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A theorem on regularly varying functions

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A THEOREM ON REGULARLY VARYING FUNCTIONS *

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1. INTRODUCTION

In his recent book [1] Feller introduces the theory of regularly varying functions, a topic which has been completely disregarded in most, if not all, previous books on probability. By means of this theory a number of probabilistic theorems and their interconnections can be clarified considerably, notably in the general area of stable laws and their domains of attraction. The proofs of these theorems usually contain rather messy arguments to show the existence of sequences of numbers with certain desirable asymptotic properties (See e.g. [2], §35). The proofs given in [1], though greatly simplified by the systematic use of the theory of regularly varying functions, still need the same or similar sequences, but, whenever this need arises, there is just a flat statement to the effect that one can find a sequence with the required properties ([1], pp. 271, 304, 425, 545).

* Report S 368, Stat. Dept., Mathematisch Centrum, Amsterdam.

These facts constitute the raison d'être of the theorem below, which may be known but certainly is not well known, and which settles the indicated existence problems in one stroke.

2. THE THEOREM

For easy reference here is the definition of regular variation.

DEFINITION: A strictly positive function f on $(0,\infty)$ varies regularly (at infinity) iff $f(tx) \sim x^{\gamma} f(t)$ as $t \to \infty$, for some finite γ (the exponent of f) and all $x > 0$.

LEMMA: If a monotone and strictly positive function f on $(0,\infty)$ varies regularly, then $f(x-0) \sim f(x) \sim f(x+0)$ as $x \to \infty$.

PROOF: Replacing f by 1/f if necessary, we may assume that f is nondecreasing. Then, for any $\rho > 1$ and all $x > 0$,

$$1 \leqq \frac{f(x)}{f(x-0)} \leqq \frac{f(x+0)}{f(x-0)} \leqq \frac{f(x\rho)}{f(x/\rho)} .$$

Since f varies regularly,

$$\lim_{x \to \infty} \frac{f(x\rho)}{f(x/\rho)} = \rho^{2\gamma} ,$$

where γ is the exponent of f, and the lemma follows, $\rho > 1$ being arbitrary.

<u>THEOREM</u>: <u>Let</u> $h(x) = f(x)g(x)$, <u>where</u> f <u>and</u> g <u>are monotone and strictly positive functions on</u> $(0,\infty)$, <u>varying regularly with exponents</u> α <u>and</u> β <u>respectively</u>.

<u>Then</u>

(i) $h(\infty) = 0$ <u>if</u> $\alpha+\beta < 0$;

(i)' $h(\infty) = \infty$ <u>if</u> $\alpha+\beta > 0$.

<u>Moreover, if</u> $\{\delta_n\}$ <u>is any nondecreasing sequence with</u> $\delta_n \to \infty$ <u>as</u> $n \to \infty$, <u>then</u>

(ii) <u>if</u> $h(\infty) = 0$, <u>there exists a nondecreasing sequence</u> $\{a_n\}$ <u>such that</u> $a_n \to \infty$ <u>and</u> $\delta_n h(a_n x) \to x^{\alpha+\beta}$ <u>for all</u> $x > 0$ <u>as</u> $n \to \infty$;

(ii)' <u>if</u> $h(\infty) = \infty$, <u>there exists a nondecreasing sequence</u> $\{a_n\}$ <u>such that</u> $a_n \to \infty$ <u>and</u> $\delta_n^{-1} h(a_n x) \to x^{\alpha+\beta}$ <u>for all</u> $x > 0$ <u>as</u> $n \to \infty$.

<u>PROOF</u>: Since (i)' and (ii)' follow from (i) and (ii) applied to $1/h$, it suffices to prove (i) and (ii).

Let $\alpha+\beta < 0$ and $\rho > 1$. Then

$$h(\rho^{n+1}) \sim \rho^{\alpha+\beta} h(\rho^n) \quad \text{as } n \to \infty \tag{1}$$

and hence

$$\lim_{n \to \infty} h(\rho^n) = 0 . \tag{2}$$

If f and g are both nonincreasing, so is h, and (i) follows. f and g cannot be both nondecreasing, since then h too would be nondecreasing in violation of (1). The only remaining possibility is that one of the functions f and g, say f, is nondecreasing and the other nonincreasing. Taking $n = n(x)$ to be the unique integer such that $\rho^n \leq x < \rho^{n+1}$, we have then

$$0 < h(x) \leq f(\rho^{n+1})g(\rho^n) \sim \rho^\alpha h(\rho^n) \quad \text{as } x \to \infty ,$$

and (i) follows from (2).

Now let $\{\delta_n\}$ be a nondecreasing sequence tending to infinity and put

$$\text{(3)} \qquad a_n = \sup \{ x : h(x) \geq \frac{1}{\delta_n} \} .$$

The a_n thus defined are finite if $h(\infty) = 0$ and form a nondecreasing sequence since $\{\delta_n\}$ is nondecreasing. If the sequence $\{a_n\}$ were bounded by a finite number B, then (3) would imply that $h(x) < 1/\delta_n$ for all n and all $x > B$, contradicting the assumed strict positivity of f and g. Consequently $a_n \to \infty$ as $n \to \infty$, and thus the lemma implies

$$\delta_n h(a_n - 0) \sim \delta_n h(a_n) \sim \delta_n h(a_n + 0) \quad \text{as } n \to \infty .$$

Since (3) insures that

$$\delta_n h(a_n + 0) \leq 1 \quad \text{and} \quad \max \{\delta_n h(a_n - 0), \delta_n h(a_n)\} \geq 1$$

for all n, it follows that $\delta_n h(a_n) \to 1$ as $n \to \infty$, and (ii) becomes a direct consequence of the regular variation of h.

REFERENCES

[1] FELLER, W. (1966). An Introduction to Probability Theory and Its Applications, Vol. II. John Wiley and Sons, Inc., New York, London, Sydney.

[2] GNEDENKO, B.V., and KOLMOGOROV, A.N. (1954). Limit distributions for sums of independent random variables, translated from the Russian and annotated by K.L. Chung. Addison-Wesley Publishing Company, Inc., Cambridge, Mass.