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On convexity preserving families of probability distribution

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On convexity preserving families of probability distributions

by W. R. van Zwet *

1. Introduction

Let F_θ be a family of probability distribution functions on R^1 with parameter $\theta \in T \subseteq R^1$, and let X denote the union of the supports of these distributions. For $k \geqq 0$, let $\{g_0, g_1, \ldots, g_{k+1}\}$ be a set of real-valued finite functions on X that are integrable with respect to F_θ for all $\theta \in T$ and define

$$\chi_i(\theta) = \int g_i(x)\, dF_\theta(x), \quad i = 0, 1, \ldots, k+1. \tag{1.1}$$

Following S. Karlin and W. J. Studden in [3] with a minor modification, we shall say that $\{g_0, g_1, \ldots, g_{k+1}\}$ constitute a *weak complete Tchebycheff system* (WCT-system) if for each $0 \leqq m \leqq k+1$ and all $x_0 < x_1 < \ldots < x_m \in X$ the determinant

$$\det\, [g_i(x_j)]_{i,j=0,\ldots,m} \geqq 0; \tag{1.2}$$

the system is called a *complete Tchebycheff system* (CT-system) if the inequality is always strict. The difference between this definition of a WCT-system and the one given in [3] is that we retain the case where $g_0, g_1, \ldots, g_m$ are linearly dependent on X for some $m \leqq k+1$; in that case any choice of $g_{m+1}, \ldots, g_{k+1}$ will trivially satisfy definition (1.2). We shall also express inequalities (1.2) by saying that g_{k+1} is *generalized convex* with respect to the WCT-system $\{g_0, \ldots, g_k\}$.

Our discussion of WCT-systems will involve the related concept of total positivity (cf. [1]). A function $f(x, \theta)$ on $X \times T$ is said to be *totally positive of order n* (TP_n) if for every $1 \leqq m \leqq n$, all $x_1 < x_2 < \ldots < x_m \in X$ and all $\theta_1 < \theta_2 < \ldots < \theta_m \in T$,

$$\det\, [f(x_i, \theta_j)]_{i,j=1,\ldots,m} \geqq 0. \tag{1.3}$$

The first question that comes to mind in this context is whether one can find conditions on the family F_θ that ensure that $\{\chi_0, \chi_1, \ldots, \chi_{k+1}\}$ will be a WCT-system on T whenever $\{g_0, g_1, \ldots, g_{k+1}\}$ constitutes a WCT-system on X. If the family F_θ possesses densities $p(x, \theta)$ with respect to a σ-finite measure μ with spectrum X and hence

$$\chi_i(\theta) = \int g_i(x)\, p(x, \theta)\, d\mu(x),$$

this question is easily answered. We have for each $0 \leqq m \leqq k+1$ (cf. [1])

$$\det\, [\chi_i(\theta_j)] = \int_{x_0 < x_1 < \ldots < x_m} \!\!\cdots\! \int \det\, [g_i(x_j)] \det\, [p(x_i, \theta_j)]\, d\mu(x_0) \ldots d\mu(x_m)$$

where in each determinant i and j run from 0 to m. It follows that the condition that p is TP_{k+2} is certainly sufficient; since we require that $\{\chi_0, \ldots, \chi_{k+1}\}$ will inherit the WCT-property for every WCT-system $\{g_0, \ldots, g_{k+1}\}$, the condition is essentially also

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necessary (by "essentially" is meant that for any $\theta_1 < \ldots < \theta_m$ the defining inequality (1.3) need not hold on a set of product-measure 0). We note that the fact that F_θ are probability distribution functions is not used in establishing the condition.

In view of this general result it is hardly surprising that recent discussions of convexity preserving properties (cf. [1] and [2]) have been confined to families of densities that are totally positive of the appropriate order. However, one usually does not discuss the class of all WCT-systems of a given order but restricts attention to a relatively small subclass (e.g. the case where $g_i = g_1^i$ for $i = 0, 1, \ldots, k$). Also one often imposes additional restrictions on the family F_θ to ensure that for those systems $\{g_0, \ldots, g_{k+1}\}$ that are considered, $\{\chi_0, \ldots, \chi_{k+1}\}$ will also belong to some restricted class.

