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ASYMPTOTIC NORMALITY OF NONPARAMETRIC TESTS FOR INDEPENDENCE

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0. Summary and basic notations. For each n let $(X_1,Y_1),\ldots,(X_n,Y_n)$ be a random sample from a continuous bivariate distribution function (df) $H(x,y)$ having marginal dfs $F(x)$ and $G(y)$. All samples are defined on a single probability space $(\Omega,\mathcal{U},P)$. The bivariate empirical df based on a sample of size n is denoted by H_n. With respect to the n random variables (rvs) $X_i(Y_i)$ corresponding to the first (second) coordinates, the empirical df is denoted by $F_n(G_n)$, the i-th order statistic by X_{in} (Y_{in}) and the rank of X_i (Y_i) by $R_i(Q_i)$.

To test the independence hypothesis $H(x,y) = F(x)G(y)$, most common rank statistics are of the linear type

$$T_n = n^{-1} \sum_{i=1}^{n} a_n(R_i)\ b_n(Q_i) \ ;$$

where $a_n(i)$, $b_n(i)$ are real numbers for $i = 1,\ldots,n$ (see Hájek and Šidák [6]).

By an approach analogous to that of Chernoff and Savage [3] for the two-sample problem, Bhuchongkul [1] proves asymptotic normality of the standardized statistic

$$(0.1) \qquad n^{1/2}(T_n-\mu) = n^{1/2}\left[\iint J_n(F_n)K_n(G_n)dH_n-\mu\right] \ ;$$

where

$$(0.2) \qquad J_n(s) = a_n(i) \ , \ K_n(s) = b_n(i)$$

for $(i-1)/n < s \leq i/n$ and $1 \leq i \leq n$ and where

$$(0.3) \qquad \mu = \iint J(F)K(G)dH$$

for some functions J and K. However, the conditions imposed on the weight functions (wfs) J_n and K_n are much stronger than in [3]. Jogdeo [7] proves asymptotic normality of these statistics by a different approach, but still needs conditions stronger than in [3].

The first theorem of Section 1 contains Bhuchongkul's Theorem 1 as a special case; a full proof of the non-trivial uniformity in H is included in our theorem. The proof of this theorem is based upon Hölder's inequality in the form

$$(0.4) \qquad \iint |\phi(F)\psi(G)|\,dH \leq [\int |\phi|^p dI]^{1/p}[\int |\psi|^q dI]^{1/q} ;$$

where I is the identity function on [0,1], dI denotes the Lebesgue measure restricted to that interval, ϕ and ψ are functions on (0,1) and $p > 1$, $q > 1$ satisfy $p^{-1} + q^{-1} = 1$.

The second theorem in Section 1 gives asymptotic normality under much weaker conditions on the wfs-conditions equivalent to those in [3]. The price for this is a condition on the df H, keeping it in some sense similar to the null hypothesis. This condition is

$$(0.5) \qquad dH \leq C[F(1-F)G(1-G)]^{-\delta/2} dFdG$$

with some fixed $C \geq 1$ and $\delta > 0$. Mathematically, this condition allows a direct factorization of the lefthand integral in (0.4) which is more efficient than Hölder's inequality. Intuitively, this condition prevents the large X's from occurring in the same pair as large Y's with too high a probability. Condition (0.5) always holds under the null hypothesis (with $C = 1$ and $\delta = 0$ even). It is also satisfied in the case (proposed by Lehmann [9]) where H can be written as a polynomial in its marginals F and G (with appropriately chosen $C \geq 1$ and again $\delta = 0$) and in the case (considered by Gumbel [5]) with $C = 10$. For further information on the latter class see also Runnenburg and Steutel [11]. Finally (0.5) holds for all bivariate normal distributions with a sufficiently small correlation coefficient (use Lemma 2 on page 166 of Feller [4] to see that (0.5) holds for a correlation coefficient between $-\delta/(2-\delta)$ and $\delta/(2-\delta)$).

Some results of Pyke and Shorack [10] are used in the proofs. We employ the notation

$$\rho(\phi,\psi) = \sup_{0<s<1} |\phi(s)-\psi(s)|$$

for functions ϕ, ψ on $(0,1)$.

1. Statement of the theorems. Let

$$B_{0n} = n^{1/2}\iint_{\Delta_n} [J_n(F_n)K_n(G_n)-J(F_n)K(G_n)]dH_n \tag{1.1}$$

where $\Delta_n = [X_{1n}, X_{nn}) \times [Y_{1n}, Y_{nn})$. Let

$$r = [I(1-I)]^{-1} \quad \text{on } (0,1). \tag{1.2}$$

We introduce the following assumptions (A1) - (A3).

(A1). The functions J_n, J, K_n, K and H are such that the rv $B_{0n} = o_p(1)$ uniformly for all continuous dfs H.

(A2). The functions J and K are defined and continuous on $(0,1)$. They have a derivative except for at most a finite number of points. In the open intervals between these points the derivatives are continuous. The functions J_n, J, J', K_n, K, K' satisfy

$$|J| \leq Dr^a,\ |J_n| \leq Dr^a \text{ for } n \geq 1,\ |K| \leq Dr^b,\ |K_n| \leq Dr^b \text{ for } n \geq 1$$

on $(0,1)$ and

$$|J'| \leq Dr^{a'},\ |K'| \leq Dr^{b'}$$

where defined on $(0,1)$. Here $D > 0$ and a, a', b, b' are fixed numbers satisfying

$$a(2+\delta)p_1<1,\ b(2+\delta)q_1<1,\ (a'-1/2+\delta)p_2<1,\ bq_2<1,\ ap_3<1,\ (b'-1/2+\delta)q_3<1$$

for some $0 < \delta < 1/2$ and $p_i, q_i > 1$ satisfying $p_i^{-1} + q_i^{-1} = 1$ for $i = 1,2,3$.

(A3). The bivariate df H is continuous. The class of all continuous bivariate dfs H will be denoted by $\mathcal{H}$.

THEOREM 1.1. Let J_n, J, J', K_n, K, K' and H satisfy (A1), (A2) and (A3). Then $n^{1/2}(T_n-\mu)\to_d N(0,\sigma^2)$ with finite μ and σ^2 given by (0.3) and (2.7). Moreover, the convergence is uniform for H in $\mathcal{H}$.

COROLLARY. Theorem 1.1 is true if $a = b = 1/4 - \delta$ and $a' = b' = 5/4 - \delta$ for some $\delta > 0$.

