An application of Pitman-efficiency to acceptance sampling procedures *

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Summary As an exercise the concept of Pitman-efficiency has been applied to the decision problem whether to use acceptance sampling "by attributes" or "by variables".
The Pitman-efficiency has been calculated in the two cases that the variance of the underlying normal distribution is known and that it is unknown. Rather surprisingly the difference between these two cases proves to be considerable, even asymptotically. The asymptotic result is compared with the exact values of the relative efficiency in the case that σ is unknown. The asymptotic approximation appears to be rather good. The results derived also help to determine a suitable choice of the null hypothesis in order to increase the Pitman-efficiency.

1. Introduction

The basic problem we are dealing with is to inspect a lot using a sampling procedure. For this purpose we consider the two well-known methods "by attributes" and "by variables".

Let the continuous random variable $\underline{x}$ describe a functional property of the product under consideration, such as length or breaking strength. We assume the random variable $\underline{x}$ to be normally distributed with mean μ and variance σ^2. To distinguish between effective and defective items we introduce the value A of the variable $\underline{x}$.

This means, e.g. we judge a certain item to be effective if $x > A$ and defective if $x \leqslant A$, where x is the value of $\underline{x}$ for the item considered.

In more statistical terms: We have to solve the following testing problem on the base of a random sample $\underline{x}_1, \ldots, \underline{x}_n$ of size n from a $N(\mu, \sigma^2)$ distribution:

$$H_0: \; p \overset{\text{def}}{=} P(\underline{x} \leqslant A) \leqslant p_0,$$

for given constants A and p_0 against

$$H_1: \; p > p_0.$$

In a picture (see Fig. 1) this means that we want to test whether the area p of the shaded region is smaller than or equal to a given constant p_0.

In the "attributive" case we reject the hypothesis H_0, and so the lot, if the number of defectives in the sample exceeds a critical number c.

When inspecting "by variables" we use the actual values of the sample and base our solution on the test statistic $\bar{\underline{x}} - k \cdot \underline{s}$,

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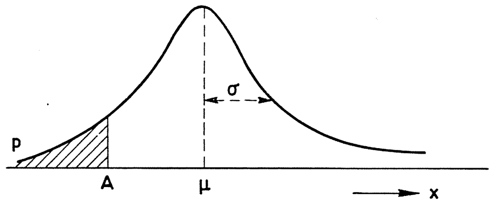

Fig. 1. The fraction defectives in the population.

where

$$\underline{\bar{x}} = \frac{1}{n}\sum_{i=1}^{n} \underline{x}_i \qquad ; \underline{\bar{x}} \text{ is an estimator of the mean } \mu,$$

$$\underline{s} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(\underline{x}_i-\underline{\bar{x}})^2}; \ \underline{s} \text{ is an estimator of the standard deviation } \sigma,$$

and k is a critical factor.

We reject the lot if

$$\underline{\bar{x}} - k \cdot \underline{s} < A. \tag{1}$$

In Appendix III one can find:

a. an intuitive justification of the choice of the critical region (1),
b. the proof that A is the only possible boundary for this critical region,
c. the proof that this critical region agrees with the critical region based on the test statistic $\sqrt{n}\,\underline{\bar{x}}/\underline{s}$ to be constructed in the classical way by the principle of invariance.

Which of the two methods has to be selected is in fact an economic decision problem. To solve this problem we need information about the costs per observation and the sample sizes n_a ("by attributes") and n_v ("by variables"), necessary to achieve an equal performance of both testing procedures. Because of the fact that we "use more information" when inspecting "by variables" we can expect n_v to be smaller than n_a.

The theory of Pitman-efficiency provides us with a method to approximate the ratio:

$$\frac{n_v}{n_a}.$$

2. Application of Pitman-efficiency

The Pitman-efficiency, or asymptotic relative efficiency (ARE), is an asymptotic approximation of the ratio n_v/n_a of the two testing methods necessary to achieve an equal performance in terms of power. For the theory we refer the reader to [5].

The results of the application will be given in this section, while the derivation of the formulas is given in Appendix I.

