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A Form of Regular Variation and Its Application to the Domain of Attraction of the Double Exponential Distribution*

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Introduction

In 1930 Karamata introduced the concept of regular variation of a positive function at infinity. He found a striking characterisation of the class of regularly varying functions. Several related forms of regular behaviour at infinity can be defined. At first a new type of regular behaviour is studied in Section 1.

The results of this section are applied to a problem in extreme value theory: we consider a sequence of independent random variables

$$X_1, X_2, X_3, \ldots$$

with the same distribution function $F(x)$ and define

$$Y_n = \max(X_1, X_2, \ldots, X_n).$$

Then

$$P\{Y_n \leqq x\} = F^n(x).$$

A distribution function F is said to belong to the domain of attraction of a non-degenerate distribution function G (notation $F \in D(G)$) when there exist constants $a_n > 0$ and b_n such that

$$F^n(a_n x + b_n) \to G(x)$$

weakly. Gnedenko [2] proved in 1943 that only three types of distribution functions have non-empty domains of attraction. In addition he gave characterizations of their domains of attraction, but he remarked that the characterization of the domain of attraction of the third type

$$\Lambda(x) = \exp(-e^{-x})$$

cannot be regarded as final and simple enough for applications. In 1949 Mezjler [6] gave another characterization in terms of the inverse function of F. In this paper we give a comparatively simple characterization of $D(\Lambda)$ involving only the distribution function itself. Furthermore we show that this criterion can be used also to characterize the domains of attraction of the two other limit types.

1. A Kind of Regular Variation

First we give the main results of Karamata's papers about regular variation ([5] and [6]).

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Definition 1. A positive function U, defined on $(0, \infty)$ is *regularly varying* at infinity when

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^{\rho} \tag{1}$$

for all $x>0$. The real number ρ is called the *exponent of regularity*. When $\rho=0$ U is called *slowly varying* at infinity.

Theorem 1. *For a positive function U defined on* $(0, \infty)$ *and summable on finite intervals the following assertions are equivalent.*

a) *U is regularly varying with exponent* $\rho > -1$,

b)
$$\lim_{x\to\infty} \frac{x \cdot U(x)}{\int_0^x U(t)\,dt} = \rho + 1 > 0. \tag{2}$$

c) *There exist real functions* $c(x)$ *and* $a(x)$ *with*

$$\begin{aligned} &\lim_{x\to\infty} c(x) = c \qquad (0<c<\infty) \\ &\lim_{x\to\infty} a(x) = \rho > -1 \end{aligned} \tag{3}$$

such that

$$U(x) = c(x) \cdot \exp\left\{\int_1^x \frac{a(t)}{t}\,dt\right\}. \tag{4}$$

Remark 1. For $\rho < -1$ the theorem holds with (2) replaced by

$$\lim_{x\to\infty} \frac{x\,U(x)}{\int_x^\infty U(t)\,dt} = -\rho - 1 > 0. \tag{5}$$

Remark 2. It can be proved that the summability of U on finite subintervals of some terminal interval (a, ∞) is implied by the regular variation and the measurability of U (see e.g. [1]). Hence for measurable U a slightly different form of Theorem 1 holds.

Corollary 1. *If U is regularly varying with exponent* $\rho > -1$

$$U_1(x) = \int_0^x U(t)\,dt$$

is regularly varying with exponent $\rho+1$. *If U is regularly varying with exponent* $\rho < -1$

$$U_2(x) = \int_x^\infty U(t)\,dt$$

is regularly varying with exponent $\rho+1$.

Definition 1 can be extended to $\rho = \pm\infty$. We define for $x>0$

$$x^\infty = \begin{cases} 0 & \text{for } x<1 \\ 1 & \text{for } x=1 \\ \infty & \text{for } x>1 \end{cases}$$

and

$$x^{-\infty}=\begin{cases}\infty & \text{for } x<1\\ 1 & \text{for } x=1\\ 0 & \text{for } x>1.\end{cases}$$

Definition 2. A positive function U defined on $(0, \infty)$ is *rapidly varying* at infinity when

$$\lim_{t\to\infty}\frac{U(tx)}{U(t)}=x^{\rho}$$

for all $x>0$ with $\rho=\pm\infty$.

For rapidly varying functions we have a much weaker version of Theorem 1.

Theorem 2. *A non-decreasing positive function U defined on* $(0, \infty)$ *is rapidly varying with $\rho=\infty$ iff*

$$\lim_{x\to\infty}\frac{x\,U(x)}{\int_0^x U(t)\,dt}=\infty. \tag{6}$$

Proof. a) Suppose that U is rapidly varying with $\rho=+\infty$. By Lebesque's theorem on dominated convergence

$$\lim_{x\to\infty}\frac{\int_0^x U(t)\,dt}{x\,U(x)}=\lim_{x\to\infty}\int_0^1\frac{U(xt)}{U(x)}\,dt=\int_0^1\lim_{x\to\infty}\frac{U(xt)}{U(x)}\,dt=0.$$

b) On the other hand if there exists a value of $t\,(0<t<1)$ and a sequence $x_n\to\infty$ such that

$$\lim_{n\to\infty}\frac{U(x_n t)}{U(x_n)}=c>0, \tag{7}$$

then in view of the monotonicity of the lefthand part of (7) as a function of t

$$\liminf_{n\to\infty}\int_0^1\frac{U(x_n s)}{U(x_n)}\,ds>0.$$

This contradicts (6).