PROOF. Take $p_1 = q_1 = 2$, $p_2 = q_3 = 4/3$ and $p_3 = q_2 = 4$. □

This corollary is stronger than Theorem 1 of [1] in which $a = b = 1/8 - \delta$ and $a' = b' = 1$. Moreover in [1] H is supposed to be absolutely continuous and J and K twice differentiable.

In the next theorem we shall need assumptions (A1') - (A3').

(A1'). Condition (A1) holds uniformly for all continuous dfs H satisfying (0.5) for some fixed $C \geq 1$ and $0 < \delta < 1/2$.

(A2'). The functions J_n, J, J' and K_n, K, K' satisfy (A2) on (0,1) with $a,b = 1/2 - \delta$ and $a',b' = 3/2 - \delta$ for some $0 < \delta < 1/2$.

(A3'). The bivariate df H is continuous and satisfies (0.5) for some fixed $C \geq 1$ and $0 < \delta < 1/2$. The class of all such dfs H will be denoted by $\mathcal{H}_{C\delta}$.

THEOREM 1.2. Let J_n, J, J', K_n, K, K' and H satisfy (A1'), (A2') and (A3'). Then $n^{1/2}(T_n-\mu)\to_d N(0,\sigma^2)$ with finite μ and σ^2 given by (0.3) and (2.7). Moreover, the convergence is uniform for H in $\mathcal{H}_{C\delta}$.

REMARK. Suppose $J_n(i/n) = E(X_{in})$ and $K_n(i/n) = E(Y_{in})$ where X has df F and Y has df G. Let further $J = F^{-1}$ and $K = G^{-1}$. Then (A1) holds if J and K satisfy (A2). See also [1], Theorem 2.

Suppose $J_n(i/n) = J(i/(n+1))$ and $K_n(i/n) = K(i/(n+1))$. Then (A1) holds if J and K satisfy (A2) and (A1') holds if J and K satisfy (A2').

<u>2. Proof of the theorems.</u> Define $\xi_i = F(X_i)$ and $\eta_i = G(Y_i)$. Then $\xi_1,\dots,\xi_n$ and $\eta_1,\dots,\eta_n$ are each sets of independent rvs uniformly distributed on $[0,1]$. Denote the corresponding ordered samples by $\xi_{1n},\dots,\xi_{nn}$ and $\eta_{1n},\dots,\eta_{nn}$. Let $F^{-1}(s) = \inf\{x: F(x) \geq s\}$ and $G^{-1}(t) = \inf\{y: G(y) \geq t\}$. Define the empirical processes $U_n = n^{1/2}[F_n(F^{-1})-I]$ and $V_n = n^{1/2}[G_n(G^{-1})-I]$ on $[0,1]$. These processes satisfy $U_n(F) = n^{1/2}(F_n-F)$ and $V_n(G) = n^{1/2}(G_n-G)$ on $(-\infty,+\infty)$.

We now give the basic application of the mean value theorem. The theorems will be proved under the assumption that both J en K fail to have a derivative in just one point, say in s_1 and t_1 respectively. For γ in a neighborhood of 0 define the sets

$$S_{\gamma 1} = F^{-1}([\gamma,s_1-\gamma]\cup[s_1+\gamma,1-\gamma]),\quad S_{\gamma 2} = G^{-1}([\gamma,t_1-\gamma]\cup[t_1+\gamma,1-\gamma]),$$

(2.1)

$$\Omega_{\gamma n} = \{\omega: \sup|F_n-F| < \gamma/2,\ \sup|G_n-G| < \gamma/2\} .$$

Define the set $S_\gamma = S_{\gamma 1} \times S_{\gamma 2}$ in the plane. Let $\chi(\Omega_{\gamma n})$ denote the indicator function of $\Omega_{\gamma n}$. For almost every ω in $\Omega_{\gamma n}$ the mean value theorem gives

$$(2.2)\quad n^{1/2}J(F_n)K(G_n) = n^{1/2}J(F)K(G) + U_n(F)J'(\Phi_n(F))K(\Psi_n(G))$$

$$+ V_n(G)K'(\Psi_n(G))J(\Phi_n(F))$$

for all (x,y) in $\Delta_n \cap S_\gamma$. In (2.2) the functions Φ_n and Ψ_n are defined by $\Phi_n(F) = F + \theta(F_n-F)$ and $\Psi_n(G) = G + \theta(G_n-G)$ where $\theta = \theta_{\omega,x,y,n}$ is a number between 0 and 1 given by the mean value theorem; and application of the mean value theorem is valid when ω is in $\Omega_{\gamma n}$ and (x,y) is in $\Delta_n \cap S_\gamma$ for some $\gamma > 0$. Thus

$$n^{1/2}(T_n-\mu) = \textstyle\sum_{i=1}^{3} A_{in} + B_{0n} + \sum_{i=1}^{10} B_{\gamma in}$$

where B_{0n} is defined in (1.1) and

$$A_{1n} = n^{1/2}[\iint J(F)K(G)dH_n - \iint J(F)K(G)dH],$$

$$A_{2n} = \iint U_n(F)J'(F)K(G)dH, \quad A_{3n} = \iint V_n(G)K'(G)J(F)dH,$$

$$B_{1n} = n^{1/2}\iint_{\Delta_n^c} J_n(F_n)K_n(G_n)dH_n, \quad B_{\gamma 2n} = -\chi(\Omega_{\gamma n})n^{1/2}\iint_{\Delta_n^c} J(F)K(G)dH_n,$$

$$B_{\gamma 3n} = \chi(\Omega_{\gamma n})n^{1/2}\iint_{\Delta_n \cap S_\gamma^c}[J(F_n)K(G_n)-J(F)K(G)]dH_n,$$

$$B_{\gamma 4n} = \chi(\Omega_{\gamma n}^c)n^{1/2}[\iint_{\Delta_n} J(F_n)K(G_n)dH_n - \iint J(F)K(G)dH_n],$$

$$B_{\gamma 5n} = -\chi(\Omega_{\gamma n})\iint_{\Delta_n^c} U_n(F)J'(F)K(G)dH,$$

$$B_{\gamma 6n} = -\chi(\Omega_{\gamma n})\iint_{\Delta_n \cap S_\gamma^c} U_n(F)J'(F)K(G)dH,$$

$$B_{\gamma 7n} = \chi(\Omega_{\gamma n})\iint_{\Delta_n \cap S_\gamma} U_n(F)[J'(\Phi_n(F))K(\Psi_n(G))-J'(F)K(G)]dH_n,$$

