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THE RATE OF CONVERGENCE OF SIMPLE LINEAR RANK STATISTICS UNDER THE HYPOTHESIS

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Let $X_1,\ldots,X_N$ be independent random variables with the same continuous distribution function F. Let R_{iN} be the rank of X_i among $X_1,\ldots,X_N$, let

(I) ϕ be a nonconstant function defined on $(0,1)$ with a bounded second derivative and $\int_0^1 \phi(u)du = 0$.

Let $c_{1N},\ldots,c_{NN}$ be constants satisfying

(II) $$\sum_{\nu=1}^{N} c_{\nu N} = 0, \quad \sum_{\nu=1}^{N} c_{\nu N}^2 = 1, \quad \max_{1\le\nu\le N} |c_{\nu N}| \longrightarrow 0$$

We shall investigate the rate of convergence of the statistics

(1) $$S_N = \sum_{\nu=1}^{N} c_{\nu N} a_N(R_{\nu N}),$$

where $a_N(i)$ are scores given in either of the following ways:

(2) $$a_N(i) = \phi\left(\frac{i}{N+1}\right), \qquad 1 \le i \le N,$$

(3) $$a_N(i) = E\phi(U_N^{(i)}), \quad 1 \le i \le N,$$

where $U_N^{(i)}$ denotes the i-th order statistic in a sample of size N from uniform distribution on $(0,1)$.

JUREČKOVÁ [4] showed (using another method) that if assumption (II) is satisfied and if

(III) ϕ is a nonconstant function defined on $(0,1)$ with a bounded first derivative

then there exist constants $D_1(\varepsilon)$, $D_2(\varepsilon)$ (not depending on N) such that

$$\sup_x \left|P\left(S_N < x\left(\int_0^1 \phi^2(u)du\right)^{\frac{1}{2}}\right) - \Phi(x)\right| \leqq D_1(\varepsilon)\sum_{\nu=1}^{N} |c_{\nu N}|^3 + D_2(\varepsilon)N^{-\frac{1}{2}+\delta},$$
$$\delta > 0.$$

But consistent with the rate of convergence of the distribution of a sum of independent random variables of the type $\sum_{\nu=1}^{N} c_{\nu N}X_{\nu}$ we expect that the rate of convergence of the distribution of S_N is $\sum_{\nu=1}^{N} |c_{\nu N}|^3$; e.g. in the two sample problem with $n = \lambda_1 N$ and $m = (1-\lambda_1)N_1$, $0 < \lambda_1 < 1$, the expected rate is $N^{-\frac{1}{2}}$ but applying Jurečková's assertion we get $N^{-\frac{1}{2}+\varepsilon}$, $\varepsilon > 0$.

During my stay at the Mathematical Centre in Amsterdam, Prof. W.R. van Zwet suggested that the author use Bjerve's method to derive the "natural" rate of convergence of the distribution of S_N, i.e. $\sum_{\nu=1} |c_{\nu N}|^3$. The author was succesful, but it was necessary to replace assumption (III) by assumption (I).

Bjerve's method (also explained by BICKEL [1]) consists of proving the following inequalities

$$\int_{|t|<\varepsilon_1(\sum_{\nu=1}^{N}|c_{\nu N}|^3)^{-1}} |Ee^{itT} - \Phi^*(t)|\ |t|^{-1}dt \leqq D_1 \sum_{\nu=1}^{N} |c_{\nu N}|^3,$$

$$\int_{|t|<\varepsilon_2(\sum_{\nu=1}^{N}|c_{\nu N}|^3)^{-1}} |Ee^{itS} - Ee^{itT}|\ |t|^{-1}dt \leqq$$

$$\int_{|t|<\varepsilon_2(\sum_{\nu=1}^{N}|c_{\nu N}|^3)^{-1}} \left\{ \left|\sum_{j=1}^{2k-1} \frac{(it)^j}{j!} E(S-T)^j e^{itT}\right|\ |t|^{-1} + \right.$$

$$\left. + \frac{E(S-T)^{2k}}{(2k)!} |t|^{2k-1} \right\} dt \leqq D_2 \sum_{\nu=1} |c_{\nu N}|^3,$$

where S is the statistic under consideration, T is a suitable sum of independent random variables, $\Phi^*(t)$ is the characteristic function of a normal distribution and ε_1, ε_2, D_1, D_2 are some constants not depending on N, and making use of the Berry-Essen theorem. The inequalities will follow by several lemmas; two of them (3,4) perhaps could be useful in other problems, too.

The main theorem of this paper is the following:

<u>THEOREM</u>. *Consider the statistic* S_N *given by (1) with scores given either by (2) or (3). Then under assumption (I) and (II) there exists a constant* D *(not depending on* N*) such that*

$$(4)\qquad \sup_x \left|P\left(S_N < x\left(\int_0^1 \phi^2(u)du\right)^{\frac{1}{2}}\right) - \Phi(x)\right| \leq D \sum_{i=1}^{N} |c_i|^3,$$

where $\Phi(x)$ *is the distribution function of* $N(0,1)$.

To begin we shall prove several lemmas. In the following we shall omit indices N in c_{iN}, R_{iN} etc. Denote by

$$(5)\qquad T_N = \sum_{\nu=1}^{N} c_\nu\, \phi(F(X_\nu))$$

$$(6)\qquad T_N^* = \sum_{\nu=1}^{N} c_\nu \frac{R_\nu - E(R_\nu|X_\nu)}{N+1}\, \phi'(F(X_\nu)),$$

$$(7)\qquad \sup_{u\in(0,1)} |\phi^{(i)}(u)| = k_i,\ i = 0,1,2,$$

where $\phi^{(i)}$ stands for the i-th derivative of ϕ.

