stichting mathematisch centrum

AFDELING MATHEMATISCHE STATISTIEK SW 6/71 MARCH

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THE RATE OF GROWTH OF SAMPLE MAXIMA

Prepublication

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

The rate of growth of sample maxima*)

by

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Introduction. Suppose $X_1, X_2, X_3, \ldots$ are independent real-valued random variables with common distribution function F. Suppose F has a positive derivative F'(x) for all sufficiently large x. We define

$$Y_n = \max(X_1, X_2, \ldots, X_n).$$

From Von Mises' work [4] we know that weak convergence properties of $\{Y_n\}$ are closely related to the behaviour of the function f defined by

$$f(x) = \frac{1-F(x)}{F'(x)} \qquad (1)$$

for $x \to \infty$. It will be shown that much about the sample behaviour of $\{Y_n\}$ can be concluded from the behaviour of the function g defined by

$$g(x) = \frac{\{1-F(x)\} \log \log\{1/1-F(x)\}}{F'(x)} \qquad (2)$$

for $x \to \infty$.

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*) Report SW 6/71 of the Department of Mathematical Statistics of the Mathematical Centre, Amsterdam.

Our exposition is based on a few lemmas of an analytic nature which are proved in section 1. In section 2 first we give conditions under which almost surely

$$0 < \liminf_{n\to\infty} Y_n/b_n \leq \limsup_{n\to\infty} Y_n/b_n < \infty$$

with b_n defined by $F(b_n) = 1 - 1/n$. For the special case that $\lim_{n\to\infty} Y_n/b_n$ exists almost surely, a more refined result is proved which previously is stated by J. Pickands III [5]. However the proof given there seems to contain an error.

Most of our conditions imply that

$$\lim_{n\to\infty} P\{\frac{Y_n-b_n}{f(b_n)} \leq x\} = \exp(-e^{-x}).$$

In section 3 we give a large deviations result in connection with this weak convergence property.

Section 1. In this section we give some lemmas which we need afterwards. The lemmas 1 and 3 play a basic role in our attack.

Lemma 1. Suppose ψ is a real-valued function with positive derivative ψ' and $\lim_{x\to\infty} \psi(x) = \infty$. If for some constant c $(0 \leq c \leq \infty)$

$$(3) \qquad \lim_{t\to\infty} \frac{\log \psi(t)}{t.\psi'(t)} = c,$$

then for all positive x

$$(4) \qquad \lim_{t\to\infty} \frac{\psi(tx)-\psi(t)}{\log \psi(t)} = \frac{\log x}{c}.$$

Proof. First suppose $0 < c \leq \infty$. Without loss of generality we assume $\psi(1) = 2$. Define the function p by

$$p(t) = \frac{t.\psi'(t)}{\log \psi(t)},$$

then

$$\int_1^t \frac{p(s)}{s}\, ds = \int_1^t \frac{\psi'(s)ds}{\log \psi(s)} = \int_2^{\psi(t)} \frac{ds}{\log s}.$$

If we denote the function $\int_2^x \frac{ds}{\log s}$ by I(x) and its inverse function by K, we get

$$(5) \qquad \psi(t) = K\Big(\int_1^t \frac{p(s)}{s}\, ds\Big).$$

Applying de l'Hospital's rule one sees that

$$\log I(y) \sim \log y \qquad \text{for } y \to \infty.$$

Substitution of x for I(y) gives

$$\log K(x) \sim \log x \qquad \text{for } x \to \infty.$$

Hence

$$(6) \qquad K'(x) = \log K(x) \sim \log x \qquad \text{for } x \to \infty.$$

We now calculate the limit (4). Using (5) we have

$$\frac{\psi(tx)-\psi(t)}{\log \psi(t)} = \frac{K\Big(\int_1^{tx} \frac{p(s)}{s}\, ds\Big) - K\Big(\int_1^t \frac{p(s)}{s}\, ds\Big)}{\log \psi(t)} =$$

$$= \frac{K\Big(\int_1^x \frac{p(ts)}{s} ds + \int_1^t \frac{p(s)}{s}\, ds\Big) - K\Big(\int_1^t \frac{p(s)}{s}\, ds\Big)}{\log K\Big(\int_1^t \frac{p(s)}{s}\, ds\Big)} .$$