In sections 3 and 4 of this paper we investigate how far the TP_{k+2} condition for p can be relaxed in two such restricted cases that seem to be important in practice. Like section 1, the second section is of an expository character.

2. Convexity of order k

Let f be a real-valued finite function defined on an arbitrary set $Y \subseteq R^1$. For $k \geqq 0$ we shall say that f is *convex of order* k (C_k) if f is generalized convex with respect to the CT-system $\{1, y, y^2, \ldots, y^k\}$, i.e. if for all $y_1 < y_2 < \ldots < y_{k+2}$

$$D_f(y_1, \ldots, y_{k+2}) = \begin{vmatrix} 1 & y_1 & y_1^2 & \ldots y_1^k & f(y_1) \\ 1 & y_2 & y_2^2 & \ldots y_2^k & f(y_2) \\ \vdots & & & & \vdots \\ 1 & y_{k+2} & y_{k+2}^2 & \ldots y_{k+2}^k & f(y_{k+2}) \end{vmatrix} \geqq 0. \tag{2.1}$$

For $k = 0,1$, (2.1) reduces to the ordinary definitions of non-decreasing or (measurable) convex functions. Generally speaking (2.1) is an extension of the concept of non-negative $(k+1)$-th derivative. C_k functions were extensively studied by T. Popoviciu in [6]. We note that S. Karlin [1] refers to C_k functions as convex of order $(k+1)$.

If P_m denotes a polynomial of degree at most m, then equivalent definitions of the C_k property are obviously:

(A) (cf. [1]). For every $P_k, f-P_k$ changes sign at most $(k+1)$ times on Y. If it does have $(k+1)$ sign-changes, the signs occur in the order $(-)^{k+1}, (-)^k, \ldots, +, -, +$ for increasing values of the argument.

(B) For every $y_1 < y_2 < \ldots < y_{k+2} \in Y$, the P_{k+1} having $P_{k+1}(y_i) = f(y_i)$, $i = 1, 2, \ldots, k+2$, has non-negative coefficient for its $(k+1)$-th degree term.

There is also a close connection with differences. Let

$$\Delta_h^1 f(y) = f(y+h) - f(y)$$
$$\Delta_h^m f(y) = \Delta_h^1 \Delta_h^{m-1} f(y) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(y+jh) \tag{2.2}$$

and generally

$$\Delta^m_{h_1,\ldots,h_m} f(y) = \Delta^1_{h_m}\Delta^{m-1}_{h_1,\ldots,h_{m-1}} f(y) = \sum_{j=0}^{m}(-1)^{m-j} \sum_{1\leq i_1<i_2<\ldots<i_j\leq m} f(y+\sum_{\nu=1} h_{i_\nu}). \quad (2.3)$$

Furthermore let

$$D^*_f(y_1,\ldots,y_{k+2}) = \frac{D_f(y_1,\ldots,y_{k+2})}{\prod_{1\leq i<j\leq k+2}(y_j-y_i)}; \quad (2.4)$$

since the denominator is positive for $y_1 < y_2 < \ldots < y_{k+2}$, D_f may be replaced by D^*_f in definition (1.1). The following relation between D^*_f and differences may be proved by induction on k.

Lemma 2.1

If Π denotes the set of permutations $\pi = [\pi(1), \pi(2), \ldots, \pi(k+1)]$ of the numbers $1, 2, \ldots, k+1$, then

$$\Delta^{k+1}_{h_1,\ldots,h_{k+1}} f(y) = \prod_{i=1}^{k+1} h_i \sum_{\pi\in\Pi} D^*_f\left(y, y+h_{\pi(1)}, \ldots, y+\sum_{\nu=1}^{k+1} h_{\pi(\nu)}\right). \quad (2.5)$$