LEMMA 1. *Under the assumptions of the Theorem,*

$$(8)\qquad P\left(\left|\sum_{\nu=1}^{N} c_\nu\left(a_N(R_\nu) - \phi\left(\frac{R_\nu}{N+1}\right)\right)\right| > \frac{1}{\sqrt{N}}\right) \leqq \frac{k_2^2}{N}\cdot\frac{5}{24},$$

where $a_N(i)$ *are given by (3).*

PROOF. The Lemma follows from the following considerations

$$P\left(\left|\sum_{\nu=1}^{N} c_\nu\left(a_N(R_\nu) - \phi\left(\frac{R_\nu}{N+1}\right)\right)\right| > \frac{1}{\sqrt{N}}\right) \leqq NE\left(\sum_{\nu=1}^{N} c_\nu\left(a_N(R_\nu) - \phi\left(\frac{R_\nu}{N+1}\right)\right)\right)^2 =$$

$$= N\sum_{\nu=1}^{N} c_\nu^2 \frac{1}{N-1}\sum_{k=1}^{N}\left(\phi\left(\frac{k}{N+1}\right) - a_N(k)\right)^2 = \frac{N}{N-1}\sum_{k=1}^{N}\left(\phi\left(\frac{k}{N+1}\right) - E\phi(U_N^{(1)})\right)^2 \leqq$$

$$\leqq \frac{1}{4}k_2^2\sum_{k=1}^{N}\left(E(U_N^{(k)} - \frac{k}{N+1})^2\right)^2 \leqq \frac{5}{24}\frac{k_2^2}{N}. \quad \square$$

LEMMA 2. *Under the assumptions of the Theorem*

$$(9)\qquad P\left(|S_N - T_N - T_N^*| > 3\frac{1}{\sqrt{N}}\left(\int_0^1 \phi^2(u)du\right)^{\frac{1}{2}}\right) \leqq \frac{B_1}{N},$$

where $B_1 = \frac{5}{2} k_2 + k_1^2$, T_N *and* T_N^* *are given by (5) and (6), respectively.*

PROOF. Using the Taylor expansion for $\phi(\frac{R_i}{N+1})$ we obtain

$$(10) \qquad S_N - T_N - T_N^* = \sum_{\nu=1}^{N} c_\nu \frac{R_\nu - E(R_\nu|X_\nu)}{N+1} [\phi'(F(X_\nu)\frac{N-1}{N+1} + \frac{1}{N+1}) - \phi'(F(X_\nu)) +$$

$$+ \sum_{\nu=1}^{N} c_\nu [\phi(F(X_\nu)\frac{N-1}{N+1} + \frac{1}{N+1}) - \phi(F(X_\nu))] + \frac{1}{2} \sum_{\nu=1}^{N} c_\nu \left(\frac{R_\nu - E(R_\nu|X_\nu)}{N+1}\right)^2 K(X_\nu, R_\nu),$$

where $|K(X_\nu, R_\nu)| \leq k_2$. In the following we shall use the fact that $R_\nu - E(R_\nu|X_\nu)$ given X_ν is the sum of independent identically distributed random variables:

$$(11) \qquad P(|\sum_\nu c_\nu \frac{R_\nu - E(R_\nu|X_\nu)}{N+1} [\phi'(F(X_\nu)\frac{N-1}{N+1} + \frac{1}{N+1}) - \phi'(F(X_\nu))]| \geq \frac{1}{\sqrt{N}}) \leq$$

$$\leq N \cdot E\{\sum_{\nu=1}^{N} c_\nu \frac{R_\nu - E(R_\nu|X_\nu)}{N+1} [\phi'(F(X_\nu)\frac{N-1}{N+1} + \frac{1}{N+1}) - \phi'(F(X_\nu))]\}^2 \leq \frac{2}{N^2} k_2^2.$$

Further,

$$(12) \qquad P(|\frac{1}{2} \sum_{\nu=1}^{N} c_\nu \left(\frac{R_\nu - E(R_\nu|X_\nu)}{N+1}\right)^2 k(X_\nu, R_\nu)| \geq \frac{1}{\sqrt{N}}) \leq$$

$$\leq N \frac{1}{4} E\{\sum_{\nu=1}^{N} c_\nu \left(\frac{R_\nu - E(R_\nu|X_\nu)}{N+1}\right)^2 k(X_\nu, R_\nu)\}^2 \leq \frac{k_2^2}{2N}.$$

Similarly, we estimate

$$P(|\sum_{\nu=1}^{N} c_\nu \phi(F(X_\nu)\frac{N-1}{N+1} + \frac{1}{N+1}) - \phi(F(X_\nu))]| \leq \frac{1}{\sqrt{N}}) \leq \frac{1}{N+1} k_1^2.$$

Now, the Lemma follows from the last inequality and (10-12). □

LEMMA 3. *Let* $\{Z_j\}_{j=1}^N$ *be a sequence of random variables of the form* $Z_j = c_j g(X_1, \ldots, X_j)$, *where* $\{X_i\}_{i=1}^N$ *are independent random variables, and such that*

$$(13) \qquad E(Z_j|X_1, \ldots, X_{j-1}) = 0.$$

Let $V_N = \sum_{j=1}^{N} Z_j$ *and* $m_{N,k} = \max_{1\le j\le N} E\, g_j^{2k}$, $k \ge 1$. *Then for* $k \le N$,

$$EV_N^{2k} \leqq k^k (\sum_{j=1}^{N} c_j^2)^k (4e)^k\, m_{N,k}, \tag{14}$$

where $c_1 \leqq c_2 \leqq \cdots \leqq c_N$.

<u>PROOF</u>. First we shall prove by induction on ℓ for k fixed that

$$E(\sum_{j=1}^{\ell} Z_j)^{2k} \leqq (\sum_{j=1}^{\ell} |c_j|)^{2k}\, m_{\ell,k}, \quad \ell \le k. \tag{15}$$

For $\ell = 2$ we have

$$E(\sum_{j=1}^{2} Z_j)^{2k} = \sum_{j=0}^{2k} \binom{2k}{j} EZ_1^j Z_2^{2k-j} \leqq (|c_1| + |c_2|)^{2k}\, m_{2,k}.$$

Assume that (15) is true for all ℓ', $2 \le \ell' \le \ell$, then

$$E(\sum_{j=1}^{\ell+1} Z_j)^{2k} \leqq \sum_{\nu=0}^{2k} \binom{2k}{\nu} E|(\sum_{j=1}^{\ell} Z_j)^{2k-\nu} Z_{\ell+1}^{\nu}| \leqq$$

$$\leqq \sum_{\nu=0}^{2k} \binom{2k}{\nu} (\sum_{j=1}^{\ell} |c_j|)^{2k-\nu} |c_{\ell+1}|^{\nu}\, m_{\ell+1,k} = \sum_{j=1}^{\ell+1} |c_j|^{2k}.$$