Consequently

$$\lim_{t\to\infty} \frac{\psi(tx)-\psi(t)}{\log \psi(t)} = \lim_{y\to\infty} \frac{K(y+a(y))-K(y)}{\log K(y)} ,$$

where

$$\lim_{y\to\infty} a(y) = \lim_{t\to\infty} \int_1^x \frac{p(ts)}{s} ds = \frac{\log x}{c} .$$

By the mean value theorem of differential calculus we get for some $0 \leq \theta(y) \leq 1$

$$\lim_{t\to\infty} \frac{\psi(tx)-\psi(t)}{\log \psi(t)} = \lim_{y\to\infty} \quad a(y) \quad \frac{K'(y+\theta(y).a(y))}{\log K(y)} =$$

$$= \lim_{y\to\infty} a(y) \frac{\log K(y+\theta(y).a(y))}{\log K (y)} = \lim_{y\to\infty} a(y) \frac{\log(y+\theta(y).a(y))}{\log(y)} = \frac{\log x}{c} .$$

For $c = 0$, the same procedure shows (4) for $x > 1$. Suppose (4) does not hold for $x < 1$. Then for some $x_0 > 1$ and sequence $t_n \to \infty$ we have

$$\limsup_{n\to\infty} \frac{\psi(t_n x_0)-\psi(t_n)}{\log \psi(t_n x_0)} < \infty.$$

On the other hand

$$\lim_{n\to\infty} \frac{\psi(t_n x_0)-\psi(t_n)}{\log \psi(t_n)} = \infty,$$

hence

$$\lim_{n\to\infty} \frac{\log \psi(t_n x_0)}{\log \psi(t_n)} = \infty.$$

As clearly for $0 < \xi < \eta$

$$\xi(\log \eta - \log \xi) < \eta - \xi,$$

we have

$$\frac{\psi(t_n)\{\log \psi(t_n x_0)-\log \psi(t_n)\}}{\log \psi(t_n x_0)} < \frac{\psi(t_n x_0)-\psi(t_n)}{\log \psi(t_n x_0)} .$$

As for $n \to \infty$ the lefthand member tends to infinity and the righthand member is bounded, by contradiction we have (4) for all positive x. □

Remark. With the aid of theorem 1.4.2 from section 1.4 of [3] one can prove that for non-decreasing ψ with $\lim_{x\to\infty} \psi(x) = \infty$ and $0 < c < \infty$ relation (4) is equivalent to

$$\lim_{x\to\infty} \frac{\psi(x) - \frac{1}{x}\int_0^x \psi(t)dt}{\log \psi(x)} = \frac{1}{c} .$$

Lemma 2. Suppose f is a positive differentiable function and

$$\lim_{t\to\infty} f'(t) = 0.$$

Then

$$\lim_{t\to\infty} \frac{f(t)}{f(t+x\, f(t))} = 1$$

uniformly on each bounded x-interval.

Proof. By the mean value theorem of differential calculus for some $0 \leq \theta(t,x) \leq 1$

$$f(t+xf(t)) = f(t) + x\, f(t)\, f'(t+\theta(t,x)x\, f(t)).$$

From $f'(t) \to 0$ for $t \to \infty$ we get $t^{-1}f(t) \to 0$ and hence $t + \theta(t,x)x\, f(t) \to \infty$ for all x.Now the statement of the lemma follows as

$$\lim_{t\to\infty} f'(t+\theta(t,x)x\, f(t)) = 0$$

uniformly on each bounded x-interval. □

Lemma 3. Suppose ψ is a twice differentiable real-valued function with positive derivative ψ' and $\lim_{x\to\infty} \psi(x) = \infty$. Define the funtion q by

(7) $$q(t) = \frac{\log \psi(t)}{\psi'(t)}$$

and suppose

$$\lim_{t\to\infty} q'(t) = 0,$$

then for all real x

(8) $$\lim_{t\to\infty} \frac{\psi(t+x.q(t)-\psi(t))}{\log \psi(t)} = x.$$

<u>Proof</u>. We proceed in the same way as in the proof of lemma 1. Again we suppose $\psi(1) = 2$ and get

$$\psi(t) = K\left(\int_1^t \frac{ds}{q(s)}\right).$$

Now

$$\frac{\psi(t+x\ q(t))-\psi(t)}{\log \psi(t)} = \frac{K\left(\int_t^{t+xq(t)} \frac{ds}{q(s)} + \int_1^t \frac{ds}{q(s)}\right) - K\left(\int_1^t \frac{ds}{q(s)}\right)}{\log K\left(\int_1^t \frac{ds}{q(s)}\right)}$$