Remark 3. For $\rho=-\infty$ Theorem 2 holds with (6) replaced by

$$\lim_{x\to\infty}\frac{x^{-1}U(x)}{\int_x^\infty U(t)\frac{dt}{t^2}}=\infty.$$

A related form of Karamata's theorem (Theorem 1) is given in the next theorem.

Theorem 3. *For a real-valued function V defined on* $(0, \infty)$ *and summable on finite intervals the following assertions are equivalent.*

a) *For every* $a>0$

$$\lim_{x\to\infty}\{V(ax)-V(x)\}=\rho\log a \tag{8}$$

where ρ *is a real constant.*

b)

$$\lim_{x\to\infty}\left\{V(x)-\frac{1}{x}\int_0^x V(t)\,dt\right\}=\rho. \tag{9}$$

c) *There exist real functions* $c(x)$ *and* $a(x)$ *with*

$$\begin{aligned}&\lim_{x\to\infty}c(x)=c \qquad (-\infty<c<\infty)\\ &\lim_{x\to\infty}a(x)=\rho\end{aligned} \tag{10}$$

such that

$$V(x)=c(x)+\int_0^x\frac{a(t)}{t}\,dt. \tag{11}$$

Proof. Relation (8) holds iff

$$U(x)=\exp\{V(x)\}$$

is regularly varying at infinity with exponent ρ. Hence the equivalence of a) and c) is contained in Theorem 1. The implication c) $\Rightarrow$ b) is a matter of standard calculation. For the proof of the implication b) $\Rightarrow$ c) we define

$$g(x)=V(x)-\frac{1}{x}\int_0^x V(t)\,dt. \tag{12}$$

Then

$$\begin{aligned}\int_1^x\frac{g(t)}{t}\,dt&=\int_1^x\frac{V(t)}{t}\,dt+\frac{1}{x}\int_0^x V(t)\,dt-\int_0^1 V(t)\,dt-\int_1^x\frac{V(t)}{t}\,dt\\ &=-\int_0^1 V(t)\,dt+V(x)-g(x),\end{aligned}$$

hence

$$V(x)=\int_0^1 V(t)\,dt+g(x)+\int_1^x\frac{g(t)}{t}\,dt. \tag{13}$$

Remark. If (8) holds with $\rho>0$, V is slowly varying at infinity.

The next theorem can be seen as an attempt to characterize a subclass of the class of slowly varying functions with functions which behave even more regularly.

Theorem 4. *For a real-valued strictly increasing function* V *which is defined on* $(0,\infty)$ *the following assertions are equivalent.*

a) *For every positive* a *and* $b\neq 1$

$$\lim_{x\to\infty}\frac{V(ax)-V(x)}{V(bx)-V(x)}=\frac{\log a}{\log b}. \tag{14}$$

b) *The function*

$$V(x)-\frac{1}{x}\int_0^x V(t)\,dt \tag{15}$$

is slowly varying at infinity.

c) *There exists a slowly varying function g such that*

$$V(x)=c+g(x)+\int_1^x \frac{g(t)}{t}\,dt. \tag{16}$$

d) *For every* $a>0$

$$\lim_{x\to\infty}\frac{V(a\,x)-V(x)}{V(x)-\frac{1}{x}\int_0^x V(t)\,dt}=\log a. \tag{17}$$

Proof. a) ⇒ b). Writing $(b>0, 0<a<1)$

$$\frac{V(b\,x)-V(a\,b\,x)}{V(x)-V(a\,x)}=\frac{V(x)-V(a\,b\,x)}{V(x)-V(a\,x)}-\frac{V(x)-V(b\,x)}{V(x)-V(a\,x)}$$

and using (14) we see that the function

$$h(x)=V(x)-V(a\,x)$$

is slowly varying for every $0<a<1$.

By Theorem 1 this implies

$$\lim_{x\to\infty}\frac{\frac{1}{x}\int_0^x V(t)\,dt-\frac{1}{a\,x}\int_0^{a\,x} V(t)\,dt}{V(x)-V(a\,x)}=1,$$

hence

$$\lim_{x\to\infty}\left\{\frac{V(x)-\frac{1}{x}\int_0^x V(t)\,dt}{V(x)-V(a\,x)}-\frac{V(a\,x)-\frac{1}{a\,x}\int_0^{a\,x} V(t)\,dt}{V(x)-V(a\,x)}\right\}=0. \tag{18}$$

By Fatou's lemma we have

$$\liminf_{x\to\infty}\frac{V(x)-\frac{1}{x}\int_0^x V(t)\,dt}{V(x)-V(a\,x)}\geq\int_0^1 \liminf_{x\to\infty}\frac{V(x)-V(t\,x)}{V(x)-V(a\,x)}\,dt=\int_0^1\frac{\log t}{\log a}\,dt>0. \tag{19}$$

