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An Inequality for Expected Values of Sample Quantiles.

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AN INEQUALITY FOR EXPECTED VALUES OF SAMPLE QUANTILES

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1. Introduction. Let $F$ be a continuous distribution function on $\mathbb{R}$ that is strictly increasing on the (finite or infinite) open interval $I$ where $0 < F < 1$, and let $G$ denote the inverse of $F$. For $n = 1, 2, \ldots$ and $0 < \lambda < 1$, let

$$\gamma_n(\lambda) = \left[ (n+1)/\Gamma(\lambda(n+1)) \Gamma((1-\lambda)(n+1)) \right] \int_0^1 G(y)y^{\lambda(n+1)-1}(1-y)^{(1-\lambda)(n+1)-1}dy.$$ 

Obviously, if $X_{i:n}$ denotes the $i$th order statistic of a sample of size $n$ from the parent distribution $F$, then

$$\gamma_n(i/(n+1)) = EX_{i:n}, \quad i = 1, 2, \ldots, n.$$ 

We shall call $\gamma_n(\lambda)$ the expected value of the $\lambda$-quantile of a sample of size $n$ from $F$, even though this interpretation is meaningless when $\lambda(n+1)$ is not an integer. We shall assume that for some $\lambda$ the integral converges for sufficiently large $n$, which ensures that the same will hold for every $0 < \lambda < 1$. By making minor changes in W. Hoeffding's proof in [2], one shows that $\gamma_n$ converges to $G$ on $(0, 1)$ for $n \to \infty$.

Consider another continuous distribution function $F^*$ that is strictly increasing on the interval $I^*$ where $0 < F^* < 1$, and let $G^*, \gamma^*_n$ and $X^*_{i:n}$ be defined for $F^*$ analogous to $G, \gamma_n$ and $X_{i:n}$ for $F$. Furthermore let

$$\phi(x) = G^*F(x), \quad x \in I.$$ 

In [5] the author studied the following order relations between $F$ and $F^*$:

$$\phi$$ is convex on $I$;

$$F$$ and $F^*$ represent symmetric distributions and $\phi$ is concave-convex on $I$.

If $x_0$ denotes the median of $F$, relation (1.4) implies that $\phi$ is antisymmetric about $x_0$ (i.e., $\phi(x_0 + x) + \phi(x_0 - x) = 2\phi(x_0)$) and hence that $\phi$ is concave for $x < x_0$ and convex for $x > x_0$.

Let $\phi_n$ be the function that maps the expected value of the $\lambda$-quantiles of a sample of size $n$ from $F$ on the corresponding quantities for $F^*$:

$$\phi_n(x) = \gamma^*_n\gamma^{-1}_n(x).$$
For \( n \to \infty \), \( \phi_n \) will converge to the function \( \phi \) on \( I \) that maps the population quantiles of \( F \) on those of \( F^* \). It is shown in this note that if relations (1.3) or (1.4) hold, \( \phi_n \) shares the convexity or concave-convexity of \( \phi \), and the convergence of \( \phi_n \) to \( \phi \) is monotone. The convexity property yields a theorem on the behavior of the ratio of expected values of spacings of consecutive order statistics from \( F \) and \( F^* \). Simple applications are given in Section 3.

2. The results.

**Theorem 2.1.** If condition (1.3) holds, \( \phi_n(x) \) is convex in \( x \) for fixed \( n \), and non-increasing in \( n \) for fixed \( x \).