$$B_{\gamma 8n} = \chi(\Omega_{\gamma n})\iint_{\Delta_n \cap S_\gamma} U_n(F)J'(F)K(G)d(H_n-H),$$

$$B_{\gamma 9n} = -\chi(\Omega_{\gamma n}^c)\iint U_n(F)J'(F)K(G)dH,$$

$$B_{\gamma 10n} = \chi(\Omega_{\gamma n})\iint_{\Delta_n \cap S_\gamma} V_n(G)K'(\Psi_n(G))J(\Phi_n(F))dH_n$$

$$- \iint V_n(G)K'(G)J(F)dH.$$

Since $\sum_{i=5}^{9} B_{\gamma in} = \chi(\Omega_{\gamma n})\iint_{\Delta_n \cap S_\gamma} U_n(F)J'(\Phi_n(F))K(\Psi_n(G))dH_n$ $- \iint U_n(F)J'(F)K(G)dH$ is symmetric to $B_{\gamma 10n}$, we will not treat $B_{\gamma 10n}$ in the sequel.

We now proceed by proving the asymptotic normality of the first order terms.

The rv A_{1n} can be written in the form

(2.3) $$A_{1n} = n^{-1/2} \sum_{i=1}^{n} A_{1in},$$

where the $A_{1in} = J(F(X_i))K(G(Y_i)) - \mu$ are independent and identically distributed (iid). Applying (0.4) with $p = p_1$ and $q = q_1$ (applying (0.5)) it is seen that under (A_2) (under A2') the rv A_{1in} has a finite absolute moment of order $2 + \delta_1$ for some $\delta_1 > 0$. Moreover this moment will be uniformly bounded for H within $\mathcal{H}$ ($\mathcal{H}_{C\delta}$). Finally the mean of this rv is 0.

Because $U_n(F) = n^{-1/2} \sum_{i=1}^{n} [\phi_{X_i}(F)-F]$, where

(2.4) $$\phi_{X_i}(F(x)) = 0 \text{ if } x < X_i \text{ and } \phi_{X_i}(F(x)) = 1 \text{ if } x \geq X_i,$$

we have

(2.5) $$A_{2n} = n^{-1/2} \sum_{i=1}^{n} A_{2in},$$

where the $A_{2in} = \iint [\phi_{X_i}(F)-F]J'(F)K(G)dH$ are idd. For δ as in (A2) (or (A2')) we have

$|A_{2in}| \leq \max\{\xi_i^{-1/2+\delta/4}, (1-\xi_i)^{-1/2+\delta/4}\}.D^2.\iint r^{a'-1/2+\delta/4}(F)r^b(G)dH.$

The first random factor possesses an absolute moment of order $2 + \delta_2$ for some $\delta_2 > 0$. Applying (0.4) with $p = p_2$ and $q = q_2$ (applying (0.5)) we see that the second non-random factor is uniformly bounded for H in H ($H_{C\delta}$) under the assumptions of Theorem 1.1 (Theorem 1.2). Hence A_{2in} has a finite absolute moment of order $2 + \delta_2$, which is uniformly bounded for H in H ($H_{C\delta}$). The mean of this rv is 0.

Analogously we can write

$$(2.6) \qquad A_{3n} = n^{-1/2} \sum_{i=1}^{n} A_{3in},$$

with $A_{3in} = \iint [\phi_{Y_i}(G)-G]K'(G)J(F)dH$. The rv A_{3in} has a finite absolute moment of order $2 + \delta_2$ which is uniformly bounded for H in H ($H_{C\delta}$). The mean again is 0.

Combining (2.3), (2.5) and (2.6) we get $A_{1n} + A_{2n} + A_{3n} \to_d N(0,\sigma^2)$ uniformly for H within H ($H_{C\delta}$) by Esseen's theorem (see e.g. [3], section 4). The variance σ^2 is, as in [1], given by

$$(2.7) \quad \sigma^2 = \mathrm{Var}[J(F(X))K(G(Y)) + \iint [\phi_X(F)-F]J'(F)K(G)dH + \iint [\phi_Y(G)-G]K'(G)J(F)dH]$$

with ϕ as defined in (2.4).

We now turn to the second order terms. The term B_{0n} is dealt with in (A1) and (A1'). The asymptotic negligibility of the other second order terms will be given in Section 4 as corollaries to the lemmas of Section 3.

3. Some lemmas. We start with a number of lemmas that are useful for both theorems.

LEMMA 3.1. For any $\tau \geq 0$ the function r^{τ} is symmetric about 1/2, decreasing on (0,1/2] and has the property that for each β in (0,1) there exists a constant $M = M_{\beta}$ such that $r^{\tau}(\beta s) \leq Mr^{\tau}(s)$ for $0 < s \leq 1/2$ and $r^{\tau}(1-\beta(1-s)) \leq Mr^{\tau}(s)$ for $1/2 < s < 1$.

PROOF. On (0,1/2] we have $r^{\tau}(\beta s) = (\beta s)^{-\tau}(1-\beta s)^{-\tau} \leq \beta^{-\tau} r^{\tau}(s)$. A similar argument applies to the interval (1/2,1). □

CONVENTION 3.1. Let Ω_n denote a subset of Ω and let $S = S_1 \times S_2$ denote a product set in the x,y-plane. The symbol $\theta = \theta_n$ will denote a function defined for ω in Ω_n and (x,y) in $\Delta_n \cap S$ that satisfies $0 \leq \theta \leq 1$. (The function θ will typically arise in various applications of the mean value theorem; as in (2.2).) For such θ we let $\Phi_n(F) = F + \theta(F_n - F)$ and $\Psi_n(G) = G + \theta(G_n - G)$.