Thus (15) is true. In particular

$$E(\sum_{j=1}^{k} Z_j)^{2k} \leqq (\sum_{j=1}^{k} |c_j|)^{2k}\, m_{N,k} \leqq (k \sum_{1}^{k} c_j^2)^k\, m_{N,k}.$$

Now, we can follow the proof of Lemma 6.1 of BICKEL [1]. Assuming that (14) is true for $N = \ell + 1$:

$$EV_{\ell+1}^{2k} = E(\sum_{j=1}^{\ell+1} Z_j)^{2k} = \sum_{\nu=0}^{2k} \binom{2k}{\nu} E(\sum_{j=1}^{\ell} Z_j)^{2k-\nu} Z_{\ell+1}^{\nu} \leqq$$

$$\leqq E(\sum_{j=1}^{\ell} Z_j)^{2k} + \sum_{\nu=2}^{2k} \binom{2k}{\nu} (E(\sum_{j=1}^{\ell} Z_j)^{2k})^{\frac{2k-\nu}{2k}} (EZ_{\ell+1}^{2k})^{\frac{\nu}{2k}} \leqq$$

$$\leq k^k(\sum_{j=1}^{\ell} c_j^2)^k(4e)^k[1 + \sum_{\nu=2}^{2k}\binom{2k}{\nu}\Big(\frac{|c_{\ell+1}|}{(4ek\sum_{i=1}^{\ell} c_i^2)^{\frac{1}{2}}}\Big)^j]m_{\ell+1,k} \leq$$

$$\leq k^k(\sum_{j=1}^{\ell} c_j^2)^k(4e)^k[1 + \frac{|c_{\ell+1}|4k^2}{4ek\sum_{i=1}^{\ell} c_i^2}\Big(1 + \frac{|c_{\ell+1}|}{(4ek\sum_{i=1}^{\ell} c_i^2)^{\frac{1}{2}}}\Big)^{2k-2}]m_{\ell+1,k} \leq$$

$$\leq k^k(\sum_{j=1}^{\ell} c_j^2)^k(4e)^k[1 + \frac{|c_{\ell+1}|^2 k}{\sum_{\nu=1}^{\ell} c_\nu^2}]m_{\ell+1,k} \leq (4ek)^k(\sum_{i=1}^{\ell+1} c_i^2)^k\, m_{\ell+1,k}. \quad \square$$

LEMMA 4. *Let assumption (II) be satisfied and let* $U_1,\ldots,U_N$ *be random variables such that for any permutation* $(j_1,\ldots,j_N)$ *of* $(1,\ldots,N)$

$$\text{(16)}\qquad E\prod_{i=1}^{N} U_i^{\alpha_i} = E\prod_{i=1}^{N} U_{j_i}^{\alpha_i},$$

where $\sum_{i=1}^{N}\alpha_i = 2k$, *for integers* $\alpha_i \geq 0$. *Then for* $2k \leq d(\max_{1\leq i\leq N}|c_i|)^{-1}$

$$\text{(17)}\qquad E(\sum_{i=1}^{N} c_iU_i)^{2k} \leq k^k(4e)^{2k+1}\, EU_1^{2k}\, d^{2k}.$$

PROOF. Using the multinomial expansion we get

$$\text{(18)}\qquad E(\sum_{i=1}^{N} c_iU_i)^{2k} = \sum_{\alpha=1}^{2k}\ \sum_{(k_1,\ldots,k_\alpha)\in A(\alpha)} \frac{(2k)!}{\Pi_{v=1}^{\alpha}(v!)^{k_v}\Pi_{v=1}^{\alpha}(k_v)!}\ \cdot$$

$$\cdot \sum_{(i_1,\ldots,i_{\Sigma_{v=1}^{\alpha}k_v})\in B(\Sigma_{v=1}^{\alpha}k_v)} \Big(\prod_{j=1}^{k_1} c_{i_j}U_{i_j}\Big)\prod_{j=k_1+1}^{k_1+k_2}(c_{i_j}U_{i_j})^2\cdots\prod_{j=\Sigma_{v=1}^{\alpha-1}k_v+1}^{\Sigma_{v=1}^{\alpha}k_v}(c_{i_j}U_{i_j})^{\alpha},$$

where $A(\alpha) = \{(k_1,\ldots,k_\alpha)$, for integers $k_\nu \geq 0$, $\sum_{v=1}^{\alpha} v\,k_v = 2k\}$ and $B(\sum_{v=1}^{\alpha}k_v) = \{(i_1,\ldots,i_{\Sigma_{v=1}^{\alpha}k_v});\ 1 \leq i_j \leq N$ integer, $i_j \neq i_v$ for $j \neq v\}$.

First, we estimate $EZ_N(k_1,\ldots,k_\alpha)$ where

$$Z_N(k_1,\ldots,k_\alpha) = \sum_{(i_1,\ldots,i_{\Sigma_{v=1}^{\alpha}k_v})\in B(\Sigma_{v=1}^{\alpha}k_v)} \Big(\prod_{j=1}^{k_1} c_{i_j}U_{i_j}\Big)\cdots\prod_{j=\Sigma_{v=1}^{\alpha-1}k_v+1}^{\Sigma_{v=1}^{\alpha}k_v}(c_{i_j}U_{i_j})^{\alpha}.$$

Jensen's inequality and (16) imply

$$(19) \qquad |EZ_N(k_1,\ldots,k_\alpha)| \leqq EU_1^{2k}\, V(k_1,\ldots,k_\alpha),$$

where

$$V(k_1,\ldots,k_\alpha) = \sum_{(i_1,\ldots,i_{\sum_{\nu=1}^{\alpha} k_v}) \in B(\Sigma_{v=1}^{\alpha} k_v)} \prod_{j=1}^{k_1} c_{i_j} \prod_{j=k_1+1}^{k_1+k_2} c_{i_j}^2 \cdots \prod_{j=\Sigma_{v=1}^{\alpha-1} k_v+1}^{\Sigma_{v=1}^{\alpha} k_v} (c_{i_j} U_{i_j})^\alpha$$