When σ is unknown we have

$$\text{ARE (1)} = \frac{1+\dfrac{\theta_0^2}{2}}{2\pi p_0(1-p_0)e^{\theta_0{}^2}} \tag{2}$$

and when σ is known

$$\text{ARE (2)} = \frac{1}{2\pi p_0(1-p_0)e^{\theta_0{}^2}}, \tag{3}$$

where θ_0 is determined by

$$p_0 = \Phi(-\theta_0) = \int_{-\infty}^{-\theta_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \mathrm{d}x. \tag{4}$$

When $\theta_0 = 0$, which means that $p_0 = \Phi(0) = \frac{1}{2}$, we can easily verify that

$$\text{ARE (1)} = \text{ARE (2)} = \frac{2}{\pi}. \tag{5}$$

Indeed this is the well-known Pitman-efficiency of the sign test with respect to Student's t-test.

We remark that the asymptotic results (2) and (3) show a great difference for σ unknown compared with σ known: from (2) and (3) we have

$$\frac{\text{ARE (1)}}{\text{ARE (2)}} = 1 + \frac{\theta_0^2}{2}, \text{ which is an increasing} \tag{6}$$

function of $|\theta_0|$.

For instance if $\theta_0 = 2$, the ratio (6) is equal to 3, so that the Pitman-efficiency in the case that σ is unknown is three times the Pitman-efficiency in the case that σ is known! In some other cases, like Student's test in comparison with the normal test for a mean, knowledge of the value σ has no influence on asymptotic results ($n \to \infty$).

We further note that SCHAAFSMA and WILLEMZE [7], using a result of HAMAKER [2] in 1954 already derived formula (2) and (3) on the base of other arguments.

For the relations between the methods of SCHAAFSMA, WILLEMZE and HAMAKER and the Pitman-concept we refer to Appendix II.

3. Comparison of the ARE with exact relative efficiency

In practice σ usually is unknown and therefore we restrict ourselves to this case in comparing the exact relative efficiency (RE) with the ARE. The exact values have been calculated numerically on the Electrologica-X8 of the Mathematical Centre in Amsterdam. A part of the results is given in Table 1 to demonstrate the reasonable approximation of Pitman-efficiency to exact efficiency, especially considering the discrete character of n_a, given n_v. For a more elaborate list of results and the way the exact values of the ratio n_v/n_a have been derived, we refer to Appendix V.

One can see from Table 1 and Appendix V that the approximation deteriorates when p_0 is small.

Table 1. Exact and asymptotic relative efficiency for σ unknown, $\alpha = .05$ and $n_v = 8$

power	p_0	n_a	RE (1)	RE (2)	ARE (1)
.90	.20	14	.571	.643	.663
.60	.20	14	.571	.643	.663
.90	.10	13	.615	.692	.623
.60	.10	14	.571	.643	.623
.90	.05	14	.571	.643	.526
.60	.05	15	.533	.667	.526
.60	.01	21	.381	.571	.265

4. Influence of the choice of A

The value A of the variable $\underline{x}$ – the bound between an effective and defective item – is in general chosen rather arbitrarily. From (2) and (3) it can be derived that in both cases the Pitman-efficiency converges to zero as p_0 converges to zero. This means that there will often be an opportunity to increase the power of the test "by attributes" relative to the test "by variables" by choosing A such that the tolerated fraction defectives p_0 in the lot is not too small.

To demonstrate this we have tabulated (Table 2) the Pitman-efficiency for different values of p_0.

Table 2. Pitman-efficiency for different values of p_0

p_0	ARE (1)	ARE (2)	p_0	ARE (1)	ARE (2)
.01	.265	.072	.15	.655	.426
.025	.409	.140	.20	.663	.490
.05	.526	.224	.25	.661	.539
.10	.623	.342			

The table shows for instance that for increasing values of p_0 the ARE (1) of the tests increases rather rapidly until $p_0 \approx .10$.

This effect is weakened somewhat if one considers the exact values of the RE (relative efficiency) from Table 1 and Appendix V, but the general trend remains the same.

This means that with increasing p_0 the inspection method "by attributes" is becoming less bad in relation to the inspection method "by variables" and that in using "attributes" we should avoid choosing A such that p_0 falls below e.g. 10%.

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Appendix I Derivation of the formulas for the Pitman-efficiency

Let $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n$ be n observations from a normal distribution with unknown mean μ and unknown variance σ^2.