LEMMA 3.2. For $\kappa = 0$ or $\kappa = 1$ and for each $\zeta \geq 0$, $\tau \geq 0$ and $\varepsilon > 0$, there exists a positive constant $M = M_{\kappa,\zeta,\tau,\varepsilon}$ (not depending on n) such that $P(\Omega_{\varepsilon n}) > P(\Omega_n) - \varepsilon$ for all n and uniformly for H within $\mathcal{H}$; where

$$\Omega_{\varepsilon n} = \{\omega \text{ in } \Omega_n : |U_n(F)|^{\kappa}\, r^{\zeta}(\Phi_n(F))\, r^{\tau}(\Psi_n(G))$$

$$\leq M\, r^{\zeta-\kappa(1/2-\delta/4)}(F)\, r^{\tau}(G) \text{ for all } (x,y) \text{ in } \Delta_n \cap S\} .$$

PROOF. It is well-known that for each $\varepsilon > 0$ there exists a constant $\beta = \beta_{\varepsilon}$ in (0,1) such that $P(\Omega_{\varepsilon 1n}) > 1 - \varepsilon/3$ for all n where

$$\Omega_{\varepsilon 1n} = \{\omega : \beta s \leq F_n(F^{-1}(s)) \leq 1 - \beta(1-s) \text{ for } \xi_{1n} \leq s < \xi_{nn}\} ;$$

and such that $P(\Omega_{\varepsilon 2n}) > 1 - \varepsilon/3$ for all n where

$$\Omega_{\varepsilon 2n} = \{\omega: \beta t \leq G_n(G^{-1}(t)) \leq 1 - \beta(1-t) \text{ for } \eta_{1n} \leq t < \eta_{nn}\} .$$

See e.g. Lemma 2.5 of Pyke and Shorack[10]. Because of the definition of $\Phi_n(F)$ (especially because $0 \leq \theta \leq 1$) on $\Omega_n \cap \Omega_{\varepsilon 1n}$ we have $\beta F \leq \Phi_n(F) \leq 1 - \beta(1-F)$ for x in $[X_{1n}, X_{nn}) \cap S_1$. By Lemma 3.1 this implies that for some constant $M_{\zeta,\varepsilon}$ we have $r^{\zeta}(\Phi_n(F)) \leq M_{\zeta,\varepsilon}\, r^{\zeta}(F)$ for x in $[X_{1n}, X_{nn}) \cap S_1$ on the set $\Omega_n \cap \Omega_{\varepsilon 1n}$. Analogously we have $r^{\tau}(\Psi_n(G)) \leq M_{\tau,\varepsilon}\, r^{\tau}(G)$ for y in $[Y_{1n}, Y_{nn}) \cap S_2$ on the set $\Omega_n \cap \Omega_{\varepsilon 2n}$. If $\kappa = 0$ we have $|U_n(F)|^{\kappa} = 1$ for all x and each ω. If $\kappa = 1$, Lemma 2.2 of[10] implies for each $\varepsilon > 0$ the existence of a constant M_{ε} such that $P(\Omega_{\varepsilon 3n}) > 1 - \varepsilon/3$ for all n where

$$\Omega_{\varepsilon 3n} = \{\omega: |U_n(F)| \leq M_{\varepsilon}\, r^{-1/2+\delta/4}(F) \text{ for all } x\} .$$

Hence $\Omega_{\varepsilon n} \supset (\Omega_n \cap \Omega_{\varepsilon 1n} \cap \Omega_{\varepsilon 2n} \cap \Omega_{\varepsilon 3n})$ where $P(\Omega_{\varepsilon 1n} \cap \Omega_{\varepsilon 2n} \cap \Omega_{\varepsilon 3n}) > 1-\varepsilon$. □

Some special choices will illustrate the use of this lemma. Taking $\theta = 0$ on $\Omega \times \Delta_n$ we have $\Phi_n(F) = F$ and $\Psi_n(G) = G$. Taking $\theta = 1$ on $\Omega \times \Delta_n$ we have $\Phi_n(F) = F_n$ and $\Psi_n(G) = G_n$. By taking in the latter case moreover $\kappa = 0$, $\zeta = a$, $\tau = b$ we have with probability larger than $P(\Omega)-\varepsilon = 1 - \varepsilon$ that $|J(F_n)K(G_n)| \leq Mr^{a}(F)\, r^{b}(G)$ for (x,y) in Δ_n, uniformly in n and H.

<u>LEMMA 3.3</u>. For each $\varepsilon > 0$ there exists an index n_{ε} such that

$$P(|\Phi_n(F)-F| < \varepsilon,\ |\Psi_n(G)-G| < \varepsilon \text{ for all } (x,y) \text{ in } \Delta_n \cap S)$$

$$> P(\Omega_n) - \varepsilon$$

for all $n > n_{\varepsilon}$ and uniformly for H within $\mathcal{H}$.

PROOF. The definition of Φ_n gives at once that $|\Phi_n(F)-F| \leq n^{-1/2}|U_n(F)|$ for x in $[X_{1n},X_{nn})\cap S_1$, where $|U_n(F)| = O_p(1)$ uniformly in n and F by Lemma 2.2 of [10]. □

LEMMA 3.4. $\sup_{-\infty<x,y<\infty}|H_n(x,y)-H(x,y)| \to_p 0$ uniformly for H in $\mathcal{H}$ as $n \to \infty$.

PROOF. This follows from Theorem 1-m of Kiefer [8]. □

For any natural number k assign to each bounded function ϕ on [0,1] the number $h_k(\phi)$ defined by $h_k(\phi) = \rho(\phi,\phi_k)$; where ϕ_k is the step function derived from ϕ by $\phi_k(0) = \phi(0)$ and $\phi_k(s) = \phi(i/k)$ for $(i-1)/k < s \leq i/k$ and $i = 1,\ldots,k$.

LEMMA 3.5. $h_k(U_n) \to_p 0$ as $k,n \to \infty$.

PROOF. The U_n-processes converge weakly to a separable tied-down Wiener process U_0. See Billingsley [2]. In Pyke and Shorack [10] these U_n- and U_0-processes are replaced by $\tilde{U}_n$- and $\tilde{U}_0$-processes on a single new probability space $(\tilde{\Omega},\tilde{\mathcal{U}},\tilde{P})$. (See also Skorokhod [12].) They satisfy the condition $\rho(\tilde{U}_n,\tilde{U}_0) \to_{a.s.} 0$ as $n \to \infty$. Now $h_k(\tilde{U}_n) \leq \rho(\tilde{U}_n,\tilde{U}_0) + \rho(\tilde{U}_0,\tilde{U}_{0,k}) + \rho(\tilde{U}_{0,k},\tilde{U}_{n,k})$. We see that $\rho(\tilde{U}_{0,k},\tilde{U}_{n,k}) \leq \rho(\tilde{U}_n, \tilde{U}_0) \to_{a.s.} 0$ as $n \to \infty$ (independently of k). Moreover for almost every ω the function $\tilde{U}_0$ is uniformly continuous on [0,1] so that $\rho(\tilde{U}_0,\tilde{U}_{0,k}) \to_{a.s} 0$ as $k\to\infty$. This implies $h_k(\tilde{U}_n) \to_{a.s.} 0$ for $k,n \to \infty$. Because the new processes $\tilde{U}_n$ have the same finite dimensional distributions as the original processes U_n, we can translate this last result into the claim stated in the lemma. □

CONVENTION 3.2. By ν we will understand the random index $1 \leq \nu(\omega) \leq n$ such that (X_ν,Y_ν) is the observation with largest first coordinate. By λ we will understand the random index $1 \leq \lambda(\omega) \leq n$ such that (X_λ,Y_λ) is the observation with largest second coordinate.