By induction on k_1 we prove that

$$(20) \qquad |V(k_1,\ldots,k_\alpha)| \leqq 2^{k_1} d^{k_1} \left(\frac{k_1}{2}\right)! \,(\max_i |c_i|)^{\Sigma_{v=3}^{\alpha}(v-2)k_v}$$

holds for any $(k_1,\ldots,k_\alpha)$ integers, $k_v \geq 0$, $\sum_{v=1}^{\alpha} v\, k_v = 2k$. Using assumption (II) we have for $k_1 = 0$ and any integers $k_\nu \geq 0$ with $\sum_{\nu=1}^{\alpha} v\, k_\nu = 2k$ that

$$|V(0,k_2,\ldots,k_\alpha)| \leqq (\max_i |c_i|)^{\Sigma_{v=3}^{\alpha}(v-2)k_v}.$$

For $k_1 = 1$ we have

$$V(1,k_2,\ldots,k_\alpha) = -\sum_{v=2}^{\alpha-1} k_v V(0,k_2,\ldots,k_{v-1},k_{v+1}+1,\ldots,k_\alpha) - $$
$$- k_\alpha V(0,k_2,\ldots,k_\alpha - 1, 1)$$

and then by assumption (II) and $2k \leqq d(\max_i |c_i|)^{-1}$

$$|V(1,k_2,\ldots,k_\alpha)| \leqq \sum_{v=2}^{\alpha} k_v (\max_i |c_i|)^{\Sigma_{v=3}^{\alpha}(v-2)k_v+1} \leqq$$
$$\leqq \frac{d}{2} (\max_i |c_i|)^{\Sigma_{v=3}^{\alpha}(v-2)k_v}.$$

Assuming that (20) is true for $k_1 = r$ we have for $k_1 = r + 1$:

$$|V(r+1,k_2,\ldots,k_\alpha)| \leqq \left|\sum_{v=2}^{\alpha-1} k_v V(r,k_2,\ldots,k_v-1,k_{v+1}+1,\ldots,k_\alpha)\right| +$$
$$+ k_\alpha |V(r,k_2,\ldots,k_\alpha-1,1)| + r|V(r-1,k_2+1,\ldots,k_\alpha)| \leqq$$

$$\leq 2^r(\tfrac{r}{2})!\ d^r(\max_i|c_i|)^{\Sigma_{v=3}^{\alpha} vk_v+1} \sum_{v=2}^{\alpha} k_v + r2^{r-1}(\tfrac{r-1}{2})!\ d^{r-1}(\max_i|c_i|)^{\Sigma_{v=3}^{\alpha} vk_v} \leq$$

$$\leq d^{r+1}2^{r+1}(\tfrac{r+1}{2})!\ (\max_i|c_i|)^{\Sigma_{v=3}^{\alpha} vk_v}.$$

If we use (18-20) and polynomial expansion

$$(2k)^{2k} = \sum_{\alpha=1}^{2k} \sum_{(k_1,\ldots,k_\alpha)\in A(\alpha)} \frac{(2k)!}{\Pi_{v=1}^{\alpha}(v!)^{k_v}\Pi_{v=1}^{\alpha}(k_v)!} \cdot \frac{(2k)!}{(2k-\Sigma_{v=1}^{\alpha} k_v)!}$$

then to prove (17) it suffices to show that

$$(21)\qquad (2d)^{k_1}(\tfrac{k_1}{2})!(\max_i|c_i|)^{\Sigma_{v=3}^{\alpha}(v-2)k_v} \leq (4ed)^{2k+1}k^k(2k)^{-2k}\frac{(2k)!}{(2k-\Sigma_{v=1}^{\alpha} k_v)!}.$$

In view of the assumption $d(\max_i|c_i|)^{-1} \geq 2k$, inequality (21) will follow from the following one:

$$(22)\qquad (d2)^{k_1}(\tfrac{k_1}{2})!(2k)^{-\Sigma_{v=3}^{\alpha}(v-2)k_v} d^{\Sigma_{v=3}^{\alpha}(v-2)k_v} (2k-\sum_{v=1}^{\alpha} k_v)!((2k)!)^{-1} \leq$$

$$\leq (4ed)^{2k+1}\ k^k\ (2k)^{-2k}.$$

Using Stirling's inequality

$$\sqrt{2\pi}(n+\tfrac{1}{2})^{n+\frac{1}{2}}e^{-(n+\frac{1}{2})-\frac{1}{24}(n+\frac{1}{2})} < n! < \sqrt{2\pi}(n+\tfrac{1}{2})^{n+\frac{1}{2}}e^{-(n+\frac{1}{2})}$$

we obtain after some computations that the left side of inequality (22) is smaller or equal to

$$\sqrt{2\pi}\ 2^{k_1}(2k)^{-\Sigma_{v=3}^{\alpha}(v-3)k_v}(\tfrac{k_1+1}{2})^{\frac{k_1+1}{2}}(2k-\sum_{v=1}^{\alpha} k_v+\tfrac{1}{2})^{2k-\Sigma_{v=1}^{\alpha}k_v+\frac{1}{2}}d^{2k}\ \cdot$$

$$\cdot\ (2k+\tfrac{1}{2})^{-2k+\frac{1}{2}}\exp\{\sum_{v=2}^{\alpha} k_v + \frac{k_1}{2} + \frac{1}{2} + \frac{k+\frac{1}{2}}{24}\} \leq$$

$$\leq \sqrt{\pi}\ 2^{2k}\ d^{2k}(2k)^{-2k}\ \exp\{2k+1\}(2k)^{\Sigma_{v=2}^{\alpha}k_v+\frac{k_1}{2}} \leq (4de)^{2k+1}(2k)^{-2k}\ k^k.$$

Thus inequality (22) holds and Lemma 4 is proved. □

Lemma 3 and 4 easily imply the following assertion:

LEMMA 5. *Under the conditions of the Theorem there exists a constant* B(d) *not depending on* N *and* k *such that for* $2k \leqq d(\max_i |c_i|)^{-1}$, $d > 1$,

$$E(T_N^*)^{2k} \leqq (2k)^{2k} N^{-k} (B(d))^{2k}, \tag{23}$$

where T_N^* *is given by (6).*

PROOF. Putting in Lemma 4 $U_i = (N+1)^{-1}(R_i - E(R_i|X_i))\phi'(F(X_i))$, $1 \le i \le N$, we have

$$E(T_N^*)^{2k} \leqq N^{-2k} k^k (4ed)^{2k+1} E[(R_1 - E(R_1|X_1))\phi'(F(X_1))]^{2k}$$

and then applying Lemma 3 for $Z_j = (u(X_1 - X_j) - F(X_1))\phi'(F(X_1))$, $1 \le j \le N$, $u(x) = 1$, $x \ge 0$, $u(x) = 0$, $x < 0$, we obtain inequality (23). □