The testing problem was as follows:

$$H_0: \; p \overset{\text{def}}{=} P(\underset{\sim}{x} \leqslant A) \leqslant p_0,$$

for given constants A and p_0 against

$$H_1: \; p > p_0.$$

This two-dimensional problem – $p = p(\mu, \sigma)$ – with composite hypothesis can be reduced to a one-dimensional one with a simple hypothesis H_0 by introducing a new parameter θ:

$$\theta \overset{\text{def}}{=} \frac{\mu - A}{\sigma}. \tag{7}$$

Then

$$p \overset{\text{def}}{=} P(\underset{\sim}{x} \leqslant A) = P\left(\frac{\underset{\sim}{x} - \mu}{\sigma} \leqslant \frac{A - \mu}{\sigma}\right) = P\left(\frac{\underset{\sim}{x} - \mu}{\sigma} \leqslant -\theta\right) = \Phi(-\theta), \tag{8}$$

where Φ is the standard normal distribution function.

Using (8) we are able to reformulate the hypothesis H_0 that $\Phi(-\theta) \leqslant p_0$ as follows:

$$H_0: \; -\theta \leqslant \Phi^{-1}(p_0) \overset{\text{def}}{=} -\theta_0.$$

Note that in fact if an alternative value θ_1 converges to θ_0 the line $(\mu - A)/\sigma = \theta_1$ in the (μ, σ) plane converges to the line $(\mu - A)/\sigma = \theta_0$ (see Fig. 2).

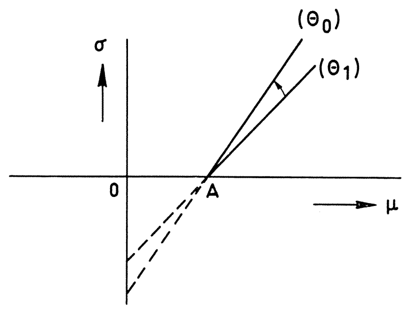

Fig. 2. θ_1 converging to θ_0.

Solving this testing problem "by attributes" we use as test statistic the random variable $\underline{T}_n$ defined by

$$\underline{T}_n \stackrel{\text{def}}{=} \sum_{i=1}^{n} \underline{a}_i \qquad \text{where} \qquad \underline{a}_i = \begin{cases} 1 & \text{if } \underline{x}_i \leqslant A \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

From (8) it can be seen that $\underline{T}_n$ has a binomial distribution with parameters n and $\Phi(-\theta)$.

When we solve the testing problem "by variables" we use as test statistic $\underline{\bar{x}} - k\underline{s}$ or equivalently the test statistic $\underline{t}_n$ defined by

$$\underline{t}_n \stackrel{\text{def}}{=} \frac{\underline{\bar{x}}}{\underline{s}} \sqrt{n}. \tag{10}$$

The random variable $\underline{t}_n$ has a non-central t-distribution with $(n-1)$ degrees of freedom and non-centrality parameter $-\theta\sqrt{n}$ (see e.g. [3]).

The theorem of PITMAN (see [6], [5]) tells us that – under certain regularity conditions which are satisfied in our case – the ARE is equal to the limit as $n \to \infty$ of the ratio of the efficacies of the two methods to be compared. So

$$\text{ARE} = \lim_{n\to\infty} \frac{\text{Eff}_a}{\text{Eff}_v}, \tag{11}$$

where

$$\text{Eff}_a = \frac{\left[\dfrac{\mathrm{d}}{\mathrm{d}\theta} E(\underline{T}_n|\theta)\right]^2_{\theta=\theta_0}}{\sigma^2(\underline{T}_n|\theta_0)} \tag{12}$$

and

$$\text{Eff}_v = \frac{\left[\dfrac{\mathrm{d}}{\mathrm{d}\theta} E(\underline{t}_n|\theta)\right]^2_{\theta=\theta_0}}{\sigma^2(\underline{t}_n|\theta_0)}. \tag{13}$$

From (9) we can see that

$$E(\underline{T}_n|\theta) = n\Phi(-\theta)$$

and

$$\sigma^2(\underline{T}_n|\theta) = n\Phi(-\theta)(1-\Phi(-\theta)).$$

So

$$\text{Eff}_a \stackrel{\text{def}}{=} \frac{\left[\dfrac{\mathrm{d}}{\mathrm{d}\theta} E(\underline{T}_n|\theta)\right]^2_{\theta=\theta_0}}{\sigma^2(\underline{T}_n|\theta_0)} = \frac{\left[n \cdot \dfrac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}\theta_0^2)\right]^2}{np_0(1-p_0)}. \tag{14}$$