Consequently

$$\lim_{t\to\infty} \frac{\psi(t+x\ q(t))-\psi(t)}{\log \psi(t)} = \lim_{y\to\infty} \frac{K(b(y)+y)-K(y)}{\log K(y)}$$

where by lemma 2

$$\lim_{y\to\infty} b(y) = \lim_{t\to\infty} \int_t^{t+xq(t)} \frac{ds}{q(s)} = \lim_{t\to\infty} \int_0^x \frac{q(t)}{q(t+s\ q(t))}\, ds = x.$$

In the same way as in the proof of lemma 1 the statement (8) follows. □

The following lemma is of a probabilistic character. The elements for this lemma can be found in [1], [2] and [5]. We consider the situation described in the introduction.

Lemma 4. Suppose $\{c_n\}$ is a sequence of positive constants, $b_n = \inf\{x \mid 1-F(x) \leq 1/n\}$ and $\{c_n x+b_n\}$ is an ultimately non-decreasing sequence for all real $x > -1$.

a) For all distribution functions F we have almost surely

$$\liminf_{n\to\infty} \frac{Y_n-b_n}{c_n} \leq 0.$$

b) Suppose c is a finite constant. We have almost surely

$$\limsup_{n\to\infty} \frac{Y_n-b_n}{c_n} = c$$

if and only if

(9) $$\sum_{n=1}^{\infty} \{1-F(c_n x+b_n)\}$$

converges for all $x > c$ and diverges for all $x < c$.

c) If for all $-1 < x < 0$

(10) $$\sum_{n=1}^{\infty} \{1-F(c_n x+b_n)\} \exp\{-n(1-F(c_n x+b_n))\} < \infty,$$

then almost surely

(11) $$\liminf_{n\to\infty} \frac{Y_n-b_n}{c_n} \geq 0.$$

Proof.

a)

$$P\{Y_n \leq b_n \text{ infinitely often}\} \geq \limsup_{n\to\infty} P\{Y_n \leq b_n\} = \limsup_{n\to\infty} F^n(b_n) \geq$$

$$\geq (1-1/n)^n = e^{-1} > 0.$$

As $\{Y_n \leq b_n$ infinitely often$\}$ is a tail event, we have

$$P\{Y_n/c_n \leq b_n/c_n \text{ infinitely often}\} = P\{Y_n \leq b_n \text{ infinitely often}\} = 1.$$

b) As $\{c_n x+b_n\}$ is a non-decreasing sequence for all real $x > -1$, we have $Y_n > c_n x + b_n$ infinitely often if and only if $X_n > c_n x + b_n$ infinitely often. As the X_n are independent, part b) is a direct consequence of the Borel-Cantelli lemmas.

c) As $\sum_{n=1}^{\infty} \{1-F(b_n)\} = \infty$, we have almost surely $Y_n > b_n$ i.o. Hence also $Y_n > c_n x + b_n$ i.o. for all $x < 0$. So for proving (11) it is sufficient to show that almost surely

$$P\{Y_n \leq c_n + b_n \text{ and } Y_{n+1} > c_{n+1}x + b_{n+1} \text{ finitely often}\} = 1.$$

or equivalently (as $\{c_n x+b_n\}$ is non-decreasing for $x > -1$)

$$P\{Y_n \leq c_n x + b_n \text{ and } X_{n+1} > c_{n+1}x + b_{n+1} \text{ finitely often}\} = 1.$$

By the first Borel-Cantelli lemma this is true if

$$(12) \quad \sum_{n=1}^{\infty} P\{Y_n \leq c_n x + b_n \text{ and } X_{n+1} > c_{n+1}x + b_{n+1}\} =$$

$$= \sum_{n=1}^{\infty} \{1-F(c_{n+1}x+b_{n+1})\}.\ F^n(c_n x+b_n)$$

converges. Now

$$1 - F(c_{n+1}x+b_{n+1}) \leq 1 - F(c_n x+b_n)$$

and

$$F^n(c_n x+b_n) = \exp\{n \log F(c_n x+b_n)\} \leq \exp\{-n(1-F(c_n x+b_n))\},$$

hence the convergence of (12) is implied by (10). □

<u>Section 2</u>. In the situation described in the introduction we prove the following statement concerning the rate of growth of $\{Y_n\}$.

