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TESTING FOR k-SAMPLE LOCATION AND SCALE ALTERNATIVES, I *)

by YVES LEPAGE **)

ABSTRACT

In a k-sample case ($k \geq 2$), the problem of testing identity of distribution versus alternatives containing both location and scale parameters is studied. A contiguous sequence of alternatives is constructed and for those alternatives, an asymptotically most powerful rank test is found.

1. INTRODUCTION

The purpose of this work is to derive an asymptotically most powerful linear rank test for the k-sample ($k \geq 2$) problem where the distributions are differing both in their location and scale parameters.

A contiguous sequence of alternatives is constructed and the asymptotic distribution of linear rank statistics under such contiguous alternatives is found by specializing the results of Beran (1970). A rank test asymptotically most powerful among all tests is also deduced in a similar way as Hájek and Šidák (1967).

2. ASYMPTOTIC DISTRIBUTION

Let N_ν ($\nu=1,2,\ldots$) be a sequence of positive integers such that $N_\nu \to \infty$ when $\nu \to \infty$. For $\nu=1,2,\ldots$, let $(A_{\nu 1},\ldots,A_{\nu k})$, $k \geq 2$, be a partition of $\{1,\ldots,N_\nu\}$ and put $n_{\nu j} = \text{card } A_{\nu j}$, $j=1,\ldots,k$. Moreover, for each ν consider

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a sequence of random variables $X_{\nu 1},\ldots,X_{\nu N_\nu}$ and denote by $R_{\nu i}$ the rank of $X_{\nu i}$ among $X_{\nu 1},\ldots,X_{\nu N_\nu}$; $i=1,\ldots,N_\nu$.

Suppose that under the hypothesis H_ν, the random variables $X_{\nu 1},\ldots,X_{\nu N_\nu}$ are independently and identically distributed according to a continuous distribution function and suppose that under the alternatives K_ν, the joint density of $X_{\nu 1},\ldots,X_{\nu N_\nu}$ is given by

$$(2.1)\qquad q_\nu = \prod_{j=1}^{k} \prod_{i\in A_{\nu j}} e^{-c_j/\sqrt{N_\nu}} f(e^{-c_j/\sqrt{N_\nu}} x_i - d_j/\sqrt{N_\nu})$$

with $\underset{\sim}{c} = (c_1,c_2,\ldots,c_k)' \in \mathbb{R}^k$, $\underset{\sim}{d} = (d_1,d_2,\ldots,d_k)' \in \mathbb{R}^k$, $c_1 = d_1 = 0$ and at least one of the vectors $\underset{\sim}{c}$ or $\underset{\sim}{d}$ non null, and a density function f which satisfies the following condition:

Condition A.

Let $\Theta \subseteq \mathbb{R}^2$ be an open subset containing (0,0) and for $\underset{\sim}{\theta}=(\theta_1,\theta_2)' \in \Theta$, put

$$(2.2)\qquad f(x,\underset{\sim}{\theta}) = e^{-\theta_1} f(e^{-\theta_1}x-\theta_2) .$$

(i) For almost all x, $f(x,\underset{\sim}{\theta})$ is continuously differentiable with respect to $\underset{\sim}{\theta}$ whenever $\underset{\sim}{\theta} \in \Theta$.

(ii) If $||\cdot||$ represents the usual Euclidean norm,

$$(2.3)\qquad \lim_{||\underset{\sim}{\theta}||\to 0} \int_{-\infty}^{\infty} \left[\left(\frac{\partial f(x,\underset{\sim}{\theta})}{\partial\theta_1}\right)^2 \Big/ f(x,\underset{\sim}{\theta})\right]dx = I_1(f) < \infty$$

and

$$(2.4)\qquad \lim_{||\underset{\sim}{\theta}||\to 0} \int_{-\infty}^{\infty} \left[\left(\frac{\partial f(x,\underset{\sim}{\theta})}{\partial\theta_2}\right)^2 \Big/ f(x,\underset{\sim}{\theta})\right]dx = I(f) < \infty$$

with

$$(2.5)\qquad I_1(f) = \int_0^1 \phi_1^2(u,f)du \quad\text{and}\quad I(f) = \int_0^1 \phi^2(u,f)du$$

where, if F is the distribution function corresponding to f,

$$\phi_1(u,f) = -1 - F^{-1}(u)\,\frac{f'(F^{-1}(u))}{f(F^{-1}(u))} \quad \text{and} \quad \phi(u,f) = -\frac{f'(F^{-1}(u))}{f(F^{-1}(u))}\,, \tag{2.6}$$

$0 < u < 1$.