We note that for $h_1 = h_2 = \ldots = h_{k+1} = h$, (2.5) reduces to

$$\Delta^{k+1}_h f(y) = (k+1)!\, h^{k+1}\, D^*_f[y, y+h, \ldots, y+(k+1)h]. \quad (2.6)$$

It follows from lemma 2.1 that if f is C_k on Y, then for all $h_1, h_2, \ldots, h_{k+1} > 0$,

$$\Delta^{k+1}_{h_1,\ldots,h_{k+1}} f(y) \geq 0 \quad (2.7)$$

whenever defined, i.e. whenever all $y + \sum_{\nu=1}^{j} h_{i_\nu} \in Y$.

In the special case that Y is an interval there is also a converse result and the following definition of the C_k property is equivalent to (2.1) in this case:

(C) f is (Lebesgue)-measurable and for $h > 0$, $y \in Y$, $y+(k+1)h \in Y$,

$$\Delta^{k+1}_h f(y) \geq 0. \quad (2.8)$$

In this case, however, the C_k property is hardly a generalization of non-negative $(k+1)$-th derivative at all. In fact, if Y is an open interval and $k \geq 1$, definition (2.1) ensures continuity of f on Y and is equivalent to

(D) f is $(k-1)$ times continuously differentiable and $f^{(k-1)}$ is convex on Y.

Finally we consider the special case where Y is a set of consecutive integers. For integer $h > 0$

$$\Delta^{k+1}_h f(y) = \sum_{h_1=0}^{h-1}\ldots\sum_{h_{k+1}=0}^{h-1} \Delta^{k+1}_1 f\left(y + \sum_{j=1}^{k+1} h_j\right). \quad (2.9)$$

Combining (2.6) and (2.9) we find that the C_k property may be defined in this case by

(E) For all y, $y+k+1 \in Y$

$$\Delta^{k+1}_1 f(y) \geq 0. \quad (2.10)$$

For further details concerning the definitions given above the reader is referred to [6].

Let f_1 and f_2 be real-valued finite functions on Y. We shall say that f_2 is C_k with respect to f_1 on Y if there exists a C_k function f on $f_1(Y)$ such that $f_2 = f(f_1)$ on Y. If f_1 is non-decreasing on Y and f_2 is constant on any set where f_1 is constant, this reduces to

$$\begin{vmatrix} 1 & f_1(y_1) & f_1^2(y_1) & \dots & f_1^k(y_1) & f_2(y_1) \\ 1 & f_1(y_2) & f_1^2(y_2) & \dots & f_1^k(y_2) & f_2(y_2) \\ \vdots & & & & & \vdots \\ 1 & f_1(y_{k+2}) & f_1^2(y_{k+2}) & \dots & f_1^k(y_{k+2}) & f_2(y_{k+2}) \end{vmatrix} \geq 0 \qquad (2.11)$$

for all $y_1 < y_2 < \dots < y_{k+2} \in Y$.

3. Preserving convexity of order k

Returning to the setup of section 1, we let g be a real-valued finite function on X that is integrable with respect to F_θ for all $\theta \in T$ and define

$$\chi(\theta) = \int g(x)\, dF_\theta(x).$$

We shall say that the family F_θ preserves convexity of order k if χ is C_k on T whenever g is C_k on X, i.e. whenever g is generalized convex with respect to $\{1, x, \dots, x^k\}$ then χ is generalized convex with respect to $\{1, \theta, \dots, \theta^k\}$. In [1] S. KARLIN has shown that if densities $p(x, \theta)$ with respect to μ exist, then a sufficient condition for F_θ to preserve convexity of order k is that p is TP_{k+2} and that whenever g is a polynomial of exact degree $m \leq k$, then χ is also a polynomial of exact degree m. According to the result of section 1 the first part of this condition ensures that χ is generalized convex with respect to the WCT-system

$$\int x^i\, dF_\theta(x), \qquad i = 0, 1, \dots, k,$$

whereas the second part ensures that this is equivalent to generalized convexity with respect to $\{1, \theta, \dots, \theta^k\}$.

