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A STRONG LAW OF LARGE NUMBERS FOR LINEAR COMBINATIONS OF ORDER STATISTICS

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A strong law of large numbers for linear combinations of order statistics.*)

by

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ABSTRACT

An elementary proof of a strong law of large numbers for linear combinations of order statistics is given. The conditions of the theorem are easy to check and apply to almost every robust estimator based on linear combinations of order statistics which may arise in practice. The relation with recent results of BICKEL & LEHMANN (1975) and WELLNER (1977) is pointed out.

KEY WORDS & PHRASES: *linear combinations of order statistics, strong law of large numbers, strong consistency.*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

Linear combinations of order statistics received much attention during the last ten years. The main reason for studying these statistics is their importance in robust estimation problems. Much is known about them including their weak convergence to a normal limit distribution under quite general conditions (see e.g. SHORACK (1972) and STIGLER (1974)).

In this note we shall establish a strong law of large numbers for linear combinations of order statistics. The conditions of the theorem are easy to check and apply to almost every robust estimator based on linear combinations of order statistics which may arise in practice. Let

$$T_n = n^{-1} \sum_{i=1}^{n} J\left(\frac{i}{n+1}\right) X_{in} + \sum_{k=1}^{K} d_{kn} X_{i_k n}$$

where X_{in}, $i = 1,2,\ldots,n$ denotes the i^{th} order statistic of a random sample $X_1,\ldots,X_n$ of size n from a distribution with distribution function (d.f.) F, J is a bounded measurable weight function on (0,1), $d_{1n},\ldots,d_{Kn}$ are given constants and the indices $i_1,\ldots,i_K$ satisfy $1 \le i_1 \le i_2 \le \ldots \le i_K \le n$. Though not indicated in our notation the indices i_k $(1 \le k \le K)$ may depend on n. We shall assume that all random variables are defined on the same probability space (Ω, A, P). The inverse of a d.f. will always be the left-continuous one.

<u>THEOREM</u>. *Suppose*

(i) J *is bounded on* (0,1) *and continuous except at possibly finitely many points*.

(ii) $E|X_1| < \infty$

(iii) $(K > 0)$. *There exist real numbers* $d_1,\ldots,d_K$ *and numbers* $0 < p_1 < \ldots < p_K < 1$ *such that* $d_{kn} \to d_k$ *and* $\frac{i_k}{n} \to p_k$, *as* $n \to \infty$, *for* $1 \le k \le K$. *The* p_k *th quantile* ξ_{p_k} *of the d.f.* F *is uniquely determined for* $1 \le k \le K$. *Then*

$$P\left(\lim_{n\to\infty} T_n = \int_0^1 F^{-1}(s)\, J(s)\, ds + \sum_{k=1}^{K} d_k\, \xi_{p_k}\right) = 1.$$

If, in addition, $J(s) = 0$ *for* $0 < s < \alpha$ *and* $\beta < s < 1$ *assumption* (ii) *can be dropped.*

This result may be used to show that estimators based on linear combinations of order statistics are strongly consistent; e.g. it is immediate from our result that symmetrically trimmed means ($K = 0$) are strongly consistent estimators for the centre of any symmetric population. The same is true for symmetrically Winsorized means ($K = 2$), with Winsorizing percentages p_1 and $p_2 = 1-p_1$, provided the p_1^{th} and $(1 - p_1)^{th}$ quantile of F are uniquely determined.

Related results where strong laws of large numbers for general linear combinations of order statistics are established are due to WELLNER (1976). His results allow quite weight functions (including some unbounded J's) but his moment condition is slightly stronger than ours for some of the statistics we consider. In particular the classical strong law of large numbers for the sample mean fails to be a corollary of WELLNER's results. We also want to mention that BICKEL & LEHMANN (1975) have given an argument (see their remark (A) on p. 1054) which immediately yields a strong law for trimmed linear combinations of order statistics.

PROOF.

