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MC

AFDELING MATHEMATISCHE STATISTIEK SW 8/71 MAY

L. DE HAAN

ON R. VON MISES' CONDITION FOR THE DOMAIN OF ATTRACTION OF EXP($-e^{-x}$)

Prepublication

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

Abstract

On R. von Mises' condition for the domain of attraction of $\exp(-e^{-x})$.

There exist well-known necessary and sufficient conditions for the domain of attraction of the double exponential distribution. For practical purposes a simple sufficient condition due to von Mises is very useful. It is shown that each distribution function F in the domain is a rather simple function of some distribution function satisfying von Mises' condition.

On R. von Mises' condition for the domain of attraction of $\exp(-e^{-x})$. *)

by Laurens de Haan

Mathematisch Centrum, Amsterdam

Suppose X_1, X_2, X_3, ... are independent real-valued random variables with common distribution function F. We say that F is in the domain of attraction of the double exponential distribution (notation $F \in D(\Lambda)$; $\Lambda(x) = \exp(-e^{-x})$) if there exist two sequences of real constants $\{b_n\}$ and $\{a_n\}$ (with $a_n > 0$ for n = 1, 2, ...) such that for all real x

$$(1) \qquad \lim_{n\to\infty} P\{\frac{\max(X_1,X_2,\ldots,X_n)-b_n}{a_n} \leq x\} = \exp(-e^{-x}).$$

Necessary and sufficient conditions for $F \in D(\Lambda)$ are well-known ([1] and [2]) but rather intricate. The following relatively simple criterion is due to R. von Mises ([3] p. 285). It is convenient for the formulation of the theorem to use the symbol x_0 for the upper bound of X_i defined by

$$x_0(F) = \sup\{x \mid F(x) < 1\}.$$

Theorem 1 Suppose F is twice differentiable and $F'(x)$ is positive for all $x < x_0$. If

$$(2) \qquad \lim_{x\uparrow x_0} \frac{F''(x)\{1-F(x)\}}{\{F'(x)\}^2} = -1,$$

then $F \in D(\Lambda)$.

A distribution function F satisfying (2) will be called a von Mises function.

Our theorem states that each F from $D(\Lambda)$ is linked to some von Mises function in a relatively simple way.

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*) Report SW 8/71, Afdeling Mathematische Statistiek, Mathematisch Centrum, Amsterdam.

Theorem 2 a) Suppose $F \in D(\Lambda)$. There exists a von Mises function F_1 and a regularly varying function U with exponent 1 such that for all $x < x_0$

$$\frac{1}{1-F(x)} = U\left(\frac{1}{1-F_1(x)}\right). \tag{3}$$

b) If F_1 is a von Mises function and U a regularly varying function with exponent 1, then any distribution function F given by (3) belongs to $D(\Lambda)$.

Proof a) We use theorem 2.5.3 of [2] which states that if $F \in D(\Lambda)$, there exist a real constant c_1 and real-valued functions c, a and f defined on $(-\infty, x_0)$ with

$$\left\{\begin{array}{l} c(x) > 0 \text{ for all } x < x_0,\ \lim_{x \uparrow x_0} c(x) = c_1 > 0, \\ \lim_{x \uparrow x_0} a(x) = 1, \\ f(x) \text{ is positive and differentiable for all } x < x_0 \\ \text{and } \lim_{x \uparrow x_0} f'(x) = 0, \\ \text{moreover } \lim_{x \uparrow x_0} f(x) = 0 \text{ if } x_0 < \infty, \end{array}\right. \tag{4}$$

such that for $x_1 < x < x_0$

$$1 - F(x) = c(x).\ \exp\left\{-\int_{x_1}^{x} \frac{a(t)}{f(t)}\, dt\right\}.$$

First suppose $x_0 = \infty$. Define the function F_1 by

$$F_1(x) = \left\{\begin{array}{ll} 0 & \text{for } x \le 1 \\ 1 - \exp\left(-\int_1^x \frac{dt}{f(t)}\right) & \text{for } x > 1. \end{array}\right.$$

Clearly this distribution function is twice differentiable and from $\lim_{x\to\infty} f'(x) = 0$ we have that F_1 satisfies (2). Denote the inverse function of $\frac{1}{1-F_1}$ by V and define U by

$$U(x) = c(V(x)).\exp\{\int_1^x \frac{a(V(t))}{t}\,dt\} \qquad \text{for } x > 1.$$

From (4) it follows by the representation theorem for regularly varying functions (see e.g. [2] theorem 1.2.2), that U varies regularly with exponent 1. It is easy to see that with these functions F_1 and U we have (3).

If $x_0 < \infty$ the proof goes through with obvious changes.

b) A well-known theorem of Gnedenko [1] states that $F \in D(\Lambda)$ if and only if for some positive function f

$$\lim_{t\uparrow x_0} \frac{1-F(t+x.f(t))}{1-F(t)} = e^{-x} \qquad \text{for all real } x.$$

By assumption this relation holds for F_1 i.e. for some positive function f_1 we have

$$(5) \qquad \lim_{t\uparrow x_0} \frac{1}{1-F_1(t+x.f(t))} \Big/ \frac{1}{1-F_1(t)} = e^{x} \qquad \text{for all real } x.$$

If U is regularly varying with exponent 1, we have

$$\lim_{s\to\infty} \frac{U(sy)}{U(s)} = y$$

uniformly on any interval of the form $0 < y_1 \leq y \leq y_2 < \infty$. Hence (5) implies

$$\lim_{t\uparrow x_0} \frac{1-F(t)}{1-F(t+x.f_1(t))} = \lim_{t\uparrow x_0} \frac{U(\frac{1}{1-F_1(t+x.f_1(t))})}{U(\frac{1}{1-F_1(t)})} = e^{x}$$

for all real x

and so $F \in D(\Lambda)$. □

References

[1] Gnedenko, B.V. (1943). Sur la distribution limite du terme maximum d'une série aléatoire. Annals of Math. 44 423-453.

[2] de Haan, L. (1970). On regular variation and its application to the weak convergence of sample extremes. MC tract 32, Mathematisch Centrum, Amsterdam.

[3] von Mises, R. (1936). La distribution de la plus grande de n valeurs. In: Selected Papers II (Am. Math. Soc.) 271-294.