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A note on a paper by D.S. Moore on chi-square statistics \*)

by

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Summary

In this note we draw attention to an elementary proof of the asymptotic negligibility of the remainder terms in a paper by D.S. Moore (1971) on the limiting distribution of chi-square statistics. The asymptotic negligibility turns out to be an immediate consequence of a modification of Lemma 1 by Bahadur (1966) in more dimensions.

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\*) This paper is not for review; it is meant for publication in a journal.

## 1. INTRODUCTION

Suppose that we are given a sequence  $X_1, X_2, \dots$  of mutually independent and identically distributed  $k$ -dimensional random vectors. All random vectors are supposed to be defined on a single probability space  $(\Omega, \mathcal{A}, P)$  and their common distribution function (df)  $F_\theta$  depends on an  $m$ -dimensional parameter  $\theta$  which is restricted to an open subset  $T$  of  $m$ -dimensional number space  $\mathbb{R}^m$ . Given any positive integer  $n$ , we define the empirical df  $F_n$  based on the first  $n$  random vectors of the sequence in the usual way.

In the context of testing goodness of fit, as described in a paper by Moore (1971),  $\mathbb{R}^k$  is partitioned into a fixed finite number of cells, where the cell boundaries are allowed to be functions of the estimated parameter values. Let us proceed along the lines of Moore's paper and define for  $i = 1, 2, \dots, k$  a non-random partition of the  $x_i$  - axis by functions of  $\theta \in T$ , satisfying

$$(1.1) \quad -\infty = \xi_{i,0}(\theta) < \xi_{i,1}(\theta) < \dots < \xi_{i,v_i-1}(\theta) < \xi_{i,v_i}(\theta) = \infty.$$

The partitions of the axes induce a partition of  $\mathbb{R}^k$  into  $v = \prod_{i=1}^k v_i$  cells. According to a specific enumeration these cells will be denoted by  $I_\sigma(\theta)$ ,  $\sigma = 1, 2, \dots, v$ . Suppose that for each positive integer  $n$  we have an estimator  $\hat{\theta}_n = \hat{\theta}_n(X_1, X_2, \dots, X_n)$  of  $\theta$ . To  $I_\sigma(\theta)$  there corresponds the random cell  $I_\sigma(\hat{\theta}_n)$  when  $\theta$  is replaced by  $\hat{\theta}_n$  in (1.1). The mass assigned to any Borel set  $B \subset \mathbb{R}^k$  by the df  $F_\theta$  will be denoted by  $F_\theta\{B\}$ , and similarly the mass assigned to  $B$  by the empirical df  $F_n$  will be denoted by  $F_n\{B\}$ . The latter, of course, equals the number of  $\{X_1, X_2, \dots, X_n\}$  contained in  $B$ , divided by  $n$ .

In the search for the asymptotic distribution of chi-square type statistics

$$(1.2) \quad T_n = \sum_{\sigma=1}^v n [F_n\{I_\sigma(\hat{\theta}_n)\} - F_{\hat{\theta}_n}\{I_\sigma(\hat{\theta}_n)\}]^2 [F_{\hat{\theta}_n}\{I_\sigma(\hat{\theta}_n)\}]^{-1},$$

one may write  $n^{\frac{1}{2}}[F_n\{I_\sigma(\hat{\theta}_n)\} - F_{\hat{\theta}_n}\{I_\sigma(\hat{\theta}_n)\}] = A_{1n} + A_{2n} + B_{1n} + B_{2n}$ , where

$$A_{1n} = n^{\frac{1}{2}} [F_n\{I_\sigma(\theta_0)\} - F_{\theta_0}\{I_\sigma(\theta_0)\}],$$

$$A_{2n} = n^{\frac{1}{2}} [F_{\theta_0}\{I_\sigma(\hat{\theta}_n)\} - F_{\hat{\theta}_n}\{I_\sigma(\hat{\theta}_n)\}],$$

