{63}\right)$$

$$\beta_{X,T}(\lambda;.05) = \beta_{X,T}\left(\lambda;\frac{5}{126}\right) = \frac{5}{126} + \frac{25}{14}\lambda^2 + 5\lambda^4 - 8\lambda^5 - \frac{100}{21}\lambda^7 + \frac{25}{2}\lambda^8 - \frac{340}{63}\lambda^9 - \frac{1}{6}\lambda^{10}.$$

Some numerical values may be found in the next table.

Table 14

Some power values by two-sided testing with fixed significance level of . 05 ; m = n = 5.

	λ									
	.00	.05	.10	.15	.20	.30	.40	.50	.60	.70
$\beta_W\left(\lambda;\ \frac{2}{63}\right)$	.0317	.0353	.0465	.0660	.0949	.186	.323	.498	.686	.850
$\beta_{X,T}\left(\lambda;\ \frac{5}{126}\right)$	.0397	.0442	.0580	.0818	.117	.221	.370	.550	.730	.877

6. Conclusions.

In the case of $m = n = 6$ (one-sided testing with significance level of $\frac{43}{924} \approx .0465$) and the Exponential distribution the (randomized) power of both van der Waerden's and Terry's test is better than that of Wilcoxon's test in the neighbourhood of the null-hypothesis. For the Rectangular distribution the (randomized) power of both van der Waerden's and Terry's test is constantly better than that of Wilcoxon's test. These results are in agreement with the result of Hodges and Lehmann [11]. For the Exponential distribution the power of Wilcoxon's test increases more rapidly for larger λ, so that for $\lambda = .5$ the power is already slightly larger than that of van der Waerden's and Terry's test. If we use non-randomized power of van der Waerden's and Terry's test with significance levels in the neighbourhood of $\frac{43}{924}$ we would draw the same conclusions.

As an illustration we give the following two tables with first derivatives :

EXPONENTIAL DISTRIBUTION

Table 15

One-sided testing, $m = n = 6$.

Test	α	$\beta'(0 ; \alpha)$
W	$\frac{43}{924}$	.253
X and T	$\frac{42}{924}$	.260
"	$\frac{48}{924}$	.299
"	rand. $\frac{43}{924}$	.266

HOMOGENEOUS DISTRIBUTION

Table 16

One-sided testing, $m = n = 6$.

Test	α	$\beta'(0 ; \alpha)$
W	$\frac{43}{924}$	.506
X and T	$\frac{42}{924}$	.519
"	$\frac{48}{924}$	.597
"	rand. $\frac{43}{924}$	.532

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