This regularity condition on the densities is the adaptation for a location and a scale parameter alternative of Condition A of Beran (1970). One can easily verify that the normal, the logistic and the Cauchy densities satisfy Condition A but the exponential, the double exponential and the double quadratic ($f(x) = \frac{1}{2}(1+|x|)^{-2}$) densities don't since from Nickerson, Spencer and Steenrod (1959), p.146, the continuous differentiability of $f(x,\underset{\sim}{\theta})$ is equivalent to the existence and continuity of the column vector of first partial derivatives with respect to $\underset{\sim}{\theta}$, $(\partial f(x,\underset{\sim}{\theta})/\partial\theta_1, \partial f(x,\underset{\sim}{\theta})/\partial\theta_2)'$. Also, if f satisfies Condition A, we conclude from Lemma 3.3 of Beran (1970), that

$$\int_0^1 \phi_1(u,f)du = \int_0^1 \phi(u,f)du = 0\ . \tag{2.7}$$

For simplicity of notation, let for $i \in A_{\nu j}$, $j=1,\ldots,k$,

$$\underset{\sim}{\theta}_{\nu i} = (c_j/\sqrt{N_\nu},\ d_j/\sqrt{N_\nu})'\ , \qquad \nu=1,2,\ldots\ , \tag{2.8}$$

and

$$\overline{\underset{\sim}{\theta}}_\nu = \frac{1}{N_\nu}\sum_{i=1}^{N_\nu} \underset{\sim}{\theta}_{\nu i}\ . \tag{2.9}$$

Consider now the linear rank statistics

$$S_\nu = \sum_{i=1}^{N_\nu} \underset{\sim}{\gamma}'_{\nu i}\,\underset{\sim}{a}_\nu(R_{\nu i}) \tag{2.10}$$

where $\underset{\sim}{\gamma}_{\nu 1},\ldots,\underset{\sim}{\gamma}_{\nu N_\nu}$ are vectors and $\underset{\sim}{a}_\nu(1),\ldots,\underset{\sim}{a}_\nu(N_\nu)$ are the values of a vector score function $\underset{\sim}{a}_\nu(\cdot)$.

We will say that a sequence of vector score functions $\underset{\sim}{a}_\nu(\cdot)$, $\nu=1,2,\ldots$, is *generated* by a vector valued function $\underset{\sim}{\phi}(u)$, $0 < u < 1$, if

(i) $\int_0^1 \underset{\sim}{\phi}'(u)\underset{\sim}{\phi}(u)du < \infty$ and $\int_0^1 (\underset{\sim}{\phi}(u)-\bar{\underset{\sim}{\phi}})'(\underset{\sim}{\phi}(u)-\bar{\underset{\sim}{\phi}})du > 0$ where $\bar{\underset{\sim}{\phi}} = \int_0^1 \underset{\sim}{\phi}(u)du$.

(ii) $\lim_{\nu\to\infty} \int_0^1 \|\underset{\sim}{a}_\nu(1+[uN_\nu]) - \underset{\sim}{\phi}(u)\|^2 du = 0$ with $[uN_\nu]$ denoting the largest integer not exceeding uN_ν.

In Beran (1970), one can find methods for constructing vector score functions that are generated by a given vector function $\underset{\sim}{\phi}(u)$, $0 < u < 1$.