However, this condition is not necessary. For $k = 0$ a condition that is necessary as well as sufficient was given by J. KRZYZ in [4]:

Lemma 3.1

χ is non-decreasing on T whenever g is non-decreasing on X if and only if the family F_θ is stochastically increasing (i.e. $F_\theta(x)$ is non-increasing in θ for every fixed x).

Since the TP_2 property of p is equivalent to monotone likelihood ratio, KRZYZ's condition is weaker than KARLIN's for $k = 0$ (cf. [5]).

For general k it is also easy to find a necessary and sufficient condition, provided that we restrict attention to those C_k functions g that can be extended to a C_k function on an open interval containing X. Since the convex functions constitute a convex cone spanned by the linear functions and functions of the form

$$h(x) = \begin{cases} 0 & \text{for} \quad x \leq x_0 \\ x - x_0 & \text{for} \quad x > x_0, \end{cases}$$

we find from definition D of section 2 that the convex cone of C_k functions is spanned by the polynomials P_k of degree at most k and functions of the form

$$h_k(x) = \begin{cases} 0 & \text{for} \quad x \leq x_0 \\ (x - x_0)^k & \text{for} \quad x > x_0. \end{cases}$$

For $k = 0$ this is obviously also true. It follows that it is sufficient as well as necessary to require that χ be C_k whenever g is of one of the forms mentioned above. However, if g is a polynomial of degree at most k, then so is $-g$ and as a result both χ and $-\chi$ are required to be C_k, which implies that χ is also a polynomial of degree at most k. Hence we have proved

Lemma 3.2

χ is C_k on T whenever g is C_k on an open interval containing X, if and only if for every x_0

$$\int (x - x_0)^k \, dF_\theta(x)$$

is C_k on T and whenever g is a polynomial of degree at most k the same holds for χ.

We note that for $k \leq 1$ the condition that the C_k function g can be extended to a C_k function on an open interval containing X is always satisfied. For $k = 0$ the lemma reduces to lemma 3.1.

Although for $k \geq 1$ lemma 3.2 seems to be fairly useless for practical purposes, the results obtained so far do seem to indicate that there exists a large class of C_k preserving families that do not possess any total positivity properties. The results in the remainder of this section exhibit a number of these families.

Theorem 3.1

Let F_0 and F be distribution functions with characteristic functions φ_0 and φ respectively, and suppose that F is infinitely divisible and has $F(-0) = 0$. If for $t \geq 0$, F_t denotes the distribution function corresponding to $\varphi_0 . \varphi^t$, then the family F_t, $0 \leq t < \infty$, preserves convexity of all orders.

Proof

Let G_t denote the distribution function corresponding to φ^t and let X_t (t ≥ 0) be a stochastic process with non-negative stationary independent increments for which X_0, $X_{s+t} - X_s$ and X_t ($s, t \geq 0$) have distribution functions F_0, G_t and F_t respectively.

For fixed $t \geq 0$ and $h > 0$ define

$$Z_i = X_{t+ih} - X_{t+(i-1)h}, \quad i = 1, 2, \ldots, k+1.$$

$Z_1, Z_2, \ldots, Z_{k+1}$ are independent and identically distributed random variables that are also independent of X_t. Hence, because of the exchangeability of $Z_1, \ldots, Z_{k+1}$,

$$E\left[\sum_{j=0}^{k+1}(-1)^{k+1-j}\binom{k+1}{j}g(X_{t+jh})\,\Big|\,X_t=x\right]=$$
$$=E\left[\sum_{j=0}^{k+1}(-1)^{k+1-j}\binom{k+1}{j}g(x+Z_1+\ldots+Z_j)\right]=$$
$$=E\left[\sum_{j=0}^{k+1}(-1)^{k+1-j}\sum_{1\leqq i_1<\ldots<i_j\leqq k+1}g(x+Z_{i_1}+\ldots+Z_{i_j})\right]=$$
$$=E\,[\Delta^{k+1}_{Z_1,\ldots,Z_{k+1}}\,g(x)].$$

Since $Z_1, \ldots, Z_{k+1} \geqq 0$ with probability 1, the last expression is non-negative for every C_k function g and all x by (2.7). As a result

$$\Delta_h^{k+1}\,\chi(t)=E\left[\sum_{j=0}^{k+1}(-1)^{k+1-j}\binom{k+1}{j}g(X_{t+jh})\right]\geqq 0$$

for all $t \geqq 0$ and $h > 0$. As χ is a measurable function defined on the interval $[0, \infty)$, it is C_k by definition C of section 2.