LEMMA 6. *Under the conditions of the Theorem there exists* $\varepsilon_1 > 0$ *and* D_1 *(not depending on* N*) such that*

$$\int_{|t|<\varepsilon_1(\Sigma_{\nu=1}^N |c_\nu|^3)^{-1}} |E \exp\{itT_N\} - \exp\{-\frac{t^2}{2}\int_0^1 \phi^2(u)du\}|\ |t|^{-1}dt \leqq \tag{24}$$

$$\leqq D_1 \sum_{\nu=1}^N |c_\nu|^3.$$

PROOF. We shall prove it in the usual way. We start with estimating

$$\int_{|t|<-D^*\log(\Sigma_\nu |c_\nu|^3)} |E \exp\{itT_N\} - \exp\{-\frac{t^2}{2}\int_0^1 \phi^2(u)du\}|\ |t|^{-1}dt, \tag{25}$$

where D^* will be chosen suitably. Notice that $-\log(\Sigma_\nu |c_\nu|^3) < 2(\max_i |c_i|)^{-1}$. For the characteristic function of $c_\nu\phi(F(X_\nu))$ we obtain

$$E \exp\{itc_\nu\phi(F(X_\nu))\} = 1 - \frac{c_\nu^2 t^2}{2}\int_0^1 \phi^2(u)du + \frac{|t|^3}{3!}|c_\nu|^3 \int_0^1 |\phi^3(u)|du\,\theta_\nu, \tag{26}$$

$$|\theta_\nu| \leqq 1,\ 1 \le \nu \le N.$$

Thus for $|t|<-(\log(\sum_\nu|c_\nu|^3))D^*$, where $D^* = \frac{1}{2}\{\int_0^1\phi^2(u)du + \frac{1}{3}\int_0^1|\phi(u)|^3du\}^{-1}$ the Taylor expansion for $\log[E\exp\{itc_\nu\phi(F(X_\nu))\}]$ can be applied

(27) $$\log[E\exp\{itc_\nu\phi(F(X_\nu))\}] = -\frac{t^2}{2}c_\nu^2\int_0^1\phi^2(u)du + \frac{|t|^3}{3!}|c_\nu|^3\int_0^1|\phi(u)|^3du\,\theta_\nu^*,\quad |\theta_\nu^*| \leqq 1,\ 1 \le \nu \le N$$

and then

$$\log[E\exp\{it\sum_{\nu=1}^N c_\nu\phi(F(X_\nu))\}] = -\frac{t^2}{2}\int_0^1\phi^2(u)du + \frac{|t|^3}{3!}\sum_{\nu=1}^N|c_\nu|^3\theta_\nu^* \cdot$$
$$\cdot\int_0^1|\phi(u)|^3du,\quad E\exp\{itT_N\} = \exp\{-\frac{t^2}{2}\int_0^1\phi^2(u)du\}\cdot$$
$$\cdot\left(1+\frac{2}{3}|t|\sum_{\nu=1}^N|c_\nu|^3\theta_\nu^{**}\int|\phi(u)|^3du\,(\int_0^1\phi^2(u)du)^{-1}\right),\quad |\theta_\nu^{**}| \leqq 1,\ 1\le\nu\le N.$$

Putting the last expression in (25) we obtain

(28) $$\int_{|t|<-D^*\log(\sum_\nu|c_\nu|^3)}|E\exp\{itT_N\} - \exp\{-\frac{t^2}{2}\int_0^1\phi^2(u)du\}|\ |t|^{-1}dt \leqq$$
$$\leqq \int_{|t|<-D^*\log(\sum_\nu|c_\nu|^3)}\frac{2}{3}(\sum_\nu|c_\nu|^3)\int_0^1|\phi(u)|^3du(\int_0^1\phi^2(u)du)^{-1}\cdot$$
$$\cdot\exp\{-\frac{t^2}{2}\int_0^1\phi^2(u)du\}dt \leqq \frac{2}{3}\int_0^1|\phi(u)|^3du(\int_0^1\phi^2(u)du)^{-1}\sum_{\nu=1}^N|c_\nu|^3.$$

Now we turn to the integral

$$\int_{-D^*\log\Sigma|c_\nu|^3<|t|<\varepsilon_1(\Sigma|c_\nu|^3)^{-1}}\frac{|E\exp\{itT_N\} - \exp\{-\frac{t^2}{2}\int_0^1\phi^2(u)du\}|}{|t|}dt.$$

Using (26) we obtain

$$E\exp\{itc_\nu\phi(F(X_\nu))\} \leqq \exp\{-\frac{t^2}{2}\int_0^1\phi^2(u)du + \frac{|t|^3}{3!}\sum_\nu|c_\nu|^3\theta_\nu\int_0^1|\phi(u)|^3du\}.$$

Then for $|t| \le \frac{3}{2} (\sum_{\nu}|c_\nu|^3)^{-1} \int_0^1 \phi^2(u)du \, (\int_0^1 |\phi^3(u)|du)^{-1}$ the inequality

(29) $$E \exp\{itT_N\} \le \exp\{-\frac{t^2}{4}\}$$

holds. The last inequality implies

(30) $$\int\limits_{-D^* \log(\sum_\nu |c_\nu|^3) < |t| < \varepsilon_1 (\sum_\nu |c_\nu|^3)^{-1}} \frac{|E \exp\{itT_N\} - \exp\{-\frac{t^2}{2}\int_0^1 \phi^2(u)du\}|}{|t|} dt \leqq$$

$$\leqq 2 \int\limits_{-D^* \log(\sum_\nu |c_\nu|^3) < |t| < \varepsilon_1 (\sum_\nu |c_\nu|^3)^{-1}} |t|^{-1} \exp\{-\frac{t^2}{4} \int_0^1 \phi^2(u)du\} dt \le$$

$$\le 2(\sum_\nu |c_\nu|^3)^k,$$

when $\varepsilon_1 = \frac{3}{2} \int_0^1 \phi^2(u)du \, (\int_0^1 |\phi(u)|^3 du)^{-1}$ and $k > 0$ arbitrarily. Our lemma follows from (28) and (30). □