Let, for each $n \geq 1$, $U_1,\ldots,U_n$ be independent uniform (0,1) random variables and let, for $1 \leq i \leq n$, U_{in} denote the i^{th} order statistic of $U_1,\ldots,U_n$. It is well-known that, for each $n \geq 1$, the joint distribution of $(X_1,\ldots,X_n)$ is the same as that of $(F^{-1}(U_1),\ldots,F^{-1}(U_n))$ for any d.f. F. Since the validity of a strong law depends only on J, F and the d_{kn}'s we may and shall identify X_i with $F^{-1}(U_i)$ and also X_{in} with $F^{-1}(U_{in})$ for $1 \leq i \leq n$. Let Γ_n denote the empirical d.f. of $U_1,\ldots,U_n$. χ_A denotes the indicator of a set A. The Lebesgue measure is denoted by λ.

We give the proof first for $K = 0$. The idea of the proof is to use Moore's representation of T_n in terms of the empirical d.f. Γ_n (see MOORE (1968)):

$$(1) \qquad T_n = \int_0^1 F^{-1}(s)\, J\left(\frac{n}{n+1}\,\Gamma_n(s)\right) d\,\Gamma_n(s).$$

Introduce random variables A_n and B_n by

$$(2)\qquad A_n = \int_0^1 F^{-1}(s)\, J(s)\, d\,\Gamma_n(s) = n^{-1} \sum_{i=1}^n F^{-1}(U_i)J(U_i)$$

and

$$(3)\qquad B_n = \int_0^1 F^{-1}(s)\Big(J\big(\tfrac{n}{n+1}\,\Gamma_n(s)\big) - J(s)\Big)\, d\,\Gamma_n(s)$$

so that

$$(4)\qquad T_n = A_n + B_n.$$

Since A_n is the mean of n independent identically distributed random variables and the boundedness of J (see assumption (i)) and assumption (ii) imply that $E\,|\,F^{-1}(U_1)\ J(U_1)\,| \le \sup_{0<s<1} |\,J(s)|\,.\ E|X_1| < \infty$ we can apply Kolmogorov's strong law of large numbers to find that

$$(5)\qquad P\Big(\lim_{n\to\infty} A_n = \int_0^1 F^{-1}(s)\ J(s)ds\Big) = 1.$$

It remains to show that

$$(6)\qquad P(\lim_{n\to\infty} B_n = 0) = 1.$$

We shall prove (6), by way of an example, for the case that J is discontinuous at only one point $s_1 \in (0,1)$. Now new difficulties will be encountered when treating the other cases.

Remark first that, for any $0 < \delta < \frac{2}{3}\min\,(s_1, 1 - s_1)$, we can decompose the r.v. B_n as follows:

$$(7)\qquad B_n = \sum_{m=1}^5 B_{nm\delta}$$

where the r.v.'s $B_{nm\delta}$, m = 1,2,3,4,5 are given by the integral on the right of (3), provided we restrict the region of integration to the interval $(0,\delta], (\delta, s_1 - \frac{\delta}{2}], (s_1 - \frac{\delta}{2}, s_1 + \frac{\delta}{2}], (s_1 + \frac{\delta}{2}, 1-\delta]$, and $(1-\delta, 1)$ respectively. We shall show first that, given any $\varepsilon > 0$, there exists a sufficiently small number $\delta(\varepsilon,F)$ such that

(8) $$P\left(\limsup_{n\to\infty} |B_{n1\delta}| \le 2\,\varepsilon \sup_{0<s<1} |J(s)| \right) = 1$$

for all $0<\delta<\delta(\varepsilon,F)$. To prove this we note first that for any $0 < \delta < 1$

(9) $$|B_{n1\delta}| \le 2 \sup_{0<s<1} |J(s)| \int_0^\delta |F^{-1}(s)|\, d\,\Gamma_n(s).$$