If we consider only integer values of t in theorem 3.1, we may drop the assumption that F is infinitely divisible without affecting the proof. The C_k character of χ on the integers now follows from $\Delta_1^{k+1}\,\chi \geqq 0$ by definition E of section 2. Specializing to the case where F_0 is degenerate at 0 we obtain

Corollary 3.1

Every family F_n, $n = 1, 2, \ldots$, of n-fold convolutions of a distribution function F_1 having $F_1(-0) = 0$ preserves convexity of every order.

We note that the fact that F_n preserves convexity or order k was proved by S. KARLIN and F. PROSCHAN in [2] under the additional assumption that F_1 possesses a density p that is a Pólya frequency density of order $k+2$ (i.e. $p(x-y)$ is TP_{k+2} in x and y).

Another special case of theorem 3.1 is obtained by assuming F to be degenerate at 1, in which case the theorem reduces to:

Every location parameter family $F_\theta(x) = G(x-\theta)$, $-\infty < \theta < \infty$, preserves convexity of every order.

This result is of course rather trivial. Without invoking theorem 3.1, it follows at once from

$$\Delta_h^{k+1}\,\chi(\theta)=\Delta_h^{k+1}\int g(x+\theta)\,dG(x)=\int\Delta_h^{k+1}\,g(x+\theta)\,dG(x).$$

In the same manner one easily verifies:

Every scale parameter family $F_\theta(x) = G(x/\theta)$, $0 < \theta < \infty$, preserves convexity of every odd order. If moreover $G(-0) = 0$, then the family preserves convexity of all orders.

4. Invariant convexity preserving families

Let g_1, g_2, χ_1 and χ_2 be defined as in section 1. We shall say that the family F_θ is invariant convexity preserving if, whenever g_1 is non-decreasing and g_2 is convex with respect to g_1 on X, then χ_1 is non-decreasing and χ_2 is convex with respect to χ_1 on T. In terms of WCT-systems we may express this property by requiring that for every WCT-system of the form $\{1, g_1, g_2\}$ the corresponding system $\{1, \chi_1, \chi_2\}$ is also a WCT-system.

In the first place this definition asserts that the family F_θ preserves the monotonicity of g_1 and hence by lemma 1 the family is stochastically increasing; F_θ also preserves convexity (of order 1) provided that the parameter is subjected to a suitable non-decreasing transformation

$$\eta = \eta(\theta) = \int x \, dF_\theta(x).$$

Moreover, this convexity preserving property is invariant under non-decreasing transformations g_1 of the random variable, the appropriate monotone transformation of θ then becoming χ_1. It is precisely because of this invariance that we do not require that F_θ be convexity preserving with respect to θ itself ,i.e. that η be linear in θ. This property would be destroyed by non-linear transformations g_1 anyway and would only result in fixing a possibly awkward parametrization.

From the general result of section 1 it follows that F_θ is invariant convexity preserving if the density p is TP_3. The following theorem provides a necessary and sufficient condition.

Theorem 4.1

Define $\bar{F}_\theta(x) = 1 - F_\theta(x)$. The family F_θ is invariant convexity preserving if and only if $\{1, \bar{F}_\theta(x_1), \bar{F}_\theta(x_2)\}$ is a WCT-system on T for every fixed pair $x_1 < x_2$.