<u>LEMMA 7</u>. *Under the assumptions of the Theorem there exists a constant* A *(not depending on* N*) such that*

(31) $$\int\limits_{|t| \leqq D^* \log(\sum_\nu |c_\nu|^3)^{-1}} |E(T_N^* \exp\{itT_N\})| \le A \min(\sum_\nu |c_\nu|^5, N^{-1})$$

<u>PROOF</u>. $E[T_N^* \exp\{itT_N\}]$ can be rewritten in the following form:

$$(N+1)ET_N^* \exp\{itT_N\} = \sum_{\nu=1}^{N} \sum_{\substack{j=1 \\ j \ne \nu}}^{N} c_\nu E[u(X_\nu - X_j) - F(X_\nu)] \, \phi'(F(X_\nu)) \cdot$$

$$\cdot \exp\{itT_N\} = E \exp\{it \sum_{\alpha=1}^{N} c_\alpha \phi(F(X_\alpha))\} \cdot \sum_{\nu=1}^{N} \sum_{\substack{j=1 \\ j \ne \nu}}^{N} c_\nu E[u(X_\nu - X_j) - F(X_\nu)] \cdot$$

$$\cdot \phi'(F(X_\nu)) \exp\{it(c_\nu \phi(F(X_\nu)) + c_j \phi(F(X_j)))\} \{E \exp\{it[c_\nu \phi(F(X_\nu)) +$$

$$+ c_j(F(X_j))]\}\}^{-1}.$$

To estimate this expectation we use the Taylor expansion for $\exp\{it[c_\nu\phi(F(X_\nu)) + c_j\phi(F(X_j))]\}$, and for $(E\exp\{itc_\nu\phi(F(X_\nu))\})^{-1}$ we apply (27) and the Taylor expansion. In the sequel we use a shorter notation

$$Z_{\nu j} = (u(X_\nu - X_j) - F(X_\nu))\,\phi'(F(X_\nu)),\quad \nu \neq j,\ 1 \le \nu, j \le N,$$
$$Z_\nu = \phi(F(X_\nu)),\qquad 1 \le \nu \le N,$$

Notice that $E(Z_{\nu j}|Z_\nu) = 0$. $E[\exp\{itT_N\}T_N^*]$ can be rewritten:

$$(N+1)ET_N^*\exp\{itT_N\} = E\exp\{itT_N\}\sum_{\nu=1}^{N}\sum_{\substack{j=1\\ j\neq\nu}}^{N} c_\nu\, E\, Z_{\nu j}[1 + itc_\nu Z_\nu + itc_j Z_j -$$
$$-\frac{t^2}{2}c_\nu^2 Z_\nu^2 - t^2 c_\nu c_j \cdot Z_\nu Z_j - \frac{t^2}{2}c_j^2 Z_j^2 + \frac{|t|^3}{3!}|c_\nu Z_\nu + c_j Z_j|^3 \eta_{\nu j}]\,\cdot$$
$$\cdot\,[1 + \frac{t^2}{2}(c_\nu^2 + c_j^2)\int_0^1 \phi^2(u)du - \frac{|t|^3}{3!}(|c_\nu|^3 + |c_j|^3)\int_0^1 |\phi(u)|^3 du\, \gamma_{\nu j} +$$
$$+ \{\frac{|t|^2}{2}(c_\nu^2 + c_j^2)\int_0^1 \phi^2(u)du + \frac{|t|^3}{3!}(|c_\nu|^3 + |c_j|^3)\int_0^1 |\phi^3(u)|du\, \gamma_{\nu j}\}^2\, \gamma^*],$$

where $|\gamma_{\nu j}| \leqq 1$, $|\eta_{\nu j}| \le 1$, $|\gamma^*| \leqq 1$. After long computations we arrive at

$$|ET_N^*\exp\{itT_N\}| \leqq |E\exp\{it\sum_{\alpha=1}^{N} c_\alpha\phi(F(X_\alpha))\}k_1$$

$$\{[t^2k_0^4\sum_{\nu=1}^{N}|c_\nu|^5 + O(\sum_\nu c_\nu^6)] + \frac{1}{\sqrt{N}}[\sum_\nu|c_\nu|^3\,\frac{7}{3}\,t^3k_0^3 + \sum_\nu c_\nu^4(k_0^4t^4\,\frac{29}{12}) +$$

$$+ O(\sum_\nu|c_\nu|^5)] + \frac{1}{N}[k_1 t + \sum_\nu|c_\nu|^3 k_0^4 + t^4\,\frac{9}{4} + O(\sum_\nu c_\nu^4)]\}.$$

Next we make use of the fact that

$$\int_{-\infty}^{+\infty}|t|^k\exp\{-\frac{t^2}{2}\int\phi^2(u)du\}dt \leqq 2\left(\frac{2}{\int_0^1\phi^2(u)du}\right)^{\frac{k+1}{2}}\Gamma\left(\frac{k+1}{2}\right).$$

The last two inequalities imply

$$\int_{|t|<-\log(\sum_\nu |c_\nu|^3)} |ET_N^* \exp\{itT_N\}|dt \leq (\int_0^1 \phi^2(u)du)^{\frac{1}{2}} k_1 \cdot$$

$$\cdot [\sum_\nu |c_\nu|^5 k_0^4 \Gamma(\tfrac{3}{2}) 2^{\frac{1}{2}}(\int_0^1 \phi^2(u)du)^{-\frac{1}{2}} - N^{-\frac{1}{2}} \sum_\nu |c_\nu|^3 \cdot \frac{14}{3} k_0^3 + \frac{k_0}{N} + 0(\sum_\nu |c_\nu|^6)].$$

□

<u>LEMMA 8</u>. *Under the assumptions of the Theorem there exists a constant* C *(not depending on* m *and* N*) such that*

$$(32) \qquad |E(T_N^*)^m \exp\{itT_N\}| \leq \exp\{-\frac{t^2}{4}\} m^m C^m \exp\{\frac{mt^2 \max_\nu |c_\nu|^2}{2}\}$$

for

$$|t| \leq \varepsilon_2^*(\sum_\nu |c_\nu|^3)^{-1}, \ \varepsilon_2^* = \frac{3}{2} \int_0^1 \phi^2(u)du \ (\int_0^1 |\phi(u)|^3 du)^{-1}.$$