<u>Theorem 1</u>. Suppose F is a distribution function with positive derivative $F'(x)$ for all real x. If for some constant c ($0 \leq c < \infty$)

$$\lim_{t\to\infty} \frac{g(t)}{t} = c \tag{13}$$

(with g defined by (2)), then almost surely

$$\begin{cases} \liminf_{n\to\infty} Y_n/b_n = 1 \\ \limsup_{n\to\infty} Y_n/b_n = e^c. \end{cases} \tag{14}$$

Here b_n is defined by $F(b_n) = 1 - 1/n$.

If (13) holds with $c = \infty$, then almost surely $\limsup_{n\to\infty} Y_n/b_n = \infty$.

Remark. For $c = 0$ the theorem has been proved by Geffroy [2].

Proof. We use lemma 1 with $\psi(x) = \log 1/1-F(x)$. Then

$$\frac{\log \psi(t)}{t\,\psi'(t)} = \frac{\{1-F(t)\}\log\log\{1/1-F(t)\}}{t\,F'(t)} = \frac{g(t)}{t} \to c \text{ for } t \to \infty$$

and hence for $x > 0$

$$\lim_{t\to\infty} \log\{\frac{1-F(tx)}{1-F(t)}\} \cdot \{\log\log 1/1-F(t)\}^{-1} = -\frac{\log x}{c}$$

or equivalently

$$1 - F(tx) = \{1-F(t)\}\,\{\log 1/1-F(t)\}^{c(t)}$$

with

$$\lim_{t\to\infty} c(t) = -\frac{\log x}{c}\,.$$

Substitution of b_n for t gives

$$(15) \qquad 1 - F(b_n x) = \{1-F(b_n)\}\,\{\log 1/1-F(b_n)\}^{r_n} = \frac{(\log n)^{r_n}}{n}$$

with

(16) $$\lim_{n\to\infty} r_n = -\frac{\log x}{c} .$$

First we prove the statement concerning the lim sup for $0 \le c \le \infty$. As the righthand side of (16) is less than -1 for $x > e^c$ and larger than -1 for $x < e^c$, we have proved

$$\sum_{n=1}^{\infty} \{1-F(b_n x)\} < \infty \qquad \text{for } x > e^c$$

$$\sum_{n=1}^{\infty} \{1-F(b_n x)\} = \infty \qquad \text{for } x < e^c$$

and by part b) of lemma 4 (with $c_n = b_n$) we have almost surely

$$\limsup_{n\to\infty} Y_n/b_n = e^c.$$

To prove the statement concerning the lim inf for $0 \le c < \infty$ we verify condition (10) of lemma 4 with $c_n = b_n$. Using (15) we have for $0 < x < 1$

$$\sum_{n=1}^{\infty} \{1-F(b_n x)\} \exp\{-n(1-F(b_n x))\} =$$

$$= \sum_{n=1}^{\infty} n^{-1}(\log n)^{r_n} \exp\{-(\log n)^{r_n}\}.$$

Take $M \geq \frac{-2c}{\log x} + 1$, then

$$\sum_{n=1}^{\infty} \{1-F(b_n x)\} \exp\{-n(1-F(b_n x))\} << \sum_{n=1}^{\infty} n^{-1}(\log n)^{r_n} (\log n)^{-Mr_n} <<$$

$$<< \sum_{n=1}^{\infty} n^{-1}(\log n)^{-3/2} < \infty$$

and we have almost surely

$$\liminf_{n\to\infty} Y_n/b_n \geq 1.$$

By part a) of lemma 4 (with $c_n = b_n$) the proof is complete. □

<u>Remark</u>. In the usual way (see e.g. [2] p. 121) the result can be translated as follows: if $g(x) \to c$ ($0 \leq c \leq \infty$), then $P\{\limsup_{n\to\infty}(Y_n - b_n) = c\} = 1$; moreover $P\{\liminf_{n\to\infty} Y_n - b_n = 0\} = 1$ for $0 \leq c < \infty$.

