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NEGATIVE FACTORIAL MOMENTS OF POSITIVE RANDOM VARIABLES

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NEGATIVE FACTORIAL MOMENTS OF POSITIVE RANDOM VARIABLES *)

by Yves LEPAGE **) 

SUMMARY

This paper is concerned with the problem of finding a technique to obtain the expected value of functions of the form \((X+A) \ldots (X+A+k-1)\)^{-1} where \(X\) is a random variable with \(X+A > 0\) almost surely and \(k\) is a positive integer. The technique is applied to the Poisson, geometric and negative binomial distributions.

1. INTRODUCTION

Let \(X\) be a random variable defined on a probability space \((\Omega, A, P)\) and suppose that \(X+A > 0\) a.s. \((P)\). The purpose of this paper is to find a technique to obtain the expected value of functions of the random variable \(X\), of the form

\[ [(X+A) \ldots (X+A+k-1)]^{-1} \]

where \(k\) is a positive integer.

A technique of successive integration of the factorial moment-generating function was suggested by Chao and Strawderman (see [1]) to obtain the expected value of functions of the random variable \(X\), of the form

\[ (X+A)^{-n} \]

where \(n\) is a non-negative integer. In section 2, this technique is modified to obtain the basic result and in section 3, it is applied to three special

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cases: the Poisson, geometric and negative binomial distributions. The utility of negative factorial moments arises specially in life testing problems.

2. THE BASIC RESULT

Define the factorial moment-generating function of $X+A-1$ by

$$g_1(t;A) = E(t^{X+A-1})$$

where $0 \leq t \leq 1$ and $A \in \mathbb{R}$.

Then, for $k = 1, 2, \ldots$, define

$$g_{k+1}(t;A) = \int_0^t g_k(u;A) du$$

where $0 \leq t \leq 1$ and $A \in \mathbb{R}$.

**Theorem.** For $0 \leq t \leq 1$ and $A \in \mathbb{R}$, we have

$$E \left[ \frac{t^{X+A+k-1}}{(X+A) \ldots (X+A+k-1)} \right] = \int_0^t g_k(u;A) du$$

where $k = 1, 2, \ldots$.

**Proof.** Since $X+A > 0$, we have the following equality

$$\frac{t^{X+A+k-1}}{(X+A) \ldots (X+A+k-1)} = \int_0^t \left( \int_0^{t_k} \ldots \left( \int_0^{t_2} t_1^{X+A-1} dt_1 \right) \ldots dt_{k-1} \right) dt_k.$$

Consequently, by taking the expectation on each side of the equality, we obtain the desired result. □

**Corollary.**

$$E \left[ \frac{1}{(X+A) \ldots (X+A+k-1)} \right]^{-1} = \int_0^1 g_k(u;A) du .$$
Proof. The result is obtained by setting \( t = 1 \) in the preceding theorem. \( \square \)

3. APPLICATIONS

3.1. Poisson distribution

Let \( X \) be Poisson distributed with parameter \( \lambda \). We know from Parzen (see [2], p.219), that

\[
g_1(t;\lambda) = t^{\lambda-1} e^{\lambda t - \lambda}.
\]

Suppose \( \lambda = 1 \). Then, we have

\[
g_2(t;1) = \int_0^t e^{u} u^{-1} du = \frac{e^{-\lambda}}{\lambda} (e^{\lambda t} - 1).
\]

Thus, after successive integrations, we obtain

\[
g_k(t;1) = \frac{e^{-\lambda}}{\lambda} \left( \frac{e^{\lambda t}}{\lambda^{k-2}} - \sum_{r=0}^{k-2} \frac{t^r}{r! \lambda^{k-2-r}} \right).
\]

Consequently, for \( k = 2,3,\ldots \), it follows that

\[
E([(X+1) \ldots (X+k)]^{-1}) = \frac{1-e^{-\lambda}}{\lambda^k} - \sum_{r=0}^{k-2} \frac{e^{-\lambda}}{(r+1)! \lambda^{k-1-r}}.
\]

In particular, we have

\[
E([(X+1)(X+2)]^{-1}) = \frac{1}{\lambda^2} \left[ 1-e^{-\lambda} (1+\lambda) \right].
\]