$$B_{1n} = n^{\frac{1}{2}} [F_n\{I_\sigma(\hat{\theta}_n) - I_\sigma(\theta_0)\} - F_{\theta_0}\{I_\sigma(\hat{\theta}_n) - I_\sigma(\theta_0)\}],$$

$$B_{2n} = n^{\frac{1}{2}} [F_{\theta_0}\{I_\sigma(\theta_0) - I_\sigma(\hat{\theta}_n)\} - F_n\{I_\sigma(\theta_0) - I_\sigma(\hat{\theta}_n)\}],$$

and where  $\theta_0$  is the true parameter value. The expression on the left of Moore's formula (2.2) equals  $B_{1n} + B_{2n}$ , but we have arranged the terms somewhat differently for purposes that will become clear below.

An essential part of the proof of Theorem 1 by Moore (1971) consists of showing that

$$(1.3) \quad B_{1n} + B_{2n} \xrightarrow{P} 0,$$

as  $n \rightarrow \infty$ . Moore derives this result by appealing to rather advanced papers by Dudley (1966) and Neuhaus (1969). It is the purpose of this note to draw attention to a more elementary proof of (1.3), by showing that it is an immediate consequence of a modification of Lemma 1 by Bahadur (1966) in more dimensions. In this form Bahadur's lemma has been given by W.R. van Zwet. For completeness we shall formulate the lemma, a proof of which may

be found in Ruymgaart (1972, 1973) for  $k = 2$ . (The proof for  $k > 2$  is completely similar.) Suppose that for each  $n = 1, 2, \dots$  we are given a random sample of size  $n$  from an arbitrary fixed  $k$ -variate df  $F$  (continuous or not). The corresponding  $k$ -variate empirical df will be denoted by  $F_n$ . By an interval  $I$  in  $\mathbb{R}^k$  we understand the product set of  $k$  intervals on the real line.

LEMMA (van Zwet). Let  $I_1, I_2, \dots$  be a sequence of intervals in  $\mathbb{R}^k$  and let  $I_n^* = \{I_n^* : I_n^* \text{ is an interval contained in } I_n\}$ ,  $n = 1, 2, \dots$ . Then, as  $n \rightarrow \infty$ ,

$$\sup_{I_n^* \in I_n} |F_n\{I_n^*\} - F\{I_n^*\}| = O_P([n^{-1} F\{I_n\}]^{\frac{1}{2}}),$$

uniformly in all sequences of intervals  $I_1, I_2, \dots$  and all  $k$ -variate dfs  $F$  (continuous or not).

Let us for the moment restrict attention to regularity conditions on  $F_{\theta_0}$ , although some other conditions will also be needed (see Section 2). Using only continuity of  $F_{\theta_0}$  it follows almost immediately from the lemma that  $B_{1n} + B_{2n} = o_P(1)$ , as  $n \rightarrow \infty$ , which is Moore's result. In Moore's paper it is assumed, for other purposes, that  $F_{\theta_0}$  has a continuous density. Under the latter stronger assumption we deduce from the lemma in quite the same way that  $B_{1n} + B_{2n} = O_P(n^{-1/4})$ , as  $n \rightarrow \infty$ . The above illustrates once more the usefulness of (this modification of) Bahadur's lemma, which has also proved essential for handling some of the second order terms occurring in the proofs of asymptotic normality, under fixed alternatives, of certain nonparametric test statistics (Sen (1970), Ruymgaart (1972, 1973)).

## 2. PROOF OF THE ASYMPTOTIC NEGLIGIBILITY

The first assumption needed for the proof of (1.3) is that the function

$$(2.1) \quad \partial \xi_{i,j}(\theta) / \partial \theta_1$$

exists and is continuous for  $\theta \in T$  and  $i = 1, 2, \dots, k$ ,  $j = 1, 2, \dots, v_i - 1$ ,  $i = 1, 2, \dots, m$ .