Proof

The condition asserts that for $x_1 < x_2$ and $\theta_0 < \theta_1 < \theta_2$,

$$\begin{vmatrix} 1 & \bar{F}_{\theta_0}(x_1) \\ 1 & \bar{F}_{\theta_1}(x_1) \end{vmatrix} \geqq 0, \quad \begin{vmatrix} 1 & \bar{F}_{\theta_0}(x_1) & \bar{F}_{\theta_0}(x_2) \\ 1 & \bar{F}_{\theta_1}(x_1) & \bar{F}_{\theta_1}(x_2) \\ 1 & \bar{F}_{\theta_2}(x_1) & \bar{F}_{\theta_2}(x_2) \end{vmatrix} \geqq 0. \tag{4.1}$$

The first inequality means that F_θ is stochastically increasing and we have already remarked that this is necessary and sufficient for χ_1 to be non-decreasing whenever g_1 is. We may therefore assume that $\bar{F}_\theta(x)$ is non-decreasing in θ for every fixed x and restrict attention to the second inequality.

Let g_1 be non-decreasing and let $g_2 = f(g_1)$ where f is convex on $g_1(X)$. Since a convex function can be extended to a convex function on an interval, the same reasoning that we used in the proof of lemma 3.2 shows that we need only be concerned with functions f that are linear and functions f of the form

$$f(y) = \begin{cases} 0 & \text{for} \quad y \leqq y_0 \\ y - y_0 & \text{for} \quad y > y_0. \end{cases} \tag{4.2}$$

Without loss of generality we may assume that $y_0 = g_1(x_0) \in g_1(X)$. For linear f, χ_2 is linear and hence convex with respect to χ_1. Only functions f of the form (4.2) remain to be considered and as a result we have the following necessary and sufficient condition for a stochastically increasing family F_θ to be invariant convexity preserving:

For every non-decreasing g_1 and every $x_0 \in X$,

$$\chi_2(\theta) = \int_{x_0}^{\infty} [g_1(x) - g_1(x_0)]\, dF_\theta(x)$$

is convex with respect to $\chi_1(\theta)$.

By an approximation argument one shows that it is sufficient to consider only those functions g_1 that are left-continuous, non-decreasing step-functions assuming finitely many values. But then the above condition becomes:
For all $m = 1, 2, \ldots,$ all $x_1 < x_2 < \ldots < x_m$, all $\alpha_i > 0$, $i = 1, 2, \ldots, m$, all $1 \leq i_0 \leq m$ and all c,

$$\sum_{i=i_0}^{m} \alpha_i \bar{F}_\theta(x_i) \tag{4.3}$$

is convex with respect to

$$\sum_{i=1}^{m} \alpha_i \bar{F}_\theta(x_i) + c. \tag{4.4}$$

Since (4.4) is non-decreasing in θ and (4.3) is constant on any set where (4.4) is constant, the determinantal convexity definition (2.11) for $k = 1$ applies. By subtracting from the second column in this determinant we find that convexity of (4.3) with respect to (4.4) is equivalent to

$$\begin{vmatrix} 1 & \sum_{i=1}^{i_0-1} \alpha_i \bar{F}_{\theta_0}(x_i) & \sum_{i=i_0}^{m} \alpha_i \bar{F}_{\theta_0}(x_i) \\ \vdots & & \vdots \\ 1 & \sum_{i=1}^{i_0-1} \alpha_i \bar{F}_{\theta_2}(x_i) & \sum_{i=i_0}^{m} \alpha_i \bar{F}_{\theta_2}(x_i) \end{vmatrix} = \tag{4.5}$$

$$= \sum_{i=1}^{i_0-1} \sum_{j=i_0}^{m} \alpha_i \alpha_j \begin{vmatrix} 1 & \bar{F}_{\theta_0}(x_i) & \bar{F}_{\theta_0}(x_j) \\ \vdots & & \vdots \\ 1 & \bar{F}_{\theta_2}(x_i) & \bar{F}_{\theta_2}(x_j) \end{vmatrix} \geq 0.$$

By choosing $i_0 = m = 2$ we find that condition (4.1) is necessary; since every term in (4.5) has $x_i < x_j$ it is also sufficient. This completes the proof of the theorem.