Combining (18) and (19) we obtain

$$\lim_{x\to\infty}\frac{\left\{V(x)-\frac{1}{x}\int_0^x V(t)\,dt\right\}-\left\{V(a\,x)-\frac{1}{a\,x}\int_0^{a\,x} V(t)\,dt\right\}}{V(x)-\frac{1}{x}\int_0^x V(t)\,dt}=0.$$

b) ⇒ c). Defining

$$g(x)=V(x)-\frac{1}{x}\int_0^x V(t)\,dt$$

we get (see (13))

$$V(x)=c+g(x)+\int_1^x \frac{g(t)}{t}\,dt.$$

c) ⇒ d). For $a>1$ we have

$$\frac{V(a\,x)-V(x)}{V(x)-\frac{1}{x}\int_0^x V(t)\,dt}=\frac{g(a\,x)}{g(x)}-1+\int_1^a \frac{g(t\,x)}{g(x)}\,\frac{dt}{t}. \tag{20}$$

Since the relation

$$\lim_{x\to\infty}\frac{g(t\,x)}{g(x)}=1.$$

holds uniformly on $[1, a]$ (see [1]), we obtain (17) by letting $x\to\infty$ in (20).
d) ⇒ a). Trivial.

Remark. The requirement that V is strictly increasing is only used to ensure the finiteness of the expression to the right of the lim sign in (14) and may be replaced by the requirement: for each $a>1$ there exists a $M(a)$ such that for $x>M(a)$

$$V(a\,x)-V(x)>0.$$

Remark. It is not difficult to show that a positive function V satisfying (14) is slowly varying at infinity. It is not true that (14) implies (8) for a $\rho\in[0, \infty]$.

Corollary 2. *Theorem* 4 *remains valid if we replace everywhere* $\lim\limits_{x\to\infty}$ *by* $\lim\limits_{x\downarrow 0}$ *(of course now "slowly varying" has to be read as "slowly varying at* $x=0$*").*

Proof. If $h(x)$ is slowly varying at $x=0$ i.e. if for each $x>0$

$$\lim_{t\downarrow 0}\frac{h(t\,x)}{h(t)}=1,$$

then $x^{-2}\,h\left(\frac{1}{x}\right)$ is regularly varying with $\rho=-2$ and by Remark 1

$$\lim_{x\downarrow 0}\frac{x\,h(x)}{\int_0^x h(t)\,dt}=\lim_{y\to\infty}\frac{h(y)}{y\int_y^\infty h\left(\frac{1}{t}\right)\frac{dt}{t^2}}=1.$$

The remainder is easy.

2. Preliminaries

First we list some well-known results on the domain of attraction of the double exponential law used in the sequel (cf. [3]).

Lemma 1. *Let* $\{F_n\}$ *be a sequence of distribution functions. Suppose that there exist sequences of real numbers* $\{a_n\}$ *and* $\{b_n\}$ *with*

$$a_n>0 \qquad \textit{for}\;\; n=1, 2, 3, \dots,$$

such that

$$\lim_{n\to\infty} F_n(a_n x + b_n) = G(x) \tag{21}$$

weakly where G is non-degenerate distribution function.

Then

$$\lim_{n\to\infty} F_n(\alpha_n x + \beta_n) = G^*(x) \tag{22}$$

holds with G^* non-degenerate and real numbers $\alpha_n > 0$ and β_n iff

$$\lim_{n\to\infty} \frac{\alpha_n}{a_n} = A > 0, \qquad \lim_{n\to\infty} \frac{\beta_n - b_n}{a_n} = B \tag{23}$$

and

$$G^*(x) = G(Ax + B). \tag{24}$$

We say that a distribution function F belongs to the domain of attraction of a non-degenerate distribution function G (notation $F \in D(G)$) if for suitably chosen constants $a_n > 0$ and b_n

$$\lim_{n\to\infty} F^n(a_n x + b_n) = G(x) \tag{25}$$

for all continuity points x of G.

Theorem 5. *A distribution function F can belong only to the domain of attraction of one of the following types of distribution functions:*

$$\phi_\alpha(x) = \begin{cases} 0 & \text{for } x \leqq 0 \\ \exp(-x^{-\alpha}) & \text{for } x > 0. \end{cases} \tag{26}$$

$$\psi_\alpha(x) = \begin{cases} \exp(-(-x)^\alpha) & \text{for } x \leqq 0 \\ 1 & \text{for } x > 0. \end{cases} \tag{27}$$

$$\Lambda(x) = \exp(-e^{-x}). \tag{28}$$

In (26) *and* (27) α *is a positive constant.*

In the sequel we use the notation

$$x_0 = x_0(F) = \sup\{x \mid F(x) < 1\} \leqq \infty. \tag{29}$$

Theorem 6. a) *The distribution function F belongs to the domain of attraction of $\phi_\alpha(x)$ iff $1 - F(x)$ is regularly varying with exponent $-\alpha$.*

b) *The distribution function F belongs to the domain of attraction of $\psi_\alpha(x)$ iff $x_0 < \infty$ and $1 - F\left(x_0 - \frac{1}{x}\right)$ is regularly varying with exponent $-\alpha$.*