The second assumption is that the sequence of estimators  $\hat{\theta}_1, \hat{\theta}_2, \dots$  satisfies

$$(2.2) \quad |\hat{\theta}_n - \theta_0| = O_p(n^{-\frac{1}{2}}),$$

as  $n \rightarrow \infty$ , where  $\theta_0$  is the true parameter value.

These assumptions guarantee for each  $\varepsilon > 0$  the existence of a constant  $M_1 = M_{1\varepsilon}$  such that the set

$$(2.3) \quad \Omega_{1n} = \bigcap_{i=1}^k \bigcap_{j=1}^{v_i-1} \{ |\xi_{i,j}(\hat{\theta}_n) - \xi_{i,j}(\theta_0)| \leq M_1 n^{-\frac{1}{2}} \}$$

has probability  $P(\Omega_{1n}) \geq 1 - \varepsilon/2$  for all  $n = 1, 2, \dots$

By symmetry we need only consider  $B_{1n}$ . Let us fix  $\sigma$  and introduce for all  $i = 1, 2, \dots, k$  and  $j = 1, \dots, v_i - 1$  the intervals

$$I_{n,i,j} = \mathbb{R}^{i-1} \times [\xi_{i,j}(\theta_0) - M_1 n^{-\frac{1}{2}}, \xi_{i,j}(\theta_0) + M_1 n^{-\frac{1}{2}}] \times \mathbb{R}^{k-i},$$

$$I_{n,i,j}^* = I_{n,i,j} \cap \{ I_\sigma(\hat{\theta}_n) - I_\sigma(\theta_0) \}.$$

Note that for all  $\omega \in \Omega_{1n}$  we have  $\{ I_\sigma(\hat{\theta}_n) - I_\sigma(\theta_0) \} = \bigcup_{i=1}^k \bigcup_{j=1}^{v_i-1} I_{n,i,j}^*$ .

If  $F_{\theta_0}$  is given to be only continuous it follows that

$\max_{i,j} F_{\theta_0} \{I_{n,i,j}\} = c_n$ , where  $c_n \rightarrow 0$  as  $n \rightarrow \infty$ . The lemma of Section 1 ensures the existence of a number  $M_2 = M_{2\varepsilon}$ , such that the set

$$(2.4) \quad \Omega_{2n} = \cap_{i=1}^k \cap_{j=1}^{v_i-1} \{ |F_n \{I_{n,i,j}^*\} - F_{\theta_0} \{I_{n,i,j}^*\}| \leq M_2 n^{-\frac{1}{2}} c_n^{\frac{1}{2}} \}$$

has probability  $P(\Omega_{2n}) \geq 1 - \varepsilon/2$  for all  $n = 1, 2, \dots$ . Denoting the characteristic function of the set  $\Omega_{1n} \cap \Omega_{2n}$  by  $\chi(\Omega_{1n} \cap \Omega_{2n})$  it follows that

$$(2.5) \quad \chi(\Omega_{1n} \cap \Omega_{2n}) \mid B_{1n} \mid \leq \left[ \sum_{i=1}^k (v_i - 1) \right] M_2 c_n^{\frac{1}{2}} \rightarrow 0,$$

as  $n \rightarrow \infty$ . Because  $P(\Omega_{1n} \cap \Omega_{2n}) \geq 1 - \varepsilon$  for all  $n = 1, 2, \dots$  and  $\varepsilon > 0$  is arbitrary we may conclude from (2.5) that  $B_{1n} = o_p(1)$ .

In the case where  $F_{\theta_0}$  has a continuous density with respect to Lebesgue measure we find that  $\max_{i,j} F_{\theta_0} \{I_{n,i,j}\} \leq M_3 n^{-\frac{1}{2}}$  for some constant  $M_3$  and all  $n = 1, 2, \dots$ . The lemma applies in the same way so that for some constant  $M_2' = M_{2\varepsilon}'$  we may use (2.4) and (2.5) with  $M_2$  replaced by  $M_2'$  and  $c_n$  by  $M_3 n^{-\frac{1}{2}}$ . Consequently we now have that  $B_{1n} = o_p(n^{-1/4})$ .



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