<u>PROOF</u>. During the proof we shall assume $\int_0^1 \phi^2(u)du = 1$. $E[(T_N^*)^m \exp\{itT_N^*\}]$ can be rewritten in a suitable form:

$$ET_N^{*m} \exp\{itT_N\} = (N+1)^{-m}$$

$$(33) \qquad E\{\sum_{\nu_1} \cdots \sum_{\nu_m} \sum \cdots \sum_{(j_1,\ldots,j_m)\in A_1} h(j_1,\ldots,j_m,\nu_1,\ldots,\nu_m) +$$

$$+ \sum_{\nu_1} \cdots \sum_{\nu_m} \sum \cdots \sum_{(j_1,\ldots,j_m)\in A_2} h(j_1,\ldots,j_m,\nu_1,\ldots,\nu_m) + \ldots$$

....

$$\sum_{\nu_1} \cdots \sum_{\nu_m} \sum \cdots \sum_{(j_1,\ldots,j_m)\in A_m} h(j_1,\ldots,j_m,\nu_1,\ldots,\nu_m)\},$$

where $h(j_1,\ldots,j_m,\nu_1,\ldots,\nu_m) = \prod_{\alpha=1}^m c_{\nu_\alpha} (u(X_{\nu_\alpha} - X_{j_\alpha}) - F(X_{\nu_\alpha})) \phi'(F(X_{\nu_\alpha}))$ $\cdot \exp\{it\Sigma_{\beta=1}^N c_\beta \phi(F(X_\beta))$, A_γ, $\gamma = 1,\ldots,m$, is the set of $(j_1,\ldots,j_m)$, $1 \leq j_\alpha \leq N$, $1 \leq \alpha \leq m$, among which just γ are different. We use (29) as an estimation of $E \exp\{itT_N\}$. First, we estimate sums of type

$$(34)\qquad \sum_{\nu_1}\cdots\sum_{\nu_m}\ \sum\cdots\sum_{(j_1,\ldots,j_m)\in A_\gamma} E\ h(j_1,\ldots,j_m,\nu_1,\ldots,\nu_m),\ \gamma\le\lceil\tfrac{m}{2}\rceil,$$

$$\Big|\sum_{\nu_1}\cdots\sum_{\nu_m}\ \sum\cdots\sum_{(j_1,\ldots,j_m)\in A_\gamma} E\ h(j_1,\ldots,j_m,\nu_1,\ldots,\nu_m)\Big| \le$$

$$\le \exp\{-\frac{t^2}{4}\}\sum_{\gamma=1}^{[\frac{m}{2}]}\Big(\sum_{\nu=1}^{N}|c_\nu|\exp\{\frac{t^2c_\nu^2}{4}\}\Big)^m\Big(\sum_{\nu=1}^{N}\exp\{\frac{t^2c_\nu^2}{4}\}\Big)^\gamma m^m k_1^m.$$

Clearly,

$$N^{-\frac{1}{2}}\sum_{\nu=1}^{N}|c_\nu|\exp\{\frac{t^2c_\nu^2}{4}\}\le\exp\{\frac{t^2\max_\nu|c_\nu|^2}{4}\},$$

$$N^{-1}\sum_{\nu=1}^{N}\exp\{\frac{t^2c_\nu^2}{4}\}\le\exp\{\frac{t^2\max_\nu|c_\nu|^2}{4}\}.$$

We put these estimations in (34) and after some computations we arrive at the following inequality:

$$N^{-m}\Big|\sum_{\nu_1}\cdots\sum_{\nu_m}\sum_{\gamma=1}^{[\frac{m}{2}]}\ \sum\cdots\sum_{(j_1,\ldots,j_m)\in A_\gamma} E\ h(j_1,\ldots,j_m,\nu_1,\ldots,\nu_m)\Big|\le$$

$$\le m^m k_1^m\exp\{\frac{t^2\max_\nu|c_\nu|^2}{4}\Big(\frac{3m}{2}+1\Big)-\frac{t^2}{4}\}.$$

Now we consider the sums $\sum_{\nu_1}\cdots\sum_{\nu_m}\ \sum\cdots\sum_{(j_1,\ldots,j_m)\in A_\gamma} E\ h(j_1,\ldots,j_m,\nu_1,\ldots,\nu_m)$, $\gamma>[\frac{m}{2}]$. If $(j_1,\ldots,j_m)\in A_\gamma$, $\gamma>[\frac{m}{2}]$ then there exist at least $m-\frac{\gamma}{2}$ numbers among $(j_1,\ldots,j_m)$ which occur just once. There are just $\binom{m}{j}$ possibilities that among $(j_1,\ldots,j_m)$ j numbers exist which are not equal to others; let the index α have this property. Then α either is equal to some ν_ℓ (we estimate the corresponding number by k_1) or is not equal to any ν_ℓ, $1\le\ell\le m$, and then the corresponding member can be estimated as follows:

$$(35)\qquad |E\ Z_{\nu_\ell\alpha}\exp\{itc_\alpha\phi(F(X_\alpha))|\le|E\ Z_{\nu_\ell\alpha}(1+|t||c_\alpha|k_0\theta)|\le$$

$$\le|t||c_\alpha|k_0k_1,$$

$$|\theta|\le 1,\ Z_{\nu_\ell\alpha}=(u(X_{\nu_\ell}-X_\alpha)-F(X_{\nu_\ell}))\phi'(F(X_{\nu_\ell})).$$

Thus the complete sum can be estimated in this way:

$$N^{-m} \left|\sum_{\nu_1} \cdots \sum_{\nu_m} \sum_{\gamma=1}^{m} \sum_{(j_1,\ldots,j_m)\in A_\gamma} E\ h(j_1,\ldots,j_m,\nu_1,\ldots,\nu_m)\right| \leqq$$

$$\leqq \exp\{-\frac{t^2}{4}\}\ m^m\ N^{-m}\left(\sum_{\nu=1}^{N} |c_\nu|\ \exp\{\frac{t^2c_\nu^2}{4}\}\right)^m k_1^m \max(k_0^m,1)\ \cdot$$

$$\cdot \sum_{j=0}^{m} \binom{m}{j} \left(\sum_{\nu=1}^{N} \exp\{\frac{t^2c_\nu^2}{4}\}\right)^{\frac{m-j}{2}} \left(\sum_{\nu=1}^{N} |c_\nu t|\ \exp\{\frac{t^2c_\nu^2}{4}\}\right)^j \leqq$$

$$\leqq \exp\{-\frac{t^2}{4}\}\ m^m\ k_1^m\ 2^m \exp\{\frac{t^2\max_\nu|c_\nu|^2}{4} \cdot 2m\}. \quad \square$$