Further, from the non-central t-distribution it is well known that

$$E(\underline{t}_n|\theta) = -\frac{n-1}{n-2}\beta_{n-1}\cdot\theta\sqrt{n} \tag{15}$$

and

$$\sigma^2(\underline{t}_n|\theta) = \frac{(n-1)(1+n\theta^2)}{n-3} - \left(\frac{n-1}{n-2}\right)^2 \beta_{n-1}^2\cdot n\cdot\theta^2 \tag{16}$$

where

$$\beta_{n-1} = \sqrt{\frac{2}{n-1}}\cdot\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}.$$

Using (15) and (16), it can now easily be verified that

$$\mathrm{Eff}_v = \frac{\left(\frac{n-1}{n-2}\right)^2\cdot\beta_{n-1}^2\cdot n}{\frac{(n-1)(1+n\theta_0^2)}{n-3} - \left(\frac{n-1}{n-2}\right)^2\cdot\beta_{n-1}^2\cdot n\theta_0^2}. \tag{17}$$

Substituting (14) and (17) into (11) we get (2), as is shown in Appendix IV.

A question which may arise is the following: What happens if the variance of the underlying normal distribution is known?

From the hypothesis and (8) we can see that the inspection problem reduces to a testing problem concerning only the mean μ.

The hypothesis H_0 then becomes:

$$H_0:\ \mu \geqslant \mu_0 \overset{\mathrm{def}}{=} A+\sigma\theta_0.$$

Now the solution "by attributes" is again based on the number of defectives in the sample $\underline{T}_n$ but for the computation of the efficacy we now have to take the derivative of the expectation of $\underline{T}_n$ with respect to μ.

Since $\underline{T}_n$ has a binomial distribution with parameters

$$n \text{ and } p(\mu) \overset{\mathrm{def}}{=} \Phi\left(\frac{A-\mu}{\sigma}\right),$$

we see that

$$p'(\mu) = \frac{1}{\sqrt{2\pi}}\exp(-\tfrac{1}{2}\theta_0^2)\cdot\left(-\frac{1}{\sigma}\right) \qquad \text{and} \qquad p(\mu_0) = p_0.$$

The efficacy Eff_a in this case is equal to

$$\text{Eff}_a = \frac{\left[\frac{\mathrm{d}}{\mathrm{d}\mu} E(\underline{T}_n|\mu)\right]^2_{\mu=\mu_0}}{\sigma^2(\underline{T}_n|\mu_0)} = \frac{np'(\mu_0)}{np_0(1-p_0)} = \frac{\frac{n^2}{2\pi}\exp\left(-\frac{\theta_0^2}{2}\right)\cdot\frac{1}{\sigma^2}}{np_0(1-p_0)}. \tag{18}$$

The solution "by variables" can simply be based on the test statistic $\underline{\bar{x}}$ or equivalently on $\underline{\bar{x}}-k\cdot\sigma$.

As well-known result for this situation we mention

$$\text{Eff}_v = \frac{n}{\sigma^2}. \tag{19}$$

Substituting (18) and (19) into (11) we get (3).

Appendix II Comparison of the derivation due to Schaafsma, Willemze and Hamaker and the derivation by means of the Pitman-concept

In [2] HAMAKER introduces for the attributive case a formula (21) that relates the sample size n with the relative steepness h_0 (see (20)) in the control point p_0 (the fraction defectives in the lot corresponding with a probability of acceptance p that is equal to $\frac{1}{2}$). Here the relative steepness in the control-point is defined by:

$$h_0 = -\left(\frac{p}{P}\frac{\mathrm{d}P}{\mathrm{d}p}\right)_{p=p_0} = -2p_0\left(\frac{\mathrm{d}P}{\mathrm{d}p}\right)_{p=p_0}. \tag{20}$$

Applied to the testing procedure "by attributes" HAMAKER in fact derives for the relative steepness h_0' and p_0' the relation

$$\frac{\pi}{2}h_0'^2 \approx np_0' + 0.06. \tag{21}$$

SCHAAFSMA and WILLEMZE [7] need this result in combination with similar results for the testing procedure "by variables".