**Proof.** For each fixed \( n \) the densities

\[
\lambda = \frac{\Gamma(n + 1/\lambda + 1)}{\Gamma(n + 1)} \Gamma((1 - \lambda)(n + 1))
\]

constitute a one-parameter exponential family for \( 0 < \lambda, y < 1 \), and consequently the family is strictly totally positive of order \( \infty \) in \( \lambda \) and \( y \) (cf. [3]). According to a slight elaboration of a result due to S. Karlin that is given in [4], the convexity of \( \phi_n \) follows from the definition of \( \gamma_n \) and \( \gamma_n^* \), the total positivity of \( f_\lambda(y) \), the monotonicity of \( F \) and the convexity of \( \phi \). Also

\[
\gamma(\lambda) = \lambda \gamma_n(\lambda + (1 - \lambda)/(n + 2)) + (1 - \lambda) \gamma_n^*(\lambda - \lambda/(n + 2))
\]

and the same holds for \( \gamma_n^* \). This is easily verified by adding integrands in expression (1.1). Hence, because of the convexity of \( \phi_n \)

\[
\phi_{n+1}(\lambda) = \phi_n(\lambda \gamma_{n+1} + (1 - \lambda)/(n + 2)) + (1 - \lambda) \gamma_n^*(\lambda - \lambda/(n + 2))
\]

(2.3) for \( \lambda \in (0, 1) \) for any \( a \) and \( b \). If it does change sign three times, the signs occur in the order \((- , + , - , +)\) for increasing values of the argument. It follows from the variation diminishing property of totally positive kernels (cf. [3]) that

\[
\gamma_n^*(\lambda) - a - b \gamma_n(\lambda) = \int_a^b h(y) f_\lambda(y) dy
\]
changes sign at most three times; if it does have three sign changes, the signs occur in the order $\cdot \rightarrow \cdot \rightarrow \cdot \rightarrow \cdot$. Substituting $\gamma_n(x) = x$ we find that $\varphi_n(x) = a - bx$ possesses the same property for any $a$ and $b$. A simple geometrical argument based on the antisymmetry of $\varphi_n$ shows that this implies that $\varphi_n$ is concave-convex about $x_0$. Since for $\lambda > \frac{1}{2}$

$$(\lambda + (1 - \lambda)/(n + 2)) + (\lambda - \lambda/(n + 2)) > 1,$$

and hence by the antisymmetry of $\gamma_n$

$$\gamma_n + 1(\lambda + (1 - \lambda)/(n + 2)) + \gamma_n - 1(\lambda - \lambda/(n + 2)) > 2\sigma_0$$

the inequality of (2.3) remains valid now that $\varphi_n$ is antisymmetric and concave-convex instead of convex. This completes the proof.

We note that in the proofs of Theorems 2.1 and 2.2 we have only made use of the total positivity of $f(y)$. Exploiting the fact that the total positivity is strict, one finds that the convexity (or concave-convexity) in $x$ as well as the monotonicity in $n$ of $\varphi_n(x)$ are strict, unless $\varphi$ is linear on $I$.

The quantities $\gamma_n(\lambda)$ for non-integer $\lambda(n + 1)$ were introduced to facilitate the discussion of $\lambda$-quantiles for fixed $\lambda$ and varying $n$. However, in considering the convexity of $\varphi_n$ for fixed $n$, we may as well restrict ourselves to the case where $i = \lambda(n + 1)$ is an integer. Theorem 2.1 then states that if condition (1.3) holds, i.e. if $G^*$ is a convex function of $G$, then $EX_{i:n}$ is a convex function of $EX_{i:n}$ for varying $i$ and fixed $n$, i.e.

$$(2.4) \quad (EX_{i+1:n}^* - EX_{i:n}^*)/(EX_{i+1:n} - EX_{i:n})$$

is non-decreasing in $i$ for fixed $n$. We recall that the proof of this assertion rests solely on the fact that the family (2.1), which for $i = \lambda(n + 1)$ becomes

$$(2.5) \quad f_{i:n}(y) = [n/(i - 1)! (n - i)!]y^{i-1}(1 - y)^{n-i},$$

is totally positive of order infinity in $i$ and $y$ for fixed $n$. However, the family (2.5) is also totally positive of order infinity in $n$ and $(1 - y)$ for fixed $i$. One easily verifies that this implies that $EX_{i:n}$ is also a convex function of $EX_{i:n}$ for varying $n$ and fixed $i$. Since $EX_{i:n}$ is decreasing in $n$ for fixed $i$, it follows that

$$(2.6) \quad (EX_{i:n}^* - EX_{i+1:n}^*)/(EX_{i:n} - EX_{i+1:n})$$

is non-decreasing in $n$. Using formula (2.2) for $\lambda(n + 1) = i$, i.e.