3.2. Geometric distribution

Assume that \( X \) has a geometric distribution with parameter \( p \) \((p \in (0,1))\). We know, from Parzen (see [2], p.219), that

\[
g_1(t;0) = \begin{cases} \frac{p}{1-tq} & \text{if } 0 < t < 1, \\ 0 & \text{if } t = 0, \end{cases}
\]
where \( q = 1 - p \). Thus, we have

\[
g_2(t;0) = \int_0^t \frac{p}{1-up} du = - \frac{p}{q} \ln(1-qt)
\]

for \( 0 \leq t \leq 1 \). Consequently, by carrying the integration, one can deduce that for \( k = 3, 4, \ldots \),

\[
g_{k+1}(t;0) = \frac{(-1)^k p}{(k-1)! q^k} (1-qt)^{k-1} \ln(1-qt) + \sum_{r=1}^{k-1} a_r \frac{(-1)^{k-r+1} p t^r}{r! q^{k-r}}
\]

where \( a_1 = 1/(k-1)! \) and for \( r = 2, \ldots, k-1 \),

\[
a_r = \frac{1}{(k-r)!} + \frac{1}{(k-r-1)!} \sum_{s=2}^{r} \frac{1}{(k-s+1)!}.
\]

It follows that for \( k = 3, 4, \ldots \),

\[
E\left[ (X(X+1) \ldots (X+k-1))^{-1} \right] = \frac{(-1)^k p^k}{(k-1)! q^k} \ln p + p \sum_{r=1}^{k-1} a_r \frac{(-1)^{k-r+1} p t^r}{r! q^{k-r}}
\]

In particular, we have

\[
E\left[ (X(X+1)(X+2)(X+3))^{-1} \right] = \frac{1}{6} \frac{p^4}{q^4} \ln p + \frac{11}{36} \frac{p}{q} - \frac{5}{12} \frac{p}{q^2} + \frac{1}{6} \frac{p}{q^3},
\]

and

\[
E\left[ (X(X+1)(X+2)(X+3)(X+4))^{-1} \right] = - \frac{1}{24} \frac{p^5}{q^5} \ln p + \frac{25}{288} \frac{p}{q} - \frac{13}{72} \frac{p}{q^2} + \frac{7}{48} \frac{p}{q^3} - \frac{1}{25} \frac{p}{q^4}.
\]

3.3. **Negative binomial distribution**

Let \( X \) have a negative binomial distribution with parameters \( r \) (a positive integer) and \( p \) \((p \in (0,1))\). From Farzen (see [2], p.219), we have that

\[
g_1(t,1) = p^r/(1-qt)^r, \quad 0 \leq t \leq 1,
\]
where \( q = 1-p \). Hence,

\[
g_2(t,1) = \int_0^1 p^r/(1-qu)^r \, du = \frac{p^r}{(-r+1)(1-qt)^r-q} - \frac{p^r}{(-r+1)(-q)}
\]

if \( r \geq 2 \). One can then deduce that if \( r = k+1, k+2, \ldots \),

\[
g_{k+1}(t,1) = \frac{p^r}{(r-1)(r-2)\ldots(r-k)(1-qt)^r-k} - \sum_{s=0}^{k-1} \frac{p^r t^s}{s!(r-1)\ldots(r-k+s)q^{k-s}}.
\]

Thus, it follows that if \( r = k+1, k+2, \ldots \),

\[
E\left[ (X+1) \ldots (X+k) \right]^{-1} = \frac{p^r}{(r-1)\ldots(r-k)p^{-k}q^k} - \sum_{s=0}^{k-1} \frac{1}{s!(r-1)\ldots(r-k+s)q^{k-s}}.
\]

In particular, we find

\[
E\left[ (X+1)(X+2)(X+3)(X+4) \right]^{-1} = \frac{p^r}{(r-1)(r-2)(r-3)(r-4)p^{-4}q^4} - \frac{1}{(r-1)(r-2)(r-3)(r-4)q^4} + \frac{1}{(r-1)(r-2)(r-3)q^3} - \frac{1}{2(r-1)(r-2)q^2} - \frac{1}{6(r-1)q}
\]

if \( r = 5, 6, \ldots \).

REFERENCES