For $0 < c < \infty$ this theorem provides exact information concerning the behaviour of Y_n. For $c = 0$ we prove a refined statement.

<u>Theorem 2</u>. Suppose F is a twice differentiable distribution function and $F'(x)$ is positive for all real x. If

(17) $$\lim_{t\to\infty} g'(t) = 0$$

(with g defined by (2)), then almost surely

$$(18)\quad \begin{cases} \liminf\limits_{n\to\infty} \dfrac{Y_n - b_n}{f(b_n)\log\log n} = 0 \\[2ex] \limsup\limits_{n\to\infty} \dfrac{Y_n - b_n}{f(b_n)\log\log n} = 1 \end{cases}$$

(here f is defined by (1) and b_n defined by $F(b_n) = 1-1/n$).

Proof. The proof is similar to that of theorem 1. We use lemma 3 with $\psi(x) = \log 1/1-F(x)$. Then

$$q'(t) = g'(t) \to 0 \qquad\qquad \text{for } t \to \infty$$

and hence

$$\lim_{t\to\infty} \log\{\frac{1-F(t+xg(t))}{1-F(t)}\} \{\log\log 1/1-F(t)\}^{-1} = -x$$

or equivalently

$$1 - F(t+xg(t)) = \{1-F(t)\} \{\log 1/1-F(t)\}^{c(t)}$$

with

$$\lim_{t\to\infty} c(t) = -x.$$

Substitution of b_n for t gives

$$g(b_n) = f(b_n) \log\log 1/1-F(b_n) = f(b_n) \log\log n$$

and

$$(19) \quad 1-F(b_n+xf(b_n)\log\log n) = \{1-F(b_n)\} \{\log 1/1-F(b_n)\}^{r_n} = \frac{(\log n)^{r_n}}{n}$$

with

$$(20) \qquad \lim_{n\to\infty} r_n = -x.$$

We want to apply lemma 4 with $c_n = f(b_n) \log\log n$. By (17) for all real x the sequence $\{b_n+xf(b_n)\log\log n\} = \{b_n+xg(b_n)\}$ is ultimately non-decreasing.

As the righthand member of (20) is less than -1 for x > 1 and larger than -1 for x < 1, we have proved

$$\sum_{n=1}^{\infty} 1 - F(b_n+xf(b_n)\log\log n) < \infty \qquad \text{for } x > 1$$

$$\sum_{n=1}^{\infty} 1 - F(b_n+xf(b_n)\log\log n) = \infty \qquad \text{for } x < 1$$

and by part b) of lemma 4 we have almost surely

$$\limsup_{n\to\infty} \frac{Y_n-b_n}{f(b_n)\log\log n} = 1.$$

By part a) of lemma 4 we have

$$\liminf_{n\to\infty} \frac{Y_n-b_n}{f(b_n)\log\log n} \leq 0.$$

To prove the other statement concerning the lim inf we verify condition (10) of lemma 4. Using (19) and (20) we have for $x < 0$ with $M \geq -\frac{2}{x} + 1$

$$\sum_{n=1}^{\infty} \{1-F(b_n+xf(b_n)\log\log n)\} \exp\{-n(1-F(b_n+xf(b_n)\log\log n))\}$$

$$= \sum_{n=1}^{\infty} n^{-1}(\log n)^{r_n} \exp\{-(\log n)^{r_n}\} << \sum_{n=1}^{\infty} n^{-1}(\log n)^{r_n(1-M)} <<$$

$$<< \sum_{n=1}^{\infty} n^{-1}(\log n)^{-3/2} < \infty$$

and hence almost surely

$$\liminf_{n\to\infty} \frac{Y_n-b_n}{f(b_n)\log\log n} \geq 0. \qquad \square$$

Remark. Theorem 2 has been stated first by J. Pickands III [5] but the proof seems to contain an error: the distribution function

$$F(x) = 1 - \exp -\{\int_e^x \frac{(\log \log t)^{3/2}}{t} dt\}$$

satisfies the conditions of the theorem but the first relation in the proof does not hold (the limit actually equals infinity).