<u>PROOF OF THE THEOREM</u>. In view of Lemma 1, it suffices to consider S_N with scores given by (2). Using Lemma 2 we have

$$(36) \qquad P(S_N < x(\int_0^1 \phi^2(u)du)^{\frac{1}{2}})) \leqq P(T_N + T_N^* < (X + 3N^{-\frac{1}{2}})(\int_0^1 \phi^2(u)du)^{\frac{1}{2}}) +$$

$$+ P(|S_N - T_N - T_N^*| \geqq 3N^{-\frac{1}{2}}) \leqq P(T_N + T_N^* < (X + 3N^{-\frac{1}{2}})(\int_0^1 \phi^2(u)du)^{\frac{1}{2}}) +$$

$$+ B_1N^{-1}.$$

Similarly,

$$(37) \qquad P(S_N < x(\int_0^1 \phi^2(u)du)^{\frac{1}{2}}) > P(T_N + T_N^* < (x - 3N^{-\frac{1}{2}})(\int_0^1 \phi^2(u)du)^{\frac{1}{2}}) - B_1N^{-\frac{1}{2}}.$$

Further, we show that there exist constants $\varepsilon_1, \varepsilon_2, D_1, D_2$ such that

$$(38) \qquad \int_{|t|<\varepsilon_1(\sum_\nu |c_\nu|^3)^{-1}} |E \exp\{itT_N\} - \exp\{-\frac{t^2}{2}\int_0^1 \phi^2(u)du\}|\ |t|^{-1}dt \leqq D_1 \sum_\nu |c_\nu|^3,$$

$$(39) \qquad \int_{|t|<\varepsilon_2(\sum_\nu |c_\nu|^3)^{-1}} |E \exp\{it(T_N + T_N^*)\} - E \exp\{itT_N\}|\ |t|^{-1}dt \leqq D_2 \sum_\nu |c_\nu|^3.$$

Inequality (38) follows from Lemma 6. As for (37), we use the Taylor expansion

$$\text{(40)} \qquad E\exp\{it(T_N+T_N^*)\} = E\sum_{\nu=0}^{2k-1}\frac{(itT_N^*)^\nu}{\nu!}\exp\{itT_N\} + \frac{E(tT_N^*)^{2k}}{(2k)!}\eta_N,$$

$|\eta_N| \le 1$. Then making use of Lemmas 7 and 5 for $k = 1$ we obtain

$$\text{(41)} \qquad \int_{|t|<(\sum_\nu|c_\nu|^3)^{-\frac12}} |E\exp\{it(T_N+T_N^*)\} - E\exp\{itT_N\}|\,|t|^{-1}dt \leqq$$

$$\leqq \int_{|t|<(\sum_\nu|c_\nu|^3)^{-\frac12}} |E\,T_N^*\exp\{itT_N\}|dt + \int_{|t|<(\sum_\nu|c_\nu|^3)^{-\frac12}} \tfrac12|t|\,E(T_N^*)^2dt \leqq$$

$$\leqq A\max(\sum_\nu|c_\nu|^5, N^{-1}) + \tfrac12(B(2))^2\sum_\nu|c_\nu|^3 + C(\sum_\nu|c_\nu|^3)^{-\frac12}\exp\{-\frac{(D^*\log\Sigma_\nu|c_\nu|^3)^2}{8}\}$$

Put $k = \frac{1}{16}\log\{(2\sum_\nu|c_\nu|^3)^{-1}\}$ and $\varepsilon_2 = \min(\varepsilon_2^*, \varepsilon_2^{**})$, where ε_2^* is given in Lemma 8 and $\varepsilon_2^{**} = (B(2))^{-1}\exp\{-\frac{11}{2}\}$. By Lemma 7 we get

$$\text{(42)} \qquad \int_{(\sum_\nu|c_\nu|^3)^{-\frac12}<t<\varepsilon_2(\sum_\nu|c_\nu|^3)^{-1}} \sum_{\nu=1}^{2k-1}\frac{|t|^{\nu-1}}{\nu!}|E(T_N^*)^\nu\exp\{itT_N\}|dt \leqq$$

$$\leqq \int_{(\sum_\nu|c_\nu|^3)^{-\frac12}\le|t|\le\varepsilon_2(\sum_\nu|c_\nu|^3)^{-1}} \exp\{-\frac{t^2}{2}\}\sum_{\nu=1}^{2k-1}[\frac{|t|^{\nu-1}}{\nu!}\exp\{\frac{t^2}{2}\max|c_j|^2 2\nu\}dt$$

$$\leqq C^{2k}(\log(2\sum_\nu|c_\nu|^3)^{-1})^{\frac12}\exp\{-\frac{1}{8\sum_\nu|c_\nu|^3} + \frac14\,\frac{\log(2\sum_\nu|c_\nu|^3)\cdot\log(\varepsilon_2^{-1}\sum_\nu|c_\nu|^3)}{\sum_\nu|c_\nu|^3}\},$$

and by Lemma 5

$$\text{(43)} \qquad \int_{(\sum_\nu|c_\nu|^3)^{-\frac12}\leqq|t|\le\varepsilon_2(\sum_\nu|c_\nu|^3)^{-1}} \frac{E(T_N^*)^{2k}}{(2k)!}|t|^{2k-1}dt \leqq$$

$$\leqq \frac{(\varepsilon_2 B(2)e^{3/2})^{2k}}{k} \leqq (\sum_\nu|c_\nu|^3)\cdot\log(\sum_\nu|c_\nu|^3)^{-1}.$$

Now from (40-43) can be concluded (39). Then our Theorem will follow from inequality (38-39), the Berry-Essén argument [3] and the fact that

$$1 - 2\Phi(N^{-\frac{1}{2}}) \leq (\frac{2}{\pi N})^{-\frac{1}{2}},$$

where Φ is the distribution function $N(0,1)$. □

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