In the case σ is known they derived [7, pag. 250] for this case

$$h_0'' = 2p_0''\sqrt{n}e^{\frac{1}{2}z^2}, \tag{22}$$

where z is defined by:

$$p_0'' = \int_z^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\mathrm{d}u.$$

When σ is unknown SCHAAFSMA and WILLEMZE derived [7, pag. 260]:

$$h_0'' = \frac{2p_0''\sqrt{n}e^{\frac{1}{2}z^2}}{\sqrt{1+\frac{z^2}{2}}}. \tag{23}$$

The next step to arrive at a result concerning the sample sizes of both testing procedures was to call two testing procedures equally efficient if for the same value of the control-point p_0 the two relative steepnesses are equal; in the present case this gives:

$$h_0' = h_0'' \quad \text{for} \quad p_0' = p_0'' = p_0. \tag{24}$$

This leads to the results (2) and (3).

In [4] NOETHER explains the way to calculate the ARE by means of the limit

$$\lim_{n_v \to \infty} \frac{n_v}{n_a}, \tag{25}$$

instead of calculating the limit ($n \to \infty$) of the ratio of the efficacies of both testing procedures. The value of n_a in (25) is chosen in such a way that the slopes of the power of both testing procedures in the point p_0 (with power α under H_0) are equal.

Because of the fact that the ARE is not dependent of the actual value of α, we may choose $\alpha = \frac{1}{2}$.

Moreover, equality of relative slopes (see (24)) is equivalent with equality of absolute slopes, because the factor p_0/P_0 is the same for the two testing procedures.

This explains why the results due to SCHAAFSMA and WILLEMZE (1954) and HAMAKER (1949) are identical with the ARE derived by the Pitman-concept.

Appendix III Intuitive approach to the critical region

The hypothesis to be tested was as follows:

$$H_0\colon\ p = P(\underline{x} \leqslant A) \leqslant p_0, \text{ for given constants } A \text{ and } p_0. \tag{26}$$

If H_0 is true, the probability of rejection is not to be larger than a given level of significance α.

We can describe (26) also as follows:

$$H_0\colon\ \mu - \xi\sigma \geqslant A,$$

with ξ such that $P(\underline{u} \geqslant \xi) = p_0$, where $\underline{u}$ has a standard normal distribution.

Because

$$\underline{\bar{x}} = \frac{1}{n}\sum_{i=1}^{n} \underline{x}_i \quad \text{and} \quad \underline{s}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\underline{x}_i - \underline{\bar{x}})^2$$

are the minimal sufficient (and minimum-variance unbiased) estimators of μ and σ^2, is it reasonable to use as test statistic $\underline{\bar{x}}-k\underline{s}$, where k is a critical factor to be determined later. The critical region evidently has to be of the following form:

$$\underline{\bar{x}}-k\underline{s} < B, \tag{27}$$

with k and B such that

$$P(\underline{\bar{x}}-k\cdot\underline{s} < B|\mu-\xi\sigma = A) = \alpha \tag{28}$$

holds for every pair of values of μ and σ with $\mu-\xi\sigma = A$.

When we write for B in (27): $B = \mu-\eta\sigma$, then (28) becomes:

$$P(\underline{\bar{x}}-k\underline{s} < \mu-\eta\sigma|\mu-\xi\sigma = A) = \alpha \tag{29}$$

or

$$P\left(\frac{\dfrac{\underline{\bar{x}}-\mu}{\sigma}\sqrt{n}+\eta\sqrt{n}}{\underline{s}/\sigma} < k\sqrt{n}|\mu-\xi\sigma = A\right) = \alpha. \tag{30}$$

The random variable in the left hand side of the inequality has a non-central t-distribution with $(n-1)$ degrees of freedom and non-centrality parameter $\delta = \eta\sqrt{n}$.

When η is known the bound $k\sqrt{n}$ – and so k – can be found from the tables of the non-central Student distribution. Now μ and σ are unknown, but have to satisfy two equations:

$$\begin{cases} \mu-\xi\sigma = A \\ \mu-\eta\sigma = B, \end{cases} \tag{31}$$

where A and ξ are fixed already and B can still be chosen freely.

When we solve η from (31) we find

$$\eta = \xi + \frac{A-B}{\sigma}, \tag{32}$$

and from (32) σ disappears only if we choose $B = A$.

Thus only when choosing $B = A$, η will be known because it does not depend on the unknown σ, and only then k can be found from (30).