Theorem 7. *The distribution function F belongs to the domain of attraction of $\Lambda(x)$ iff*

$$\lim_{n\to\infty} F^n(a_n x + b_n) = \Lambda(x)$$

or equivalently

$$\lim_{n\to\infty} n\{1 - F(a_n x + b_n)\} = e^{-x} \tag{30}$$

with

$$b_n = \inf\left\{x \mid 1-F(x) \geqq \frac{1}{n}\right\}$$
$$a_n = \inf\left\{x \mid 1-F(x) \geqq \frac{1}{n\,e}\right\} - b_n. \tag{31}$$

Remark. It can be seen (cf. [4]) that the choice (31) for the stabilizing coefficients also holds for distribution functions attracted to the other limit types.

Theorem 8. *The distribution function F belongs to the domain of attraction of $\Lambda(x)$ iff it is possible to choose a function A with*

$$z\,A(z) > 0 \quad when \quad z \neq 0$$

and

$$\lim_{z \uparrow x_0} A(z) = 0 \tag{32}$$

such that for every x

$$\lim_{z \uparrow x_0} \frac{1-F(z+z\,A(z)\,x)}{1-F(z)} = e^{-x}. \tag{33}$$

Remark. In Gnedenko's paper the continuity of A is required but in his proof this property is not used.

The next theorem is due to von Mises ([8]). The theorem is extended in Gnedenko's paper.

Theorem 9. *Suppose that the distribution function F is twice differentiable. Define*

$$f(x) = \frac{F'(x)}{1-F(x)}. \tag{34}$$

If

$$\lim_{x \uparrow x_0} \frac{d}{dx}\left(\frac{1}{f(x)}\right) = 0, \tag{35}$$

then $F \in D(\Lambda)$.

3. The Domain of Attraction of Λ

Lemma 2. *If $F \in D(\Lambda)$, there exists a continuous and strictly increasing distribution function G such that*

$$1-F(x) \sim 1-G(x) \quad for \quad x \uparrow x_0. \tag{36}$$

Proof. Suppose first $x_0 = \infty$. By Theorem 8 there is a positive function A with

$$\lim_{z \to \infty} A(z) = 0.$$

such that for all x

$$\lim_{z \to \infty} \frac{1-F(z+z\,A(z)\,x)}{1-F(z)} = e^{-x}. \tag{37}$$

Both sides of (37) are monotone functions of x hence (37) holds uniformly for $0 \leqq x \leqq 1$. Taking $x(z) = z^{-1}$ we obtain

$$\lim_{z \to \infty} \frac{1 - F(z + A(z))}{1 - F(z)} = 1. \tag{38}$$

From this we see

$$\lim_{z \to \infty} \frac{1 - F(z)}{1 - F(z - 0)} = 1. \tag{39}$$

Let $\{z_n\}_{n=1}^{\infty}$ be an enumeration of the points of discontinuity of F. We define $F_1(x)$ in the following way: $F_1(z_1) = F(z_1 - 0)$, $F_1(z)$ linear on $[z_1, z_1 + A(z_1)]$ with $F_1(z_1 + A(z_1)) = F(z_1 + A(z_1))$ and $F_1(z) = F(z)$ when $z \notin [z_1, z_1 + A(z_1)]$.

Now $F_2(z)$ equals $F_1(z)$ if $z_1 < z_2 \leqq z_1 + A(z_1)$; otherwise $F_2(z)$ is constructed by putting $F_2(z_2) = F_1(z_2 - 0)$ and making F_1 linear on $[z_2, z_2 + A(z_2)]$ or $[z_2, z_1]$ if $z_2 < z_1 \leqq z_2 + A(z_2)$. In this way we construct a sequence of distribution functions F_n. As the intervals I_n where the function is changed are disjoint, $H(z) = \lim_{n \to \infty} F_n(z)$ exists and is a continuous distribution function. If $H(z) \neq F(z)$ for a z then there exists an n such that $H(z) = F_n(z)$ and $H(z) \neq F_k(z)$ for $k < n$, so

$$F(z_n - 0) \leqq H(z) \leqq F(z_n + A(z_n))$$

and

$$\frac{1 - F(z_n - 0)}{1 - F(z_n + A(z_n))} \geqq \frac{1 - H(z)}{1 - F(z)} \geqq \frac{1 - F(z_n + A(z_n))}{1 - F(z_n)}.$$