LEMMA 3.6. $P(\{\alpha_n \le \xi_\nu \le 1-\alpha_n\}\cap\{\alpha_n \le \eta_\nu \le 1-\alpha\}) \to 1$ as $n \to \infty$, uniformly for H within $\mathcal{H}$ provided only $n\alpha_n \to 0$ as $n \to \infty$.

PROOF. The probability of the complement is bounded above by $4(1-[1-\alpha_n]^n) \to 0$ as $n \to \infty$, independently of H. □

We conclude this section with some lemmas needed for Theorem 1.2.

LEMMA 3.7. P(for some index i the observation (X_i,Y_i) equals (X_{nn},Y_{nn})) $\to 0$ as $n \to \infty$, uniformly for H within $\mathcal{H}_{C\delta}$.

PROOF. This probability equals $P(\eta_\nu = \eta_{nn})$. Let $\beta_n = n^{-1}\log n$. Then $\{\eta_\nu = \eta_{nn}\} = \Omega_{1n}\cup\Omega_{2n}$ where $\Omega_{1n} = \{\eta_\nu=\eta_{nn}\}\cap[\{\xi_{nn}\le 1-\beta_n\}\cup\{\eta_{nn}\le 1-\beta_n\}]$ and $\Omega_{2n} = \{\eta_\nu=\eta_{nn}\}\cap[\{\xi_{nn}>1-\beta_n\}\cap\{\eta_{nn}>1-\beta_n\}]$. It follows that

$$P(\Omega_{1n}) \le P(\xi_{nn}\le 1-\beta_n) + P(\eta_{nn}\le 1-\beta_n) = 2(1-\beta_n)^n \to 0$$

as $n \to \infty$, and that

$$P(\Omega_{2n}) \le P(\cup_{i=1}^n\{F(X_i) + G(Y_i) \ge 2(1-\beta_n)\})$$

$$= 1 - P(\cap_{i=1}^n\{F(X_i) + G(Y_i) < 2(1-\beta_n)\})$$

$$= 1 - \{1 - \iint_\Delta dH\}^n .$$

By (0.5) and Hölder's inequality this last expression is bounded by

$$1 - \{1-C[\iint\{r(F)r(G)\}^{1/2}dFdG]^\delta[\iint_\Delta dFdG]^{1-\delta}\}^n$$

$$= 1 - \{1-C_1(2\beta_n^2)^{1-\delta}\}^n \to 0$$

as $n \to \infty$, because $2(1-\delta) > 1$ if $0 < \delta < 1/2$. In these expressions the symbol Δ denotes the region $\{F(x) + G(y) \geq 2(1-\beta_n)\}$ in 2-dimensional number space and C_1 is some constant depending on C and δ only. □

LEMMA 3.8. $P(\gamma_n \leq \eta_\nu \leq 1-\gamma_n) \to 1$ as $n \to \infty$, uniformly for H within $\mathcal{H}_{C\delta}$, if $\gamma_n = n^{-\delta}$.

PROOF. This probability equals $1 - P(0 \leq \eta_\nu < \gamma_n) - P(1 - \gamma_n < \eta_\nu \leq 1)$. Because of the independence of the sample elements and because all observations have unequal coordinates with probability 1, we have

$$P(0\leq\eta_\nu<\gamma_n) = nP([\cap_{i=1}^{n-1}\{F(X_i)\leq F(X_n)\}]\cap\{0\leq G(Y_n)<\gamma_n\})$$

$$= n\int\cdots\int_\Delta \Pi_{i=1}^n \; dH(x_i,y_i) \;;$$

where the symbol Δ now denotes the region

$$\{x_1 \leq x_n,\ldots,x_{n-1} \leq x_n, y_n \leq G^{-1}(\gamma_n)\}$$

in 2n-dimensional number space. First calculate $\int_{-\infty}^{x_n}\int_{-\infty}^{+\infty} dH(x_1,y_1) = F(x_n)$ and next $\int_{-\infty}^{x_n}\int_{-\infty}^{+\infty} F(x_n)dH(x_2,y_2) = F^2(x_n)$. After $(n-1)$ steps our original integral, taking into account the factor n, can be written as

$$n\int_{-\infty}^{+\infty}\int_{-\infty}^{G^{-1}(\gamma_n)} F^{n-1}(x_n)\; dH(x_n,y_n) \;.$$

By (0.5) this expression is bounded by

$$C\, n\left[\int_0^1 I^{n-1}\, r^{\delta/2}\, dI\right]\left[\int_0^{\gamma_n} r^{\delta/2}\, dI\right]$$

$$\leq C_1\, n\left[\Gamma(n-\delta/2)/\Gamma(n+1-\delta)\right]\left[\int_0^{\gamma_n} I^{-\delta/2}\, dI\right]$$

$$\leq C_2\, n\, n^{-1+\delta/2}\, n^{-\delta+\delta^2/2} = n^{-\delta/2+\delta^2/2} \to 0$$

as $n \to \infty$, because $-\delta/2 + \delta^2/2 < 0$ if $0 < \delta < 1$. In these expressions C_1 and C_2 are some constants depending on C and δ only. In the same way it can be shown that $P(1-\gamma_n < \eta_\nu \leq 1) \to 0$ as $n \to \infty$. All this is uniform for H in $\mathcal{H}_{C\delta}$. □

4. Uniform negligibility of the B_n terms.

COROLLARY 4.1. $B_{1n} \rightarrow_p 0$ as $n \rightarrow \infty$, uniformly for H within $\mathcal{H}$ ($\mathcal{H}_{C\delta}$).

