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Note on a paper by H.G. Tucker

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NOTE ON A PAPER BY H. G. TUCKER[1]

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0. Summary. The purpose of this note is to indicate a direct and natural way of proving theorems stated in [4] by using an explicit expression for the sequence of normalizing constants belonging to a distribution function attracted to a stable law. This results in a remark concerning a counter example given by Tucker and a slightly sharpened version of his Lemma 5.

1. Determination of normalizing constants. For a positive function f on the real line with $f(\infty) = \infty$ we define

$$f^*(x) = \inf\{y \mid f(y) \geqq x\} \qquad \text{for} \quad x > 0. \tag{1}$$

This is an extension of the concept of the inverse function. We mention the following property.

LEMMA 1. *Let ϕ_1 and ϕ_2 be measurable regularly varying functions* (see definition in [4]) *with exponent $\rho > 0$, then $\phi_1{}^*$ and $\phi_2{}^*$ are regularly varying with exponent ρ^{-1}. For any c with $0 \leqq c \leqq \infty$ we have*

$$\phi_1(x)/\phi_2(x) \to c \qquad \textit{for} \quad x \to \infty \tag{2}$$

if and only if

$$\phi_1{}^*(x)/\phi_2{}^*(x) \to c^{-1/\rho} \qquad \textit{for} \quad x \to \infty. \tag{3}$$

Following Tucker we write $F \in D(\alpha)$ when the distribution function F is in the domain of attraction of a stable law G_α of characteristic exponent α, i.e. if for suitably chosen constants $B_n > 0$ and A_n the n-fold convolutions F^{n*} of F satisfy $\lim_{n\to\infty} F^{n*}(B_n\{x + A_n\}) = G_\alpha(x)$ for every x. The numbers B_n are called *normalizing coefficients.*

LEMMA 2.[2]

(a) *If $F \in D(\alpha)$ $(0 < \alpha < 2)$ then*

$$B_n \sim c \inf\{x \mid 1 - F(x) + F(-x-0) \leqq 1/n\} \qquad \textit{for} \quad n \to \infty. \tag{4}$$

(b) *If $F \in D(\alpha)$ $(0 < \alpha \leqq 2)$ then*

$$B_n \sim c \inf\{x \mid x^{-2} \textstyle\int_{-x}^{x} t^2\, dF(t) \leqq 1/n\} \qquad \textit{for} \quad n \to \infty. \tag{5}$$

PROOF. As

$$\phi(x) = \sup_{a < s \leqq x} s^2\{\textstyle\int_{-s}^{s} t^2\, dF(t)\}^{-1} \tag{6}$$

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[2] Relation (4) is identical with (12) Chapter 7 Section 25 of [2].

which is a nondecreasing function for sufficiently large a and $x \geqq a$, satisfies (slightly generalizing the argument in [3])

$$\phi(x) \sim x^2\{\int_{-x}^{x} t^2\, dF(t)\}^{-1} \qquad \text{for} \quad x \to \infty, \tag{7}$$

ϕ is regularly varying with exponent α. From this it follows

$$\phi(x-0)/\phi(x+0) \to 1 \qquad \text{for} \quad x \to \infty. \tag{8}$$

It is easy to verify that

$$\phi(\phi^*(x)-0) \leqq x \leqq \phi(\phi^*(x)+0). \tag{9}$$

Combining (8) and (9) we obtain $\phi(\phi^*(x)) \sim x$ for $x \to \infty$. From this it follows using (7)

$$n\alpha_n^{-2} \int_{-\alpha_n}^{\alpha_n} t^2\, dF(t) \to 1 \qquad \text{for} \quad n \to \infty, \text{ with} \tag{10}$$

$$\alpha_n = \inf\{x \mid x^{-2} \int_{-x}^{x} t^2\, dF(t) \leqq 1/n\}.$$

Relation (10) is identical with (8.14) page 304 of [1] and so the α_n are normalizing constants for F. As two sequences of normalizing coefficients are asymptotically equal except for a multiplicative constant we have proved (5). The first part of the lemma is an immediate consequence of Theorem 2 page 275 of [1] and Lemma 1.

2. Correspondence between F and $\{B_n\}$. Lemma 5 of [4] states that a sequence of positive numbers is a sequence of normalizing constants for an $F \in D(\alpha)$ $(0 < \alpha < 2)$ iff

$$B_n \sim \phi(n) \qquad \text{for} \quad n \to \infty \tag{11}$$

where $\phi(x)$ is a regularly varying function with exponent α^{-1}. Clearly (11) implies

$$B_n^{-1} B_{nm} \to m^{1/\alpha} \qquad \text{for} \quad n \to \infty \quad \text{and} \quad m = 1, 2, 3 \cdots. \tag{12}$$

Tucker gives an example of a sequence $\{B_n\}$ satisfying (12) and not (11). This example might be somewhat misleading as becomes apparent from the following observation.