Hence by (38) and (39)

$$\lim_{z \to \infty} \frac{1 - H(z)}{1 - F(z)} = 1. \tag{40}$$

In an analogous way we proceed to make the function strictly increasing. Let $\{u_n\}$ be an enumeration of the initial points of intervals where H is constant and $\{v_n\}$ the corresponding endpoints. The construction of a sequence H_n is analogous to the construction of the sequence F_n using now the intervals $[u_n, v_n + A(v_n)]$ instead of $[z_n, z_n + A(z_n)]$. The function $G(z) = \lim_{n \to \infty} H_n(z)$ is a continuous and strictly increasing distribution function and as above we see

$$\lim_{z \to \infty} \frac{1 - G(z)}{1 - H(z)} = 1. \tag{41}$$

If $x_0 < \infty$ we have

$$\lim_{z \uparrow x_0} \frac{1 - F(z + z\,A(z)\,x)}{1 - F(z)} = e^{-x}$$

with

$$z\,A(z) > 0$$

and

$$\lim_{z \uparrow x_0} A(z) = 0.$$

By taking $x(z)=x_0-z$ we obtain

$$\lim_{z\uparrow x_0}\frac{1-F(z+z\,A(z)(x_0-z))}{1-F(z)}=1. \tag{42}$$

As

$$z\,A(z)(x_0-z)>0$$

and

$$\lim_{z\uparrow x_0} z\,A(z)(x_0-z)=0,$$

relation (42) implies

$$\lim_{z\uparrow x_0}\frac{1-F(z)}{1-F(z-0)}=1;$$

the remainder of the proof is analogous to the proof for the case $x_0=\infty$.

Lemma 3. *Let $\{F_t\}$ be a family of distribution functions $(-\infty<t<t_0\leqq\infty)$. Suppose that there exist real-valued functions $a(t)>0$ and $b(t)$ such that*

$$\lim_{t\uparrow t_0} F_t(a(t)\,x+b(t))=G(x)$$

weakly, where G is a non-degenerate distribution function. Then

$$\lim_{t\uparrow t_0} F_t(\alpha(t)\,x+\beta(t))=G^*(x)$$

holds with G^* non-degenerate and real-valued functions $\alpha(t)>0$ and $\beta(t)$ iff

$$\lim_{t\uparrow t_0}\frac{\alpha(t)}{a(t)}=A>0,\qquad \lim_{t\uparrow t_0}\frac{\beta(t)-b(t)}{a(t)}=B$$

and

$$G^*(x)=G(A\,x+B).$$

Proof. Analogous to the proof of Lemma 2 (see Feller [2] p. 246).

Corollary 3. *If for a distribution function F*

$$\lim_{z\uparrow x_0}\frac{1-F(z+z\,A(z)\,x)}{1-F(z)}=e^{-x} \tag{43}$$

holds with

$$z\,A(z)>0$$

and

$$\lim_{z\uparrow x_0} A(z)=0,$$

then

$$\lim_{z\uparrow x_0}\frac{1-F(f(z)+z\,B(z)\,x)}{1-F(z)}=e^{-x-b} \tag{44}$$

holds iff

$$\left.\begin{aligned}&\lim_{z\uparrow x_0}\frac{B(z)}{A(z)}=1\\ &\lim_{z\uparrow x_0}\frac{f(z)-z}{A(z)}=b.\end{aligned}\right\} \tag{45}$$

Proof. Take in Lemma 3

$$F_t(x)=1-\frac{1-F(x)}{1-F(t)} \quad \text{and} \quad G(x)=1-e^{-x}.$$

The fact that $F_t(-\infty)>0$ is immaterial.

Lemma 4[1]. *If* $F\in D(\Lambda)$ *then for all* $a>1$

$$\lim_{y\downarrow 0}\frac{V(a^{-1}y)-V(y)}{V(e^{-1}y)-V(y)}=\log a \tag{46}$$

with for $0<y<1$

$$V(y)=\inf\{x\,|\,1-F(x)\geqq y\}.$$

Proof. We use Theorem 7 and make first the remark that if a_n in (30) is replaced by

$$\alpha_n=V\left(\frac{1}{a\,n}\right)-V\left(\frac{1}{n}\right)$$

for an arbitrary $a>1$ then

$$\lim_{n\to\infty} n\{1-F(\alpha_n x+b_n)\}=a^{-x}; \tag{47}$$

this can be seen by a simple adaptation of Gnedenko's proof.

Combination of (30) and (47) gives by Lemma 1

$$\lim_{n\to\infty}\frac{V\left(\frac{1}{n\,a}\right)-V\left(\frac{1}{n}\right)}{V\left(\frac{1}{n\,e}\right)-V\left(\frac{1}{n}\right)}=\log a. \tag{48}$$

Using the monotonicity of V we see from (48)

$$\lim_{n\to\infty}\frac{V\left(\frac{1}{n+1}\right)-V\left(\frac{1}{n}\right)}{V\left(\frac{1}{n\,e}\right)-V\left(\frac{1}{n}\right)}=0. \tag{49}$$

From (47) we obtain also

$$\lim_{n\to\infty} n\{1-F(\alpha_{n+1}x+b_{n+1})\}=a^{-x}.$$

Using Lemma 1 again we get

$$\lim_{n\to\infty}\frac{V\left(\frac{1}{(n+1)\,a}\right)-V\left(\frac{1}{n+1}\right)}{V\left(\frac{1}{n\,a}\right)-V\left(\frac{1}{n}\right)}=1. \tag{50}$$

[1] Cf. Mejzler [7].