PROOF. This rv is bounded by $\sum_{i=1}^{3} B_{1in}$ where

$$B_{11n} = n^{1/2}|J_n(1)|\int_{[Y_{1n},Y_{nn})} |K_n(G_n(y))|\,dH_n(X_{nn},y) ,$$

$$B_{12n} = n^{1/2}|J_n(1)K_n(1)|\int_{\{(X_{nn},Y_{nn})\}} dH_n(x,y) ,$$

$$B_{13n} = n^{1/2}|K_n(1)|\int_{[X_{1n},X_{nn})} |J_n(F_n(x))|\,dH_n(x,Y_{nn}) .$$

Under the assumptions of Theorem 1.1 we have at once that the sum of these terms is bounded by $3D^2\, n^{-1/2}\, n^{a+b} \rightarrow 0$ as $n \rightarrow \infty$, uniformly for H in $\mathcal{H}$. (Note that $a+b < 1/2$ by (A2) and recall formula (0.2).) Under the assumptions of Theorem 1.2 consider first B_{11n}. Because the random element $G_n(y)$ is strictly bounded away from 0 and 1 on $[Y_{1n},Y_{nn})$ we see that $|K_n(G_n(y))| \leq Dr^b(G_n(y))$ on that interval. Application of Lemma 3.2 with $\theta = 1$, $\Omega_n = \Omega$, $S = (-\infty,+\infty)$, $\kappa = \zeta = 0$ and $\tau = b$ gives that $r^b(G_n(y)) \leq Mr^b(G(y))$ on $\Omega_{\varepsilon n}$; with $P(\Omega_{\varepsilon n}) > 1-\varepsilon$ uniformly in n and uniformly for H in $\mathcal{H}$. Hence $B_{11n} \leq DMn^{-1/2}|J_n(1)|r^b(\eta_\nu)$ on $\Omega_{\varepsilon n}$, using Convention 3.2. For brevity, put $n^{-1/2}|J_n(1)| = \gamma_n$ and note that by (A2') $\gamma_n \leq D_1\, n^{-\delta}$ for some constant $D_1 \geq D$, independent of H. Let $\Omega_{1n} = \{\gamma_n \leq \eta_\nu \leq 1-\gamma_n\}$. Then $B_{11n} \leq DM\gamma_n^{1-b}(1-\gamma_n)^{-b} \rightarrow 0$ as $n \rightarrow \infty$ on $\Omega_{\varepsilon n} \cap \Omega_{1n}$. Applying Lemma 3.8 we see that $P(\Omega_{1n}) \rightarrow 1$ as $n \rightarrow \infty$ uniformly for H within $\mathcal{H}_{C\delta}$. Hence $P(\Omega_{\varepsilon n} \cap \Omega_{1n}) > 1-2\varepsilon$ for n large enough, uniformly for H within $\mathcal{H}_{C\delta}$. A symmetric argument can be used for B_{13n}. For the rv B_{12n} use Lemma 3.7 to see that the set on which this rv may have a value unequal to 0 has probability converging to 0 as $n \rightarrow \infty$, uniformly for H within $\mathcal{H}_{C\delta}$. □

COROLLARY 4.2. For each γ the rv $B_{\gamma 2n} \to_p 0$ as $n \to \infty$, uniformly for H within H ($H_{C\delta}$).

PROOF. Irrespective of γ this rv is bounded by $\sum_{i=1}^{2} B_{2in}$, where

$$B_{21n} = D^2 n^{-1/2} r^a(\xi_\nu) r^b(\eta_\nu) \quad \text{and} \quad B_{22n} = D^2 n^{-1/2} r^a(\xi_\lambda) r^b(\eta_\lambda) ,$$

using Convention 3.2.

Under the assumptions of Theorem 1.1 let

$\Omega_{1n} = \{\alpha_n \le \xi_\nu \le 1-\alpha_n\} \cap \{\alpha_n \le \eta_\nu \le 1-\alpha_n\}$, with $\alpha_n = n^{a+b-3/2}$. Note that $n\alpha_n \to 0$. Then

$$B_{21n} \le D^2 n^{-1/2} \alpha_n^{-a-b} (1-\alpha_n)^{-a-b}$$

$$= (n\, \alpha_n)^{1-a-b} (1 - \alpha_n)^{-a-b} \to 0$$

as $n \to \infty$ on Ω_{1n}. Lemma 3.6 gives that $P(\Omega_{1n}) \to 1$ uniformly for H in H. For the rv B_{22n} a symmetric argument applies.

Under the assumptions of Theorem 1.2 let

$\Omega_{2n} = \{\beta_n \le \xi_\nu \le 1-\beta_n\} \cap \{\gamma_n \le \eta_\nu \le 1-\gamma_n\}$, with $\beta_n = (n \log n)^{-1}$ and $\gamma_n = n^{-\delta}$. Then $B_{21n} \le D^2 n^{-1/2} \beta_n^{-a} \gamma_n^{-b} (1-\beta_n)^{-a} (1-\gamma_n)^{-b} \to 0$ as $n \to \infty$ on the set Ω_{2n}, because $a = 1/2 - \delta$ and $b = 1/2 - \delta$ by (A2'). By Lemma 3.6 and Lemma 3.8 we see that $P(\Omega_{2n}) \to 1$ as $n \to \infty$, uniformly for H in $H_{C\delta}$. The rv B_{22n} can be treated in a similar way. □

COROLLARY 4.3. The rv $B_{\gamma 3n} \to_p 0$ as $\gamma \downarrow 0$ uniformly in all n and H within H ($H_{C\delta}$).

PROOF. For arbitrary ω in Ω, take (x,y) in Δ_n. The closed line segment from $(F(x),G(y))$ to $(F_n(x),G_n(y))$ has at most two points in common with the pair of lines $F(x) = s_1$ and $G(y) = t_1$ in the plane. On the closed segment JK is continuous, and on the open sub-segments between these points it is even continuously differentiable. Hence the mean value theorem applies step wise. Consequently for almost every ω in Ω we can write

$$(4.1)\quad n^{1/2}J(F_n)K(G_n) = n^{1/2}J(F)K(G) + \sum_{i=1}^{3} c_i[U_n(F)J'(\Phi_{in}(F))K(\Psi_{in}(G))$$

$$+ V_n(G)K'(\Psi_{in}(G))J(\Phi_{in}(F))] ,$$

for all (x,y) in Δ_n. In (4.1) the functions Φ_{in} and Ψ_{in} are defined for some θ_i similarily to equation (2.2). For each $i = 1,2,3$ the θ_i are defined for almost all ω in Ω and for (x,y) in Δ_n. Both the θ_i and the c_i lie between 0 and 1. This implies that $|B_{\gamma 3n}| \leq \sum_{i=1}^{2} B_{\gamma 3in}$; where

$$B_{\gamma 31n} = \sum_{i=1}^{3} \iint_{\Delta_n \cap S_\gamma^c} |U_n(F)| r^{a'}(\Phi_{in}(F)) r^{b}(\Psi_{in}(G))dH_n ,$$

$$B_{\gamma 32n} = \sum_{i=1}^{3} \iint_{\Delta_n \cap S_\gamma^c} |V_n(G)| r^{b'}(\Psi_{in}(G)) r^{a}(\Phi_{in}(F))dH_n .$$