A sequence of normalizing constants $\{B_n\}$ is always asymptotically equivalent to a monotone sequence of such coefficients (as is shown in Lemma 2). If we assume (12) for a sequence of positive numbers $\{B_n\}$ asymptotically equivalent to a monotone sequence $\{B_n'\}$ then (11) holds for $\phi(x) = B'_{[x]}$ (where ϕ is regularly varying with exponent α^{-1}) as can be seen from the next lemma. The sequence in Tucker's example is *not* asymptotically equivalent to a monotone sequence and this tends to confuse the point just made.

LEMMA 3. *If for a positive nondecreasing function ϕ defined on an interval (a, ∞) and a constant $\rho \geqq 0$*

$$\lim_{n\to\infty} \phi(nm)/\phi(n) = m^\rho \qquad \textit{for} \quad m = 1, 2, 3, \cdots, \tag{13}$$

ϕ is regularly varying with exponent ρ.

PROOF. We first prove

$$\phi(n+1)/\phi(n) \to 1 \qquad \text{for} \quad n \to \infty. \tag{14}$$

If (14) does not hold, we can select a sequence $\{k_r\}$ of positive integers such that $\lim_{r\to\infty} \phi(k_r+1)/\phi(k_r) = c > 1$ ($c \leqq \infty$). We choose m such that $1 \leqq ((m+1)/m)^\rho < c$; take $n_r = [k_r m^{-1}]$, then by (13)

$$c > ((m+1)/m)^\rho = \lim_{r\to\infty} \phi(n_r(m+1))/\phi(n_r m)$$
$$= \lim_{r\to\infty} \prod_{k=n_r m}^{n_r(m+1)-1} \phi(k+1)/\phi(k) \geqq \lim_{r\to\infty} \phi(k_r+1)/\phi(k_r) = c,$$

hence (14) is true.

Given $x > 0$ and $\varepsilon > 0$ we choose positive integers m and r such that

$$x-\varepsilon < m/r < x < (m+1)/r < x+\varepsilon. \tag{15}$$

Defining for real $t > 0$, $n_t = [tr^{-1}]$ we have

$$\phi(n_t m)/\phi((n_t+1)r) \leqq \phi(tx)/\phi(t) \leqq \phi((n_t+1)(m+1))/\phi(n_t r). \tag{16}$$

Combining (13), (14), (16) and (15) we find

$$(x-\varepsilon)^\rho \leqq \liminf_{t\to\infty} \phi(tx)/\phi(t) \leqq \limsup_{t\to\infty} \phi(tx)/\phi(t) \leqq (x+\varepsilon)^\rho.$$

Hence we have $\lim_{t\to\infty} \phi(tx)/\phi(t) = x^\rho$.

REMARK. It suffices to require (13) for two integers m_1 and m_2 for which $\log m_1/\log m_2$ is irrational, e.g. $m_1 = 2$ and $m_2 = 3$. The proof is simpler when one requires (13) for all m.

Using the Lemmas 1, 2 and 3 and

$$\left\{\frac{1}{1-F(x)}\right\}^{**} \sim \frac{1}{1-F(x)} \qquad \text{for} \quad x \to \infty,$$

we can restate Tucker's Lemma 3 in the following way.

LEMMA 4. (a) *Call two distribution functions F_1 and F_2 equivalent if for a c with $0 < c < \infty$*

$$1-F_1(x)+F_1(-x-0) \sim c\{1-F_2(x)+F_2(-x-0)\} \quad \textit{for} \quad x \to \infty$$

and call two sequences of positive numbers $\{B_n\}$ and $\{B_n'\}$ equivalent if for a c with $0 < c < \infty$

$$B_n \sim cB_n' \qquad \textit{for} \quad n \to \infty.$$

For each α with $0 < \alpha < 2$ there is a one-to-one correspondence between the equivalence classes of distribution functions F from $D(\alpha)$ and those equivalence classes of sequences of positive numbers $\{B_n\}$, which contain a nondecreasing sequence satisfying (12) (*then every sequence in the equivalence class satisfies* (12)). *The correspondence is: $\{B_n\}$ is a sequence of normalizing constants for F.*

(b) *Call two distribution functions F_1 and F_2 equivalent if for a c with* $0 < c < \infty$

$$\int_{-x}^{x} t^2\, dF_1(t) \sim c \int_{-x}^{x} t^2\, dF_2(t) \qquad \textit{for} \quad x \to \infty$$

and call two sequences of positive numbers $\{B_n\}$ *and* $\{B_n'\}$ *equivalent if for a c with* $0 < c < \infty$

$$B_n \sim cB_n' \qquad \textit{or} \quad n \to \infty.$$

There is a one-to-one correspondence between the equivalence classes of distribution functions F from $D(2)$ *and those equivalence classes of sequences of positive numbers* $\{B_n\}$ *which contain a nondecreasing sequence satisfying* (12) *with* $\alpha^{-1} = 0$ (*then every sequence in the equivalence class satisfies* (12)). *The correspondence is:* $\{n^{\frac{1}{2}}B_n\}$ *is a sequence of normalizing constants for F.*

It is not difficult to give a direct proof of the statement about normalizing constants contained in Tucker's Theorem 2 based on Lemmas 1 and 2 in this note.

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