Now defining

$$U(x)=V\left(\frac{1}{x}\right)$$

we see (using the entier function)

$$\frac{U(a[y])-U([y]+1)}{U(e[y])-U([y])}\leqq\frac{U(a\,y)-U(y)}{U(e[y])-U([y])}\leqq\frac{U(a([y]+1))-U([y])}{U(e[y])-U([y])}.$$

The righthand member is the same as

$$\frac{U(a([y]+1))-U([y]+1)}{U(e[y])-U([y])}+\frac{U([y]+1)-U([y])}{U(e[y])-U([y])}$$

and this by (48), (49) and (50) tends to $\log a$ as y tends to infinity. The lefthand side tends to the same limit so

$$\lim_{y\to\infty}\frac{U(a\,y)-U(y)}{U(e[y])-U([y])}=\log a \tag{51}$$

and this in combination with (48) gives the assertion of the lemma.

Lemma 5. *If* $F\in D(\Lambda)$ *then*

$$\lim_{z\uparrow x_0}\frac{1-F(z+z\,A(z)\,x)}{1-F(z)}=e^{-x} \tag{52}$$

for all x *with*

$$A(z)=\frac{\int_z^{x_0}\{1-F(t)\}\,dt}{z\{1-F(z)\}}. \tag{53}$$

Proof. By Lemma 2 and Corollary 3 we need only consider continuous and strictly increasing distribution functions.

By Lemma 4

$$\lim_{y\downarrow 0}\frac{V(a^{-1}\,y)-V(y)}{V(e^{-1}\,y)-V(y)}=\log a \tag{54}$$

for all $a>1$ with

$$V(y)=(1-F)^{-1}(y).$$

By Theorem 4 and Corollary 2 (used for $-V$) relation (54) is equivalent with

$$\lim_{y\downarrow 0}\frac{V(a^{-1}\,y)-V(y)}{\frac{1}{y}\int_0^y V(t)\,dt-V(y)}=\log a. \tag{55}$$

Taking $a=e$ and using Lemma 1 we see

$$\lim_{n\to\infty} n\{1-F(a_n\,x+b_n)\}=e^{-x}$$

with

$$b_n = V\left(\frac{1}{n}\right)$$

$$a_n = n \int_0^{1/n} V(t)\,dt - V\left(\frac{1}{n}\right).$$

Now it is not difficult to see (using (49), (51), (55) and Lemma 1) that

$$\lim_{y\downarrow 0} y^{-1}\{1-F(a(y)\,x+b(y))\} = e^{-x} \tag{56}$$

with

$$\begin{aligned} b(y) &= V(y) \\ a(y) &= \frac{1}{y}\int_0^y V(t)\,dt - V(y). \end{aligned} \tag{57}$$

Putting $y = 1 - F(z)$ in (56) and (57) we obtain (52) and (53).

Lemma 6. *If for a distribution function F*

$$\lim_{z\uparrow x_0} \frac{1-F(z+z\,A(z)\,x)}{1-F(z)} = e^{-x} \tag{58}$$

for all x with $z\,A(z) > 0$ for $z \neq 0$ and

$$\lim_{z\uparrow x_0} A(z) = 0,$$

then

$$\frac{1-F_1(z+z\,A(z)\,x)}{1-F_1(z)} = e^{-x} \tag{59}$$

where

$$F_1(x) = 1 - \int_x^{x_0} \{1-F(t)\}\,dt. \tag{60}$$

Proof. By the Lemma's 3 and 5

$$A(z) \sim \frac{\int_z^{x_0} \{1-F(t)\}\,dt}{z\{1-F(z)\}} \qquad \text{for } z\uparrow x_0. \tag{61}$$

In (58) we substitute $z = y + y\,A(y)\,s$ with s an arbitrary real number; then:

$$\lim_{y\uparrow x_0} \frac{1-F(y+y\,A(y)\,s+(y+y\,A(y)\,s)\,A(y+y\,A(y)\,s)\,x)}{1-F(y+y\,A(y)\,s)} = e^{-x}.$$

With (58) this becomes

$$\lim_{y\uparrow x_0} \frac{1-F(y+y\,A(y)\,s+(y+y\,A(y)\,s)\,A(y+y\,A(y)\,s)\,x)}{1-F(y)} = e^{-(x+s)}. \tag{62}$$

Applying Corollary 3 we obtain from (58) and (62)

$$\lim_{y\uparrow x_0} \frac{A(y+y\,A(y)\,s)}{A(y)}=1 \tag{63}$$

for all s.

Combining (58), (61) and (63) we obtain (59).