Further, for an ordered sample $U_\nu^{(1)} < \ldots < U_\nu^{(N_\nu)}$ from the uniform distribution on [0,1], we will let

$$(2.11) \qquad \underset{\sim}{a}_\nu(i,f) = \begin{bmatrix} E\ \phi_1(U_\nu^{(i)},f) \\ \\ E\ \ \phi(U_\nu^{(i)},f) \end{bmatrix} = \begin{bmatrix} a_{1\nu}(i,f) \\ \\ a_\nu(i,f) \end{bmatrix}, \quad i=1,\ldots,N_\nu \ .$$

One can easily show that if f satisfies Condition A then, the sequence of vector score functions $\underset{\sim}{a}_\nu(\cdot,f)$, $\nu=1,2,\ldots$, is generated by

$$(2.12) \qquad \underset{\sim}{\phi}(u,f) = \begin{bmatrix} \phi_1(u,f) \\ \\ \phi(u,f) \end{bmatrix}, \quad 0 < u < 1 \ .$$

More generally, if for $j=1,\ldots,k$ the sequence of score functions $a_\nu^{(j)}(\cdot)$, $\nu=1,2,\ldots$, is generated by $\phi^{(j)}(u)$, $0 < u < 1$, then the sequence of vector score functions $\underset{\sim}{a}_\nu(\cdot) = (a_\nu^{(1)}(\cdot),\ldots,a_\nu^{(k)}(\cdot))'$, $\nu=1,2,\ldots$, is generated by the vector valued function $\underset{\sim}{\phi}(u) = (\phi^{(1)}(u),\ldots,\phi^{(k)}(u))'$, $0 < u < 1$.

The usual regularity condition on the vectors of constants $\underset{\sim}{\gamma}_{\nu 1},\ldots,\underset{\sim}{\gamma}_{\nu N_\nu}$ is represented by

Condition E.

If $\bar{\underset{\sim}{\gamma}}_\nu = \frac{1}{N_\nu} \sum_{i=1}^{N_\nu} \underset{\sim}{\gamma}_{\nu i}$,

(i) for $\nu=1,2,\ldots$, $\sum_{i=1}^{N_\nu} \|\underset{\sim}{\gamma}_{\nu i} - \bar{\underset{\sim}{\gamma}}_\nu\|^2 > 0$.

(ii) $\lim_{\nu\to\infty} \sum_{i=1}^{N_\nu} \|\underset{\sim}{\gamma}_{\nu i} - \bar{\underset{\sim}{\gamma}}_\nu\|^2 \Big/ \max_{1\le i\le N_\nu} \|\underset{\sim}{\gamma}_{\nu i} - \bar{\underset{\sim}{\gamma}}_\nu\|^2 = \infty$.

The following theorem gives the asymptotic distribution of linear rank statistics under the hypothesis H_ν. The proof is omitted since it is a direct consequence of Theorem 2.3 of Beran (1970).

Theorem 2.1. *Let the sequence of vector score functions* $\underset{\sim}{a}_\nu(\cdot)$, $\nu=1,2,\ldots$, *be generated by a vector function* $\underset{\sim}{\phi}(u)$, $0 < u < 1$, *and assume that* Condition E *is satisfied. Then, under* H_ν, *the statistics* S_ν, *given by* (2.10), *are asymptotically normal* (μ_ν, σ_ν^2) *with*

$$(2.13) \qquad \mu_\nu = \sum_{i=1}^{N_\nu} \underset{\sim}{\gamma}'_{\nu i} \bar{\underset{\sim}{\Phi}}$$

and

$$(2.14) \qquad \sigma_\nu^2 = \sum_{i=1}^{N_\nu} (\underset{\sim}{\gamma}_{\nu i}-\bar{\underset{\sim}{\gamma}}_\nu)' D (\underset{\sim}{\gamma}_{\nu i}-\bar{\underset{\sim}{\gamma}}_\nu)$$

where

$$(2.15) \qquad D = \int_0^1 (\underset{\sim}{\phi}(u)-\bar{\underset{\sim}{\phi}})(\underset{\sim}{\phi}(u)-\bar{\underset{\sim}{\phi}})' du.$$

In the next theorem, the contiguity of the alternatives K_ν with respect to the hypothesis H_ν is established.