$$(2.7) \quad EX_{i:n} = [i/(n + 1)]EX_{i+1:n+1} + [(n + 1 - i)/(n + 1)]EX_{i+1:n},$$

and the corresponding expression for $EX_{i:n}^*$, we find

$$(2.8) \quad (EX_{i:n}^* - EX_{i+1:n}^*)/(EX_{i:n} - EX_{i+1:n}) = (EX_{i+1:n+1}^* - EX_{i+1:n+1}^*)/(EX_{i+1:n+1} - EX_{i+1:n}).$$

and hence (2.4) is non-decreasing in $n$.

By considering the distribution functions $1 - F^*(-x)$ and $1 - F(-x)$
stead of $F$ and $F^*$ one easily shows that

$$(2.7) \quad \frac{(EX_{n-i+1:n} - EX_{n-i:n}^*)}{(EX_{n-i+1:n} - EX_{n-i:n})}$$

is non-increasing in $i$ and non-decreasing in $n$. The former conclusion is of course equivalent to the monotonicity in $i$ of (2.4). We have proved

**Theorem 2.3.** If condition (1.3) holds, the quantities (2.4) are non-decreasing in $i$ and non-increasing in $n$, whereas (2.7) is non-decreasing in $n$.

We note that the last assertion of the theorem may also be proved directly by using the total positivity of (2.5) in $i$ and $y$ for fixed $(n - i)$ and applying (2.6).

It may be of interest to point out the similarity of Theorem 2.3 to inequalities that were recently obtained by R. E. Barlow and F. Proschan [1] for the case where $F(0) = F^*(0) = 0$ and $\phi$ is starshaped (i.e. $\phi(x)/x$ non-decreasing on $I$).

By total positivity arguments similar to those given above they show that $EX_{i:n}^*/EX_{i:n}$ is non-decreasing in $i$ and non-increasing in $n$, whereas $EX_{n-i:n}^*/EX_{n-i:n}$ is non-decreasing in $n$.

3. **Applications.** Let $F$ be the uniform distribution function on $(0, 1)$. Then

$$\gamma_n(\lambda) = \lambda \quad \text{for} \quad 0 < \lambda < 1,$$

$\phi = G^*$ and $\phi_n = \gamma_n^*$. If $F^*$ is differentiable on $I^*$, it satisfies conditions (1.3) or (1.4) if its density $F^{**}$ is non-increasing on $I^*$, or symmetric and unimodal respectively. Consequently we have:

The expected value of the $\lambda$-quantile of a sample of size $n$ from a distribution with non-increasing density is a non-increasing function of $n$; if the density is symmetric and unimodal the conclusion remains valid for $\lambda > \frac{1}{2}$. Moreover, if $F^{**}$ is non-increasing, $(n - i)(EX_{i+1:n}^* - EX_{i:n})$ is non-decreasing in $i$ and non-increasing in $n$, whereas $(n + 1)(EX_{n-i+1:n}^* - EX_{n-i:n})$ is non-decreasing in $n$.

As a second example consider the case where $F^*$ denotes the exponential distribution function. Then condition (1.3) is satisfied if the distribution $F$ has increasing failure rate

$$q(x) = F'(x)/(1 - F(x))$$

(cf. [1] or [5]). We have (cf. similar results in [1]): If $F$ has increasing failure rate, then $(n - i)(EX_{i+1:n} - EX_{i:n})$ is non-decreasing in $i$ and non-decreasing in $n$, whereas $(EX_{n-i+1:n} - EX_{n-i:n})$ is non-increasing in $n$.

For other cases where relations (1.3) or (1.4) are satisfied and the results of this paper may be applied, the reader is referred to [5].

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**REFERENCES**

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