Let R be the real line. Because $S_\gamma^c \subset ((S_{\gamma 1}^c \times R) \cup (R \times S_{\gamma 2}^c))$, we have (if $\Omega_{\varepsilon n}$ is the set arising from application of Lemma 3.2 with $\theta = \theta_i$ for ω in $\Omega_n = \Omega$ and with $\kappa = 1$, $\zeta = a'$, $\tau = b$) that

$$E(\chi(\Omega_{\varepsilon n})B_{\gamma 31n}) \leq 3D^2M\iint_{S_{\gamma 1}^c \times R} r^{a'-1/2+\delta/4}(F) r^{b}(G)dH$$

$$+ 3D^2M\iint_{R \times S_{\gamma 2}^c} r^{a'-1/2+\delta/4}(F) r^{b}(G)dH .$$

Under the assumptions of Theorem 1.1 apply (0.4) to each of the terms in the bound for this expectation. Then the first of these terms is seen to be bounded by

$$3D^2M[\iint_{S^c_{\gamma 1}\times R} r^{(a'-1/2+\delta)p_2}(F)dH]^{1/p_2}[\iint_{S^c_{\gamma 1}\times R} r^{bq_2}(G)dH]^{1/q_2}$$

$$\leq 3D^2M[\int_{S^c_{\gamma 1}} r^{(a'-1/2+\delta)p_2}(F)dF]^{1/p_2}[\int r^{bq_2}(G)dG]^{1/q_2}$$

which is less than ε^2 for $0 < \gamma < \gamma_\varepsilon$, uniformly for H within $\mathcal{H}$ by the summability of $r^{(a'-1/2+\delta)p_2}$. The same is similarily true for the second term in the bound for $E(\chi(\Omega_{\varepsilon n})B_{\gamma 31n})$. Moreover the set $\Omega_{\varepsilon n}$ has probability larger than $1-\varepsilon$, uniformly in n and H within $\mathcal{H}$. Hence $P(\chi(\Omega_{\varepsilon n})B_{\gamma 31n} > \varepsilon) \leq 2\varepsilon^2/\varepsilon = 2\varepsilon$ for all n, all H in $\mathcal{H}$ and all $0 < \gamma < \gamma_\varepsilon$ by Markov's inequality. Thus $B_{\gamma 31n} \to_p 0$ as $\gamma \downarrow 0$ uniformly in n and H in $\mathcal{H}$. Finally the entire argument can be repeated for $B_{\gamma 32n}$.

Under the assumptions of Theorem 1.2 apply (0.5) to each of the terms in the bound for this expectation. Then the first of these terms is at once seen to be bounded by

$$3D^2MC[\int_{S^c_{\gamma 1}} r^{1-\delta/4}(F)dF][\int r^{1/2-\delta/2}(G)dG] \to 0$$

as $\gamma \downarrow 0$ uniformly in n and H within $\mathcal{H}_{C\delta}$. The same holds for the second term in the bound for $E(\chi(\Omega_{\varepsilon n})B_{\gamma 31n})$. Recalling that $P(\Omega_{\varepsilon n}) > 1-\varepsilon$ uniformly in n and H within $\mathcal{H}$ and repeating the same argument for $B_{\gamma 32n}$, the corollary is proved. □

COROLLARY 4.4. For each γ the rv $B_{\gamma 4n} \to_p 0$ as $n \to \infty$, uniformly for H within $\mathcal{H}$.

PROOF. For a fixed γ, by the Glivenko-Cantelli lemma the factor $\chi(\Omega^c_{\gamma n}) = 0$ on a set whose probability approaches 1 as $n \to \infty$, uniformly in the marginals F and G for H in $\mathcal{H}$. (Note that $\sup_{-\infty<x<\infty} |F_n - F| \leq n^{-1/2} \sup_{0\leq s\leq 1} |U_n|$.) □

COROLLARY 4.5. Uniformly in γ the rv $B_{\gamma 5n} \to_p 0$ as $n \to \infty$, uniformly for H within $\mathcal{H}(\mathcal{H}_{C\delta})$.

PROOF. First remark that $|U_n(F)|r^{1/2-\delta/4}(F)$ is bounded in probability by a constant M, uniformly in n and the marginal F of H in $\mathcal{H}$ (see [10], Lemma 2.2). Because $\Delta_n^c \subset (([X_{1n},X_{nn})^c \times R) \cup (R \times [Y_{1n},Y_{nn})^c))$, with probability near one we have

$$|B_{\gamma 5n}| \leq D^2 M \iint_{[X_{1n},X_{nn})^c \times R} r^{a'-1/2+\delta/4}(F) r^b(G) dH$$

$$+ D^2 M \iint_{R \times [Y_{1n},Y_{nn})^c} r^{a'-1/2+\delta/4}(F) r^b(G) dH .$$

Under the assumptions of Theorem 1.1 apply (0.4) to each of the terms of this bound with $p = p_2$ and $q = q_2$. Then the first of these terms is seen to be bounded by

$$D^2 M [\int_{[X_{1n},X_{nn})^c} r^{(a'-1/2+\delta)p_2}(F) dF]^{1/p_2} [\int r^{bq_2}(G) dG]^{1/q_2} \to_{a.s.} 0$$

as $n \to \infty$; all this being uniform for H in $\mathcal{H}$. The corollary is proved under the assumptions of Theorem 1.1 because a similar argument holds for the second term in this bound.

Under the assumptions of Theorem 1.2 apply (0.5) to each of the terms of this bound. Then the first of these terms in turn is seen to be bounded by

$$D^2 MC [\int_{[X_{1n},X_{nn})^c} r^{1-\delta/4}(F) dF][\int r^{1/2-\delta/2}(G) dG] \to_{a.s.} 0$$

as $n \to \infty$; all this being uniform for H within $\mathcal{H}_{C\delta}$. The corollary is proved, because a similar argument holds also in this case for the second term in the bound. □

<u>COROLLARY 4.6</u>. The rv $B_{\gamma 6n} \to_p 0$ as $\gamma \downarrow 0$ uniformly in all n and H within $\mathcal{H}(\mathcal{H}_{C\delta})$.