A consequence then is $\eta = \xi$ and we have as critical region $\underline{\bar{x}}-k\underline{s} < A$, as was to be proved.

LEHMANN [3] derives the following critical region by means of the principle of invariance:

$$\frac{\underline{\bar{x}}-A}{\underline{s}}\sqrt{\frac{n}{n-1}} < C. \tag{33}$$

Taking

$$k=\sqrt{\frac{n-1}{n}}\cdot C,$$

we again arrive at (33).

Appendix IV Computation of ARE (1)

From

$$\beta_{n-1}=\sqrt{\frac{2}{n-1}}\,\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\qquad \text{[c.f. (16)]},$$

the following inequality can be derived:

$$\frac{2n-1}{2n}\leqslant\beta_{n-1}^2\leqslant\frac{2n}{2n+1},\qquad \text{for}\qquad n=1,2,\ldots \tag{34}$$

Using (34) we have

$$\lim_{n\to\infty} n\left\{\frac{n-1}{n-3}-\left(\frac{n-1}{n-2}\right)^2\frac{2n-3}{2n-2}\right\}\leqslant\lim_{n\to\infty} n\left\{\frac{n-1}{n-3}-\left(\frac{n-1}{n-2}\right)^2\beta_{n-1}^2\right\}\leqslant$$
$$\leqslant\lim_{n\to\infty} n\left\{\frac{n-1}{n-3}-\left(\frac{n-1}{n-2}\right)^2\frac{2(n-1)}{2n-1}\right\},\qquad \text{for}\qquad n=2,3,\ldots \tag{35}$$

Now

$$\lim_{n\to\infty} n\left\{\frac{n-1}{n-3}-\left(\frac{n-1}{n-2}\right)^2\frac{2n-3}{2n-2}\right\}=\tfrac{1}{2} \tag{36}$$

and

$$\lim_{n\to\infty} n\left\{\frac{n-1}{n-3}-\left(\frac{n-1}{n-2}\right)^2\frac{2(n-1)}{2n-1}\right\}=\tfrac{1}{2}. \tag{37}$$

So from (35), (36) and (37) we can conclude that

$$\lim_{n\to\infty} n\left\{\frac{n-1}{n-3}-\left(\frac{n-1}{n-2}\right)^2\beta_{n-1}^2\right\}=\tfrac{1}{2}. \tag{38}$$

Using (11), (14) and (17) we have

$$\text{ARE}(1)=\lim_{n\to\infty}\frac{\left[\frac{n}{\sqrt{2\pi}}\exp(-\frac{1}{2}\theta_0^2)\right]^2}{np_0(1-p_0)}\cdot\frac{\frac{(n-1)(1+\theta_0^2)}{n-3}-\left(\frac{n-1}{n-2}\right)^2\beta_{n-1}^2\cdot n\cdot\theta_0^2}{\left(\frac{n-1}{n-2}\right)^2\beta_{n-1}^2\cdot n}. \tag{39}$$

Using (38) and using the relation $\lim_{n\to\infty} \beta_{n-1}^2 = 1$, we can easily verify:

$$\text{ARE}(1) = \frac{1 + \frac{\theta_0^2}{2}}{2\pi p_0(1-p_0)e^{\theta_0{}^2}} \tag{40}$$

Appendix V Tables for comparing the exact ratio of the sample sizes with the ARE in the case that σ is unknown

In this section we show by means of several tables that the Pitman-efficiency is a good approximation of the exact ratio of the sample sizes, necessary to achieve an equal performance for both tests in the case that σ is unknown.

To describe the method by which we computed our tables we denote:

α	the level of significance of the tests,
p_0	the parameter which characterises the hypothesis tested,
p	the parameter which characterises the alternative hypothesis,
n_a	the sample size for inspection "by attributes",
n_v	the sample size for inspection "by variables",
$\underline{T}_n$	the test statistic "by attributes", based on n observations,
$\underline{t}_n$	the test statistic "by variables", based on n observations,
$\beta_a = \beta_a(p; n_a)$	the power of the test "by attributes", in the parameterpoint p, based on n_a observations,
$\beta_v = \beta_v(p; n_v)$	the power of the test "by variables", in the parameterpoint p, based on n_v observations,
$\beta_{\min}$	the minimum power we require for both tests in some alternative parameterpoint.