Remark. Relation (17) implies relation (13) of theorem 1 with c = 0. On the other hand for distribution functions satisfying (13)

$$\lim_{n\to\infty} \frac{f(b_n)\log \log n}{b_n} = c,$$

hence for $0 < c < \infty$ the condition (13) implies

$$\begin{cases} \liminf\limits_{n\to\infty} \dfrac{Y_n - b_n}{f(b_n)\log \log n} = 0 \\ \limsup\limits_{n\to\infty} \dfrac{Y_n - b_n}{f(b_n)\log \log n} = \dfrac{e^c - 1}{c} \end{cases}$$

almost surely.

Examples of distribution functions satisfying theorem 2 are given by Pickands. The distribution functions

$$F(x) = 1 - \exp\{- \int_e^x \frac{(\log \log t)^p}{c.t} dt\}$$

with positive p and c satisfy

$$\lim_{t\to\infty} \frac{g(t)}{t} = \lim_{t\to\infty} g'(t) = \begin{cases} 0 & \text{for } p > 1 \\ c & \text{for } p = 1 \\ \infty & \text{for } p < 1. \end{cases}$$

As all these distribution functions are in the domain of attraction of the double exponential distribution, this answers a question raised by Pickands whether theorem 2 holds for all distribution functions from this domain of attraction.

It is clear that if (18) from theorem 2 holds, then this relation is still true of we replace

$$Y_n = \max(X_1, X_2, \ldots, X_n)$$

by

$$[Y_n] + 1 = \max([X_1]+1, [X_2]+1, \ldots, [X_n]+1)$$

(here [a] is the largest integer not exceeding a). As (18) holds for the exponential distribution with $b_n = \log n$ and $f(b_n) = 1$, this relation is also true for the geometric distribution

$$F(x) = 1-e^{-[x]} \qquad \text{for } x > 0.$$

Hence the validity of (18) does not imply that F belongs to the domain of attraction of the double exponential distribution.

Section 3. Let us reconsider the condition of theorem 2.

$$g'(t) = \frac{d}{dt}\{\frac{1-F(t)}{F'(t)} \log\log 1/1-F(t)\}$$

$$= \frac{d}{dt}\{\frac{1-F(t)}{F'(t)}\} \log\log 1/1-F(t) + \{\log 1/1-F(t)\}^{-1}$$

$$= f'(t) \, . \log\log 1/1-F(t) + o(1) \qquad \text{for } t \to \infty.$$

So $g'(t) \to 0$ for $t \to \infty$ if and only if

$$(21) \qquad \lim_{t\to\infty} f'(t) \, . \log\log 1/1-F(t) = 0$$

and both imply Von Mises' condition $f'(t) \to 0$ (see [4]) for the domain of attraction of the double exponential distribution. So (21) implies

$$\lim_{n\to\infty} P\{\frac{Y_n - b_n}{f(b_n)} \leq x\} = \exp(-e^{-x}).$$

We shall prove a large deviations result related to this weak convergence property under a condition of the type (21).

<u>Theorem 3</u>. Suppose ϕ is a non-decreasing function and $\lim_{x\to\infty} \phi(x) = \infty$. If

$$(22) \qquad \lim_{t\to\infty} f'(t)\ \phi^2(1/1-F(t)) = 0$$

(with f defined by (1)), then

$$(23) \qquad \lim_{n\to\infty} \frac{1-F^n(b_n+x_n f(b_n))}{1-\exp(-e^{-x_n})} = 1$$

for all sequences of positive numbers $\{x_n\}$ with $x_n = O(\phi(n))$ for $n \to \infty$. Here b_n is defined by $F(b_n) = 1-1/n$.

<u>Proof</u>. Obviously (22) implies $f'(t) \to 0$ for $t \to \infty$ and hence by Von Mises' criterion (see [4])

$$\lim_{n\to\infty} F^n(b_n+xf(b_n)) = \exp(-e^{-x})$$

uniformly on each bounded x-interval. Hence (23) holds trivially for each bounded sequence $\{x_n\}$. Next suppose $x_n \to \infty$ for $n \to \infty$. From $-\ln y \sim 1-y$ for $y \uparrow 1$ it follows

$$1 - F^n(b_n+x_n f(b_n)) \sim n\{1-F(b_n+x_n f(b_n))\}$$

and

$$1 - \exp(-e^{-x_n}) \sim e^{-x_n}$$

for $n \to \infty$. So we have to prove

(24) $$\lim_{n\to\infty} ne^{x_n}\{1-F(b_n+x_n f(b_n))\} = 1.$$

by (1) we have

$$\frac{1}{f(t)} = \frac{F'(t)}{1-F(t)}$$

and hence

$$\int_1^x \frac{dt}{f(t)} = -\log\{1-F(x)\} + \log\{1-F(1)\}$$

or equivalently (with $c_0 = 1-F(1)$)