It may be of interest to compare the sufficient condition that F_θ possesses a TP_3 density $p(x, \theta)$ with the necessary and sufficient condition of the theorem. One easily shows that the TP_3 assumption for p implies that $\bar{F}_\theta(x)$ is TP_3, or

$$\begin{vmatrix} \bar{F}_{\theta_0}(x_0) & \bar{F}_{\theta_0}(x_1) \\ \bar{F}_{\theta_1}(x_0) & \bar{F}_{\theta_1}(x_1) \end{vmatrix} \geq 0, \quad \begin{vmatrix} \bar{F}_{\theta_0}(x_0) & \bar{F}_{\theta_0}(x_1) & \bar{F}_{\theta_0}(x_2) \\ \bar{F}_{\theta_1}(x_0) & \bar{F}_{\theta_1}(x_1) & \bar{F}_{\theta_1}(x_2) \\ \bar{F}_{\theta_2}(x_0) & \bar{F}_{\theta_2}(x_1) & \bar{F}_{\theta_2}(x_2) \end{vmatrix} \geq 0, \qquad (4.6)$$

for $x_0 < x_1 < x_2$ and $\theta_0 < \theta_1 < \theta_2$. By letting x_0 tend to $-\infty$ we see that (4.6) implies (4.1). Hence the condition that $\bar{F}_\theta(x)$ be TP_3 is also sufficient for F_θ to be invariant convexity preserving.

If we restrict ourselves to the special case where the parameter set T is an interval and $\bar{F}_\theta(x)$ is differentiable with respect to θ, it turns out that theorem 4.1 involves a TP_2 instead of a TP_3 condition.

Theorem 4.2

Let T be an interval and let $q(x,\theta) = (\partial/\partial\theta)\bar{F}_\theta(x)$ be defined on T for all x. Then the family F_θ is invariant convexity preserving if and only if q is TP_2.

Proof

The first inequality in (4.1) is equivalent to $q \geq 0$. Since $\bar{F}_\theta(x_2)$ is constant on any set where $\bar{F}_\theta(x_1)+\bar{F}_\theta(x_2)$ is constant and the latter is non-decreasing in θ, the second inequality of (4.1) asserts that $\bar{F}_\theta(x_2)$ is convex with respect to $\bar{F}_\theta(x_1)+\bar{F}_\theta(x_2)$. This in turn is equivalent to $q(x_1,\theta_1)\,q(x_2,\theta_2)-q(x_1,\theta_2)\,q(x_2,\theta_1) \geq 0$ for $x_1 < x_2$ and $\theta_1 < \theta_2$.

It is tempting to ask whether theorem 4.2 can be generalized. One conceivable generalization would deal with invariant C_k preserving families F_θ, i.e. families for which χ_1 is non-decreasing and χ_2 is C_k with respect to χ_1 whenever g_1 is non-decreasing and g_2 is C_k with respect to g_1. However, even a cursory inspection shows that only trivial examples of such families exist. The necessary requirement that χ_2 be a polynomial in χ_1 of degree at most k whenever g_2 is a polynomial in g_1 of degree at most k, can not be satisfied for every non-decreasing g_1 except in a trivial manner.

A more promising generalization is to consider families F_θ that transform WCT-systems $\{1, g_1, \ldots, g_{k+1}\}$ into WCT-systems $\{1, \chi_1, \ldots, \chi_{k+1}\}$. If one restricts attention to the case where X and T are intervals and g_i and F_θ satisfy certain regularity conditions, one shows in a fairly straightforward manner that a necessary and sufficient condition on F_θ is that q be TP_{k+1}, thus generalizing lemma 3.1 and theorem 4.2 to the case where $k \geq 2$. We may conclude that although something may be lost for $k \geq 2$, the basic reason that theorems 4.1 and 4.2 work is not the fact that $k = 1$ in that case, but that $g_0 \equiv 1$ and that F_θ are probability distribution functions.

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