For given α, p_0, $\beta_{\min}$ and n_v we calculated the tables in four steps:

Step 1

Find the alternative parameterpoint p_1, for which (see Fig. 3):

$$\beta_v = \beta_{\min}, \text{ under the condition } \beta_v(p_0; n_v) = \alpha.$$

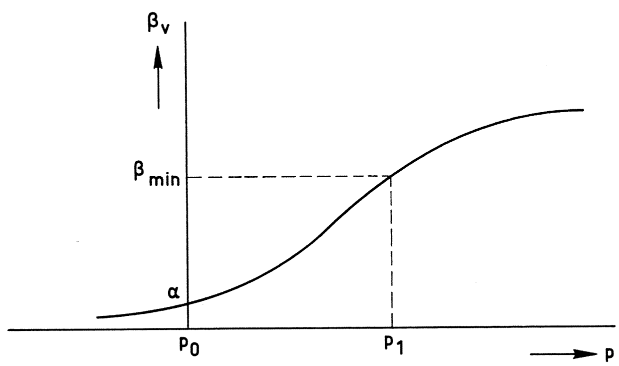

Fig. 3. The power of the test by variables.

Step 2

Find, for given α, p_0, $\beta_{\min}$ and p_1, the minimal integer n_a, for which (c.f. Fig. 4):

$$\beta_a \geqslant \beta_{\min}, \text{ under the condition } \alpha_a \stackrel{\text{def}}{=} \beta_a(p_0; n_a) \leqslant \alpha.$$

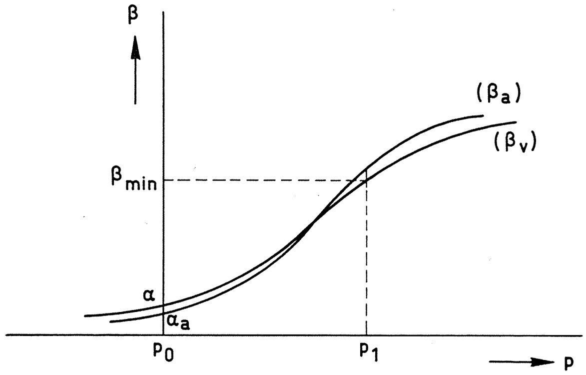

Fig. 4. The powers β_a and β_v determined by step 1 and 2.

Step 3

Find for given α_a, β_a, p_1 from step 1 and 2, and p_0 the maximal integer n_{v2} and the minimal integer n_{v3} for which resp. (c.f. Fig. 5):

$$\beta_{v2} \stackrel{\text{def}}{=} \beta_v(p; n_{v2}) < \beta_a, \text{ under } \beta_v(p_0; n_{v2}) = \alpha_a$$

and

$$\beta_{v3} \stackrel{\text{def}}{=} \beta_v(p; n_{v3}) \geqslant \beta_a, \text{ under } \beta_v(p_0; n_{v3}) = \alpha_a.$$

We remark that $n_{v3} = n_{v2} + 1$.

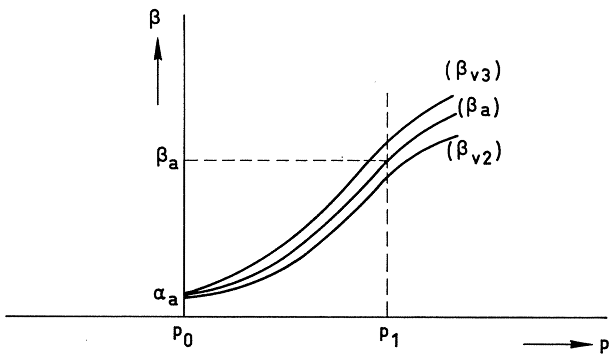

Fig. 5. The power of the tests according to step 3.

Step 4

Put $n_{v1} \stackrel{\text{def}}{=} n_v$ and compute

$$\mathrm{RE}(1) \stackrel{\text{def}}{=} \frac{n_{v1}}{n_a},$$

$$\mathrm{RE}(2) \stackrel{\text{def}}{=} \frac{n_{v2}}{n_a},$$

$$\mathrm{RE}(3) \stackrel{\text{def}}{=} \frac{n_{v3}}{n_a}.$$

We calculated the following tabels for $\alpha = .05$; $p_0 = .20$, .10, .05 and $\beta_{\min} = .90$, .60.