$$1 - F(x) = c_0 \exp\{-\int_1^x \frac{dt}{f(t)}\}.$$

Substitution in (24) gives (as $n = 1/1-F(b_n)$)

$$ne^{x_n}\{1-F(b_n+x_nf(b_n))\} = \exp\{x_n - \int_{b_n}^{b_n+x_nf(b_n)} \frac{ds}{f(s)}\} =$$

$$= \exp\{\int_0^1 -x_n(\frac{f(b_n)}{f(b_n+sx_nf(b_n))} -1)ds\}.$$

As $x_n = O(\phi(n))$, for proving the theorem it is sufficient to show

$$\lim_{n\to\infty} \phi(n)\{\frac{f(b_n)}{f(b_n+xf(b_n)\phi(n))} -1\} = 0 \tag{25}$$

uniformly on any bounded x-interval from $[0,\infty)$. Substitution of t for b_n gives $\phi(n) = \phi(1/1-F(t))$ and (25) becomes

$$\lim_{t\to\infty} \psi(t)\{\frac{f(t)}{f(t+xf(t)\psi(t))} -1\} = 0$$

with $\psi(t) = \phi(1/1-F(t))$.

Using the mean value theorem of differential calculus we get for some $0 \leq \theta(t,x) \leq 1$

$$\psi(t)\{\frac{f(t)}{f(t+xf(t)\psi(t))} - 1\} \tag{26}$$

$$= \frac{\psi(t)}{f(t+xf(t)\psi(t))} (-x)f(t)\psi(t)f'(t+\theta(t,x)xf(t)\psi(t))$$

$$= -x\{\frac{\psi^2(t)}{\psi^2(t+\theta(t,x)xf(t)\psi(t))}\} \{\frac{f(t)}{f(t+xf(t)\psi(t))}\} \cdot$$

$$\cdot \{f'(t+\theta(t,x)xf(t)\psi(t))\psi^2(t+\theta(t,x)xf(t)\psi(t))\}.$$

Now we treat the last three factors separately.

As ψ is non-decreasing the first factor is bounded by 1. By assumption the last factor tends to zero uniformly on $[0,\infty)$. As

$$\frac{f(t+xf(t)\psi(t))-f(t)}{f(t)}$$

$$= x\,\frac{\psi(t)}{\psi(t+\theta_1(t,x)xf(t)\psi(t))}\,f'(t+\theta_1(t,x)xf(t)\psi(t))\psi(t+\theta_1(t,x)xf(t)\psi(t))$$

and $\psi(t) \leq \psi^2(t)$ for sufficiently large t, it follows

$$\lim_{t\to\infty}\frac{f(t)}{f(t+xf(t)\psi(t))} = 1 \tag{27}$$

uniformly on every bounded x-interval from $[0,\infty)$ and we have proved the theorem. □

Remark. The condition of the theorem cannot be improved essentially: suppose $f'(t)\phi^2(1/1-F(t)) \to c$ with $0 < c < \infty$ and $t\phi'(t) \to 0$, then one can prove

$$\lim_{n\to\infty}\frac{1-F^n(f(b_n)\phi(n)+b_n)}{1-\exp(-e^{-\phi(n)})} = e^{c/2}.$$

As an example we consider the normal distribution. Here

$$f'(t) = te^{t^2/2} \int_t^\infty e^{-s^2/2}\, ds - 1 \sim - t^{-2} \qquad \text{for } t \to \infty.$$

As the inverse function of $1/1-F(t)$ is asymptotically equal to $\sqrt{2 \log s}$, (22) holds if

$$\lim_{t\to\infty} f'(t)\, \phi^2(1/1-F(t)) = \lim_{t\to\infty} - \frac{\phi^2(1/1-F(t))}{t^2} = \lim_{s\to\infty} - \frac{\phi^2(s)}{(2 \log s)} = 0$$

and (23) is true for sequences $\{x_n\}$ with

$$x_n = o(\sqrt{\log n}) \qquad \text{for } n \to \infty.$$

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