Theorem 2.2. *Suppose that* $\lim_{\nu\to\infty} n_{\nu j}/N_\nu = \lambda_j$ *for* $j=1,\ldots,k$. *Then, if* f *satisfies* Condition A, K_ν *are contiguous to* H_ν.

Proof. Let $p_\nu = \prod_{i=1}^{N_\nu} f(x_i)$. From Hájek and Šidák (1967), p.202, it is sufficient to show that the densities $\{q_\nu\}$ are contiguous to the densities $\{p_\nu\}$.

We have that

$$\max_{1\le i\le N_\nu} \|\underset{\sim}{\theta}_{\nu i}\|^2 = \max_{2\le j\le k}\left(\frac{c_j^2+d_j^2}{N_\nu}\right) \to 0 \quad \text{when} \quad \nu\to\infty , \tag{2.16}$$

$$\sum_{i=1}^{N_\nu} \|\underset{\sim}{\theta}_{\nu i}\|^2 = \sum_{j=2}^{k} \frac{n_{\nu j}}{N_\nu}(c_j^2+d_j^2) \le \sum_{j=2}^{k}(c_j^2+d_j^2) < \infty \qquad (\nu=1,2,\ldots) \tag{2.17}$$

and,

$$\sum_{i=1}^{N_\nu} \underset{\sim}{\theta}'_{\nu i}\left[\int_0^1 \underset{\sim}{\phi}(u,f)\underset{\sim}{\phi}(u,f)'du\right]\underset{\sim}{\theta}_{\nu i} =$$

$$= \sum_{j=2}^{k}\frac{n_{\nu j}}{N_\nu}\left(c_j^2\, I_1(f)+2c_jd_j\int_0^1 \phi_1(u,f)\phi(u,f)du+d_j^2 I(f)\right) \tag{2.18}$$

$$\to \sum_{j=2}^{k}\lambda_j\int_0^1 (c_j\phi_1(u,f)+d_j\phi(u,f))^2du < \infty \quad \text{when} \quad \nu\to\infty .$$

Thus, since by hypothesis f satisfies Condition A, we conclude from Theorem 3.1 of Beran (1970) that $\{q_\nu\}$ are contiguous to $\{p_\nu\}$ and the proof is complete. □

The last theorem of this section gives the asymptotic distribution of linear rank statistics under the contiguous sequence of alternatives K_ν.

Theorem 2.3. *Let the sequence of vector score functions* $\underset{\sim}{a}_\nu(\cdot)$, $\nu=1,2,\ldots$, *be generated by a vector function* $\underset{\sim}{\phi}(u)$, $0 < u < 1$, *and assume that* f *satisfies* Condition A, *and* Condition E *is verified. Then, under* K_ν, *the statistics* S_ν, *given by* (2.10), *are asymptotically normal* (η_ν,σ_ν^2) *with*

$$\eta_\nu = \sum_{i=1}^{N_\nu}(\underset{\sim}{\gamma}_{\nu i}-\bar{\underset{\sim}{\gamma}}_\nu)'B(\underset{\sim}{\theta}_{\nu i}-\bar{\underset{\sim}{\theta}}_\nu) + \sum_{i=1}^{N_\nu}\underset{\sim}{\gamma}'_{\nu i}\,\bar{\underset{\sim}{\Phi}} \tag{2.19}$$

where $B = \int_0^1 \underset{\sim}{\phi}(u)\underset{\sim}{\phi}(u,f)'du$ *and* σ_ν^2 *given by* (2.14).

Proof. From the proof of Theorem 2.2, we have that $\max_{1\le i\le N_\nu} ||\underset{\sim}{\theta}_{\nu i}||^2 \to 0$ when $\nu \to \infty$, $\sum_{i=1}^{N_\nu} ||\underset{\sim}{\theta}_{\nu i}||^2 < \infty$ $(\nu=1,2,\ldots)$ and, by hypothesis, the density f satisfies Condition A of Beran (1970). Thus, the result is obtained by applying Theorem 3.2 of Beran (1970). □

3. ASYMPTOTIC OPTIMALITY

The following theorem establishes an asymptotically optimum rank test among the class of all possible tests.