For $p_0 = .01$ we only computed the table for $\beta_{\min} = .60$.

In general RE(2) appears to be the best approximation of the ARE (1).

Table 3. Exact and asymptotic relative efficiency for σ unknown, $\alpha = .05$, $p_0 = .20$ and $\beta_{\min} = .90$

ARE (1) = .663			
n_v	RE (1)	RE (2)	RE (3)
4	.571	.571	.714
7	.538	.615	.692
10	.588	.647	.706
14	.583	.625	.667
18	.621	.655	.690
22	.611	.639	.667
26	.591	.659	.682
30	.625	.646	.667
40	.656	.656	.672
50	.617	.654	.667
70	.631	.658	.667
100	.645	.658	.665
120	.645	.661	.667

Table 4. Exact and asymptotic relative efficiency for σ unknown, $\alpha = .05$, $p_0 = .20$ and $\beta_{\min} = .60$

ARE (1) = .663			
n_v	RE (1)	RE (2)	RE (3)
4	.444	.667	.778
7	.500	.643	.714
10	.500	.650	.700
14	.560	.640	.680
18	.621	.621	.655
22	.550	.650	.675
26	.591	.636	.659
30	.577	.635	.654
40	.615	.646	.662
50	.588	.647	.659
70	.609	.652	.661
100	.625	.656	.663
120	.628	.654	.660

Table 5. Exact and asymptotic relative efficiency for σ unknown, $\alpha = .05$, $p_0 = .10$ and $\beta_{\min} = .90$

ARE (1) = .623			
n_v	RE (1)	RE (2)	RE (3)
4	.571	.571	.714
7	.583	.667	.750
10	.556	.667	.722
14	.560	.680	.720
18	.667	.667	.704
22	.647	.676	.706
26	.634	.659	.683
30	.625	.667	.687
40	.635	.667	.683
50	.641	.654	.667
70	.636	.655	.664
100	.629	.648	.654
120	.625	.646	.651

Table 6. Exact and asymptotic relative efficiency for σ unknown, $\alpha = .05$, $p_0 = .10$ and $\beta_{\min} = .60$

ARE (1) = .623			
n_v	RE (1)	RE (2)	RE (3)
4	.500	.625	.750
7	.538	.615	.692
10	.526	.632	.684
14	.538	.654	.692
18	.562	.656	.687
22	.564	.641	.667
26	.634	.634	.659
30	.625	.646	.667
40	.580	.638	.652
50	.633	.646	.658
70	.598	.641	.650
100	.625	.637	.644
120	.619	.634	.639

Table 7. Exact and asymptotic relative efficiency for σ unknown, $\alpha = .05$, $p_0 = .05$ and $\beta_{\min} = .90$

ARE (1) = .526			
n_v	RE (1)	RE (2)	RE (3)
4	.667	.667	.833
7	.538	.692	.769
10	.625	.687	.750
14	.583	.667	.708
18	.643	.643	.679
22	.595	.649	.676
26	.553	.638	.660
30	.588	.627	.647
40	.606	.621	.636
50	.617	.617	.630
70	.574	.607	.615
100	.585	.596	.602
120	.558	.591	.595

Table 8. Exact and asymptotic relative efficiency for σ unknown, $\alpha = .05$, $p_0 = .05$ and $\beta_{\min} = .60$

ARE (1) = .526			
n_v	RE (1)	RE (2)	RE (3)
4	.571	.714	.857
7	.500	.643	.714
10	.625	.625	.687
14	.538	.615	.654
18	.500	.611	.639
22	.564	.615	.641
26	.520	.600	.620
30	.577	.596	.615
40	.597	.597	.612
50	.538	.591	.602
70	.565	.581	.589
100	.541	.573	.578
120	.550	.569	.573

Table 9. Exact and asymptotic relative efficiency for σ unknown, $\alpha = .05$, $p_0 = .01$ and $\beta_{\min} = .60$

ARE (1) = .265			
n_v	RE (1)	RE (2)	RE (3)
4	.800	.800	1.000
7	.368	.579	.632
10	.400	.520	.560
14	.452	.516	.548
18	.321	.464	.482
22	.344	.453	.469
26	.366	.437	.451
30	.390	.429	.442
40	.333	.400	.408
50	.373	.388	.396
70	.359	.369	.374
100	.317	.349	.352
120	.312	.344	.346

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