Lemma 7. *If for a positive non-decreasing function f defined on $(-\infty, x_0)$ with*

$$\lim_{z\uparrow x_0} f(x)=\infty$$

(where $x_0 \leqq \infty$), the relation

$$\lim_{z\uparrow x_0} \frac{f(z+z\,A(z)\,x)}{f(z)}=e^x \tag{64}$$

holds for all x and a properly chosen function A with

$$z\,A(z)>0 \quad \textit{for } z\neq 0$$

and

$$\lim_{z\uparrow x_0} A(z)=0,$$

then also

$$\lim_{z\uparrow x_0} \frac{f_1(z+z\,A(z)\,x)}{f_1(z)}=e^x \tag{65}$$

for all x where

$$f_1(x)=\int_{x_1}^{x} f(t)\,dt$$

and

$$x_1=\begin{cases} x_0-1 & \text{when } x_0<\infty \\ 1 & \text{when } x_0=\infty. \end{cases}$$

Proof. Analogous to the proof of Lemma 6. First we use Theorem 4 to find

$$A(z)\sim\frac{\int_{x_1}^{z} f(t)\,dt}{z\,f(z)} \qquad \text{for } z\uparrow x_0,$$

then by Lemma 3

$$\lim_{y\uparrow x_0} \frac{A(y+y\,A(y)\,s)}{A(y)}=1$$

for all s. As in the proof of Lemma 6 the assertion of this lemma follows.

Now we are able to prove the main theorem.

Theorem 10. *A distribution function F is in the domain of attraction of the double exponential distribution iff*

$$\lim_{x\uparrow x_0} \frac{\{1-F(x)\}\left\{\int_x^{x_0}\int_y^{x_0}\{1-F(t)\}\,dt\,dy\right\}}{\left\{\int_x^{x_0}\{1-F(t)\}\,dt\right\}^2}=1. \tag{66}$$

Proof. a) Suppose $F \in D(\Lambda)$. By Lemma 5 this is equivalent to

$$\lim_{z \uparrow x_0} \frac{1 - F(z + z\,A(z)\,x)}{1 - F(z)} = e^{-x} \tag{67}$$

with

$$A(z) = \frac{\int_z^{x_0} \{1 - F(t)\}\,dt}{z\{1 - F(z)\}}. \tag{68}$$

By Lemma 6 we know that

$$F_1(x) = 1 - \int_x^{x_0} \{1 - F(t)\}\,dt \tag{69}$$

also satisfies (67) and (68). But then by Lemma 5 $F_1(x)$ also satisfies (67) with

$$A_1(z) = \frac{\int_z^{x_0} \int_y^{x_0} \{1 - F(t)\}\,dt\,dy}{z \int_z^{x_0} \{1 - F(t)\}\,dt} \tag{70}$$

and by Lemma 3

$$A(z) \sim A_1(z) \qquad \text{for } z \uparrow x_0.$$

This is equivalent with (66).

b) Suppose that (66) holds. By Theorem 8 and Theorem 9 we know that the distribution function F_2, defined for sufficiently large values of x by

$$F_2(x) = 1 - \int_x^{x_0} \int_y^{x_0} \{1 - F(t)\}\,dt\,dy \tag{71}$$

satisfies

$$\lim_{z \uparrow x_0} \frac{1 - F_2(z + z\,A_2(z)\,x)}{1 - F_2(z)} = e^{-x} \tag{72}$$

with

and

$$z\,A_2(z) > 0 \qquad \text{for } z \neq 0$$
$$\lim_{z \uparrow x_0} A_2(z) = 0. \tag{73}$$

As the function

$$f(x) = \frac{1}{1 - F_2(x)}$$

satisfies (64), by Lemma 7 the function

$$f_1(x) = \int_{x_1}^{x} f(t)\,dt$$

also satisfies (64). But by (66) this function is asymptotically equivalent to

$$f_2(x)=\int_0^x \{1-F(t)\}\left\{\int_t^{x_0}(1-F(s))\,ds\right\}^{-2} dt \sim \left\{\int_x^{x_0}(1-F(t))\,dt\right\}^{-1} \quad \text{for } x\uparrow x_0.$$

Thus $F_1(x)$ defined by (69) satisfies (72). Hence by (66)

$$\lim_{z\uparrow x_0}\frac{1-F(z+z\,A(z)\,x)}{1-F(z)}=\lim_{z\uparrow x_0}\left\{\frac{1-F_1(z+z\,A(z)\,x)}{1-F_1(z)}\right\}^2\cdot\lim_{z\uparrow x_0}\frac{1-F_2(z)}{1-F_2(z+z\,A(z)\,x)}=e^{-x}.$$

This completes the proof.

4. A Unifying Approach

For distribution functions with $x_0<\infty$ we are now able to combine the results on the domain of attraction of the two limit distributions.

Theorem 11. *Let F be a distribution function with $x_0<\infty$. The sequence*

$$F^n(a_n x+b_n)$$

tends to a non-degenerate distribution function for a proper choice of the constants $a_n>0$ and b_n iff

$$\lim_{x\uparrow x_0}\frac{\{1-F(x)\}\left\{\int_x^{x_0}\int_y^{x_0}\{1-F(t)\}\,dt\,dy\right\}}{\left\{\int_x^{x_0}\{1-F(t)\}\,dt\right\}^2}=c \quad \textit{with} \quad \tfrac{1}{2}<c\leqq 1. \tag{74}$$

F is in the domain of attraction of ψ_α with $\alpha=(1-c)^{-1}-2$ if $c<1$. F is in the domain of attraction of Λ if $c=1$.