Theorem 3.1. *Consider testing* H_ν *versus* q_ν *given by* (2.1) *with a density* f *satisfying* Condition A. *Then, if* $\lim_{\nu\to\infty} n_{\nu j}/N_\nu = \lambda_j$, $0 < \lambda_j < 1$, for $j=1,\ldots,k$, *the test based on*

$$(3.1) \qquad S_\nu^0 = \sum_{i=1}^{N_\nu} \underset{\sim}{\theta}'_{\nu i}\, \underset{\sim}{a}_\nu(R_{\nu i},f)$$

with critical region

$$(3.2) \qquad S_\nu^0 \ge k_{1-\alpha}\cdot b$$

where $k_{1-\alpha}$ *is the* $(1-\alpha)$*-quantile of the standardized normal distribution and*

$$(3.3) \qquad b^2 = \sum_{j=2}^{k} \lambda_j \int_0^1 (c_j\phi_1(u,f)+d_j\phi(u,f))^2du + \\ - \int_0^1 (\sum_{j=2}^{k} \lambda_j(c_j\phi_1(u,f)+d_j\phi(u,f)))^2du ,$$

is an asymptotically most powerful test for H_ν *versus* q_ν *at level* α. *Furthermore, the asymptotic power is given by* $1 - \Phi(k_{1-\alpha}-b)$ *where* $\Phi(\cdot)$ *is* *the distribution function of the standardized normal distribution.*

Proof. Denote by $\beta(\alpha,H_\nu,q_\nu)$ the power of the most powerful test for H_ν versus q_ν at level α, and let $p_\nu = \prod_{i=1}^{N_\nu} f(x_i)$. It is clear that

$$(3.4) \qquad \beta(\alpha,H_\nu,q_\nu) \le \beta(\alpha,p_\nu,q_\nu) .$$

Moreover, from Theorem 3.1 of Beran (1970) and since

$$(3.5) \qquad \lim_{\nu\to\infty} \sum_{i=1}^{N_\nu} (\underset{\sim}{\theta}_{\nu i}-\overline{\underset{\sim}{\theta}}_\nu)' \left[\int_0^1 \underset{\sim}{\phi}(u,f)\underset{\sim}{\phi}(u,f)'du\right](\underset{\sim}{\theta}_{\nu i}-\overline{\underset{\sim}{\theta}}_\nu) = b^2 > 0$$

because $\int_0^1 \underset{\sim}{\phi}(u,f)\underset{\sim}{\phi}(u,f)'du$ is a positive definite 2×2 matrix, we have that $\log(q_\nu/p_\nu)$ is asymptotically normal $(-\frac{1}{2}b^2,b^2)$ under p_ν and, from relation (3.40) of Beran (1970), Le Cam's third lemma (see Hājek and Šidāk (1967), p.208) and Theorem 2.2, $\log(q_\nu/p_\nu)$ is asymptotically normal $(\frac{1}{2}b^2,b^2)$ under q_ν. Consequently, the most powerful test for p_ν versus q_ν at level α has the following asymptotic power:

$$(3.6) \qquad \lim_{\nu\to\infty} \beta(\alpha,H_\nu,q_\nu) = 1-\Phi(k_{1-\alpha}-b) .$$

On the other hand, since the vectors $\underset{\sim}{\theta}_{\nu 1},\dots,\underset{\sim}{\theta}_{\nu N_\nu}$ satisfy condition E, we get from Theorem 2.3 that the statistics S_ν^0 are asymptotically normal (b^2,b^2) under q_ν. Thus, the asymptotic power of a test based on S_ν^0 with critical region (3.2) is given by $1 - \Phi(k_{1-\alpha}-b)$ and therefore

$$(3.7) \qquad \liminf_{\nu\to\infty} \beta(\alpha,H_\nu,q_\nu) \ge 1-\Phi(k_{1-\alpha}-b) .$$