PROOF. For reasons similar to those in the proof of Corollary 4.3 we have

$$|B_{\gamma 6n}| \leq D^2 M \iint_{S^c_{\gamma 1}\times R} r^{a'-1/2+\delta/4}(F)r^b(G)dH$$

$$+ D^2 M \iint_{R\times S^c_{\gamma 2}} r^{a'-1/2+\delta/4}(F)r^b(G)dH$$

on the set $\Omega_{\varepsilon n}$ arising from application of Lemma 3.2 with θ defined by formula (2.2) for ω in $\Omega_n = \Omega_{\gamma n}$ and (x,y) in $\Delta_n \cap S_\gamma$ and with $\kappa = 1$, $\zeta = a'$, $\tau = b$.

Under the assumptions of Theorem 1.1 we get the desired result in the same way as in the corresponding part of the proof of Corollary 4.3; only it is easier since dH replaces dH_n.

Under the assumptions of Theorem 1.2 we similarily copy Corollary 4.3, replacing dH_n by dH. □

COROLLARY 4.7. For each γ the rv $B_{\gamma 7n} \to_p 0$ as $n \to \infty$, uniformly for H within $\mathcal{H}$.

PROOF. Consider the set $\Omega_{\varepsilon n}$ arising from application of Lemma 3.2 with θ defined by formula (2.2) for ω in $\Omega_n = \Omega_{\gamma n}$ and (x,y) in $\Delta_n \cap S_\gamma$; but this time with $\kappa = 1$, $\zeta = \tau = 0$. Then

$$|B_{\gamma 7n}| \leq M \sup_{\Delta_n \cap S_\gamma} |J'(\Phi_n(F))K(\Psi_n(G)) - J'(F)K(G)|$$

on $\Omega_{\varepsilon n}$. Since $J'(F)K(G)$ is uniformly continuous on an open set containing $S_{\gamma/2}$, application of Lemma 3.3 gives that this bound $\to_p 0$ as $n \to \infty$. On the other hand we have $P(\Omega_{\varepsilon n}) \to 1-\varepsilon$ as $n \to \infty$. All of this is uniform for H within $\mathcal{H}$. □

<u>COROLLARY 4.8</u>. For each γ the rv $B_{\gamma 8n} \to_p 0$ as $n \to \infty$, uniformly for H within $\mathcal{H}$.

<u>PROOF</u>. We have $|B_{\gamma 8n}| \leq \sum_{i=1}^{3} B_{\gamma 8ikn}$; where (see notation above Lemma 3.5)

$$B_{\gamma 81kn} = \iint_{S_\gamma} |U_n(F)J'(F)K(G) - U_{n,k}(F)J_k'(F)K_k(G)|\,dH_n,$$

$$B_{\gamma 82kn} = \left|\iint_{S_\gamma} U_{n,k}(F)J_k'(F)K_k(G)d(H_n-H)\right| ,$$

$$B_{\gamma 83kn} = \iint_{S_\gamma} |U_n(F)J'(F)K(G) - U_{n,k}(F)J_k'(F)K_k(G)|\,dH .$$

Both $B_{\gamma 81kn}$ and $B_{\gamma 83kn}$ are bounded by the supremum of the integrand; which is in turn bounded by

$$h_k(U_n) \max_{\gamma \leq s,t \leq 1-\gamma} |J'(s)K(t)|$$

$$+ \sup_{0 \leq s \leq 1} |U_{n,k}(s)| \max_{\gamma \leq s,t \leq 1-\gamma} |J'(s)K(t) - J_k'(s)K_k(t)| .$$

Since J'K is uniformly continuous on the square $\gamma \leq s,t \leq 1-\gamma$, the first term converges in probability to 0 as $k,n \to \infty$ by Lemma 3.5. The second term also converges in probability to 0 as $k \to \infty$, even uniformly in n, because $\sup_{0\leq s\leq 1}|U_{n,k}(s)| \leq \sup_{0\leq s\leq 1}|U_n(s)|$ is bounded in probability uniformly in n by Lemma 2.2 of [10]. Hence $B_{\gamma 81kn} + B_{\gamma 83kn} \to_p 0$ as $k,n \to \infty$. All this moreover is uniform for H within $\mathcal{H}$. By the same Lemma 2.2 of [10] it is seen that the values of the step function restricted to S_γ, in $B_{\gamma 82kn}$, are bounded in probability (uniformly in n and H in $\mathcal{H}$) by a constant M. Let a_{ijkn} be the value on the rectangle

$$R_{ijk} = (F^{-1}(((i-1)/k,i/k]) \times G^{-1}(((j-1)/k,j/k]))\cap S_\gamma \ .$$

Hence with probability near one

$$B_{\gamma 82nk} = |\textstyle\sum_{i=1}^{k} \sum_{j=1}^{k} a_{ijkn} \iint_{R_{ijk}} d(H_n-H)|$$

$$\leq M4k^2 \sup |H_n-H| \to_p 0$$

for each k as $n \to \infty$, uniformly for H within $\mathcal{H}$. This convergence to 0 is seen by Lemma 3.4. These results combined prove the corollary. □

COROLLARY 4.9. For each γ the rv $B_{\gamma 9n} \to_p 0$ as $n \to \infty$, uniformly for H within $\mathcal{H}$.

PROOF. See the proof of Corollary 4.4. □

We will now show how the results of these corollaries can be combined to complete the proof of the theorems. Let therefore an arbitrary $\varepsilon > 0$ be given. First use corollaries 4.3 and 4.6 to choose a fixed γ such that $P(|B_{\gamma in}| \leq \varepsilon/10) > 1-\varepsilon/10$ uniformly in n for i = 3,6. Next use (A1) or (A1') and Corollaries 4.1, 4.2, 4.4, 4.5, 4.7 - 4.9 to choose for the above value of γ an index $n_{\varepsilon\gamma}$ such that $P(|B_{in}| \leq \varepsilon/10) > 1-\varepsilon/10$ for $n > n_{\varepsilon\gamma}$ and i = 0-1,2,4,5,7-9. This implies that $P(\sum_{i=0}^{9} B_{\gamma in} \leq \varepsilon) > 1-\varepsilon$ for all $n > n_{\varepsilon\gamma}$. □

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