Proof. For $c=1$ the content of the theorem is part of Theorem 10. Suppose (74) holds with $\frac{1}{2}<c<1$. If we write $a(x)$ for the function to the right of the lim sign in (74), then for almost all x

$$a(x)=1+\frac{d}{dx}\,\frac{\int_x^{x_0}\int_y^{x_0}\{1-F(t)\}\,dt\,dy}{\int_x^{x_0}\{1-F(t)\}\,dt}.$$

Hence

$$\lim_{x\uparrow x_0}\frac{\int_x^{x_0}\int_y^{x_0}\{1-F(t)\}\,dt\,dy}{(x_0-x)\int_x^{x_0}\{1-F(t)\}\,dt}=\lim_{x\uparrow x_0}(x_0-x)^{-1}\int_x^{x_0}\{1-a(t)\}\,dt=1-c \tag{75}$$

and by (74)

$$\lim_{x\uparrow x_0}\frac{\int_x^{x_0}\{1-F(t)\}\,dt}{(x_0-x)\{1-F(x)\}}=c^{-1}(1-c). \tag{76}$$

From Remark 1 (after a trivial transformation) we see that $1-F\left(x_0-\frac{1}{x}\right)$ is regularly varying with exponent $2-(1-c)^{-1}$.

Hence by Theorem 6 we have $F\in D(\psi_\alpha)$ with $\alpha=(1-c)^{-1}-2$. The converse is a simple application of Theorem 6, Remark 1 and Corollary 1.

For distribution functions with $x_0=\infty$ there is an additional complication. If for example $F\in D(\phi_{0.5})$ then $a(x)$ is not defined because

$$\int_0^\infty \{1-F(t)\}\,dt=\infty.$$

Our final theorem shows that this difficulty is easily overcome.

Theorem 12. *Let F be a distribution function with $x_0=\infty$. The sequence*

$$F^n(a_n x+b_n)$$

tends to a non-degenerate distribution function for a proper choice of the constants $a_n>0$ and b_n iff

$$\lim_{x\to\infty}\frac{\{1-F(x)\}\left\{\int_x^\infty\int_y^\infty\{1-F(t)\}\frac{dt}{t^3}\,dy\right\}}{x^3\left\{\int_x^\infty\{1-F(t)\}\frac{dt}{t^3}\right\}^2}=c \quad \text{with} \quad 1\leqq c<2. \tag{77}$$

F is in the domain of attraction of ϕ_α with $\alpha=(c-1)^{-1}-1$ if $c>1$. F is in the domain of attraction of Λ if $c=1$.

Proof. For $c=1$ the content of the theorem is a consequence of Theorem 10. Next suppose (77) holds with $1<c<2$. As in the proof of Theorem 11 we obtain

$$\lim_{x\to\infty}\frac{x^2\int_x^\infty\{1-F(t)\}\frac{dt}{t^3}}{1-F(x)}=c^{-1}(c-1). \tag{78}$$

From Remark 1 we see that $F\in D(\phi_\alpha)$ with $\alpha=(c-1)^{-1}-1$. The converse is again a simple application of Remark 1 and Corollary 1.

Corollary 4. *Let $F\in D(\Lambda)$. The function $1-F(x)$ is rapidly varying at infinity with $\rho=-\infty$ if $x_0=\infty$. The function $1-F\left(x_0-\frac{1}{x}\right)$ is rapidly varying at infinity with $\rho=-\infty$ if $x_0<\infty$.*

Proof. The relations (76) and (78) which are also true for $c=1$, are equivalent to the statements in this corollary (see Remark 3 and the transformation in the proof of Corollary 2).

Remark. For distribution functions with $x_0=\infty$ Corollary 4 is proved in Gnedenko's paper.

A number of related results including other characterisations of $D(\Lambda)$ will be published elsewhere.

References

1. Aardenne-Ehrenfest, T. van, Bruyn, N. G. de, Korevaar, J.: A note on slowly oscillating functions. Nieuw Arch. Wiskunde, II. R. **23**, 77–86 (1949).
2. Feller, W.: An introduction to probability theory and its applications **2**. New York: Wiley (1966).
3. Gnedenko, B. V.: Sur la distribution limite du terme maximum d'une série aléatoire. Ann. of Math., II. Ser. **44**, 423–453 (1943).
4. Haan, L. de: Quantiles and stabilizing constants. Report S 410 (SP 114) Statistische afdeling, Mathematisch Centrum, Amsterdam (1969).
5. Karamata, J.: Sur un mode de croissance régulière des fonctions. Mathematica (Cluj) **4**, 38–53 (1930).
6. – Sur un mode de croissance régulière. Théorèmes fondamentaux. Bull. Soc. math. France **61**, 55–62 (1933).
7. Mejzler, D. G.: On a theorem of B. V. Gnedenko. Sb. Trudov Inst. Mat. Akad. Nauk. Ukrain. RSR **12**, 31–35 (1949) [Russian].
8. Mises, R. von: La distribution de la plus grande de *n* valeurs. Rev. math. Union Interbalkan. **1**, 141–160 (1936).

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