The rest follows by combining (3.4), (3.6) and (3.7). □

Corollary 3.1. *In* Theorem 3.1, *the densities* q_ν *can be replaced by*

$$(3.8) \qquad q'_\nu = \prod_{j=1}^{k} \prod_{i \in A_{\nu j}} e^{-c_j/\sqrt{N_\nu}} f\left(e^{-c_j/\sqrt{N_\nu}}(x_i - d_j/\sqrt{N_\nu})\right) .$$

Proof. Define for $i \in A_{\nu j}$, $j=1,\dots,k$,

$$(3.9) \qquad \underset{\sim}{\Delta}_{\nu i} = \left(c_j/\sqrt{N_\nu},\ e^{-c_j/\sqrt{N_\nu}} d_j/\sqrt{N_\nu}\right)' .$$

One can easily verify that $\max_{1 \le i \le N_\nu} \|\underset{\sim}{\Delta}_{\nu i}\|^2 \to 0$ when $\nu \to \infty$ and

$$(3.10) \qquad \sum_{i=1}^{N_\nu} \|\underset{\sim}{\Delta}_{\nu i}\|^2 \le \sum_{j=2}^{k} (c_j^2 + d_j^2 e^{2c})$$

with $c = \max_{2 \le j \le k} |c_j|$. Thus, from Theorem 3.2 of Beran (1970), the linear rank statistics S_ν^0 given by (3.1) are, under q'_ν, asymptotically normal (b^2, b^2) since

$$(3.11) \qquad \lim_{\nu\to\infty} \sum_{i=1}^{N_\nu} (\underset{\sim}{\theta}_{\nu i} - \bar{\underset{\sim}{\theta}}_\nu)' \left[\int_0^1 \underset{\sim}{\phi}(u,f)\underset{\sim}{\phi}(u,f)'du\right] (\underset{\sim}{\Delta}_{\nu i} - \bar{\underset{\sim}{\Delta}}_\nu) = b^2 .$$

The rest follows in the same way as in Theorem 3.1. □

Corollary 3.2. *In* Theorem 3.1, *if the densities* q_ν *are replaced by*

$$(3.12) \qquad q_{\nu,\underset{\sim}{\omega}} = \prod_{j=1}^{k} \prod_{i \in A_{\nu j}} e^{-(c_j/\sqrt{N_\nu}+\omega_1)} f\left(e^{-(c_j/\sqrt{N_\nu}+\omega_1)} x_i - (d_j/\sqrt{N_\nu}+\omega_2)\right)$$

where $\underset{\sim}{\omega} = (\omega_1, \omega_2) \in \mathbb{R}^2$ *is unknown, then, the test based on* S_ν^0 *given by* (3.1) *with critical region* (3.2) *is an asymptotically uniformly most powerful* α *level test for* H_ν *versus*

$$(3.13) \qquad \{q_{\nu,\underset{\sim}{\omega}} : \underset{\sim}{\omega} \in \mathbb{R}^2\} .$$

Proof. Define for $i \in A_{\nu j}$, $j=1,\ldots,k$,

$$\underset{\sim}{\Delta}_{\nu i} = (c_j/\sqrt{N_\nu}+\omega_1,\ d_j/\sqrt{N_\nu}+\omega_2)' . \tag{3.14}$$

Since $\underset{\sim}{\Delta}_{\nu i}-\overline{\underset{\sim}{\Delta}}_\nu = \underset{\sim}{\theta}_{\nu i}-\overline{\underset{\sim}{\theta}}_\nu$, the result is deduced by an argument similar as for the Theorem 3.1. □

Corollary 3.3. *The results of* Theorem 3.1 *and* Corollaries 3.1, 3.2 *still hold if the score vector functions* $\underset{\sim}{a}_\nu(\cdot,f)$ *are replaced by score vector functions* $\underset{\sim}{a}_\nu(\cdot)$ *generated by* $\underset{\sim}{\phi}(u,f)$, $0 < u < 1$.

Proof. In view of Theorem 2.3, the result is immediate. □

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