Because $\int_0^\delta |F^{-1}(s)|\, d\,\Gamma_n(s) = n^{-1} \sum_{i=1}^n |F^{-1}(U_i)|\, \chi_{(0,\delta]}(U_i)$ and $E|F^{-1}(U_i)|.$

$\chi_{(0,\delta]}(U_i) \le E|X_1| < \infty$ by assumption (ii) we can apply Kolmogorov's strong law of large numbers to find that for any $0 < \delta < 1$

(10) $$P\left(\lim_{n\to\infty} \int_0^\delta |F^{-1}(s)|\, d\,\Gamma_n(s) = \int_0^\delta |F^{-1}(s)|\,ds\right) = 1.$$

Now, it is well-known that (see e.g. HEWITT & STROMBERG [2]. Theorem 12.34 p. 176) that assumption (ii) implies that given any $\varepsilon > 0$ there exists a sufficiently small number $\delta(\varepsilon,F)$ such that for any measurable set $A \subset (0,1)$ with $\lambda(A)<\delta(\varepsilon,F)$ we have $\int_A |F^{-1}(s)|\, ds < \varepsilon$. Application of this with $A = (0,\delta]$ and using also (9) and (10) we see that (8) is proved.
Next we shall prove that for any $0 < \delta < \frac{2}{3}s_1$

(11) $$P(\lim_{n\to\infty} B_{n2\delta} = 0) = 1.$$

To check this we note first that for any $0 < \delta < \frac{2}{3}s_1$

(12) $$|B_{n2\delta}| \le \sup_{\delta<s\le s_1-\frac{\delta}{2}} \left| J\left(\tfrac{n}{n+1}\Gamma_n(s)\right) - J(s)\right| . \int_0^1 |F^{-1}(s)|\, d\,\Gamma_n(s).$$

Now the Glivenko-Cantelli theorem and the uniform continuity of J on $[\frac{\delta}{2}, s_1 - \frac{\delta}{4}]$ ensures that

(13) $$P\left(\lim_{n\to\infty} \sup_{\delta<s\le s_1-\frac{\delta}{2}} \left|J\left(\tfrac{n}{n+1}\Gamma_n(s)\right) - J(s)\right| = 0\right) = 1.$$

Since relation (10) also holds with δ replaced by 1 we have proved (11).

To proceed now we remark that the r.v..'s B_{3n} and B_{5n} can be treated in the same manner as we did with B_{1n}. Hence we know that, given $\varepsilon > 0$, there exist a sufficiently small number $\delta(\varepsilon,F)$ such that

$$(14) \qquad P\left(\limsup_{n\to\infty} \sum_{m=1,3,5} |B_{nm\delta}| \le 6\,\varepsilon \sup_{0<s<1} |J(s)| \right) = 1$$

for all $0<\delta<\delta(\varepsilon,F)$. Using the arguments leading to (11) we find that for any $0 < \delta < \frac{2}{3}(1-s_1)$

$$(15) \qquad P(\lim_{n\to\infty} B_{n4\delta} = 0) = 1.$$

Combining all these results we find that

$$(16) \qquad P\left(\limsup_{n\to\infty} |B_n| \le 6\,\varepsilon \sup_{0<s<1} |J(s)| \right) = 1.$$

Since J is bounded on (0,1) (assumption (i)) and $\varepsilon > 0$ is arbitrary (16) proves (6). Hence the proof for K = 0 is complete.

Suppose now that K > 0. In addition to the proof for K = 0 it suffices clearly to show that assumption (iii) ensures that for $1 \le k \le K$

$$(18) \qquad P\left(\limsup_{n\to\infty} d_{kn}\, F^{-1}(U_{i_k n}) = d_k\, F^{-1}(p_k) \right) = 1$$

This, however, is straightforward from assumption (iii) (which implies that F^{-1} is continuous at the points $p_1,\ldots,p_K$) and the well-known fact that $P(\lim_{n\to\infty} U_{i_k n} = p_k) = 1$ as $\frac{i_k}{n} \to p_k$ for $1 \le k \le K$. This completes the proof for K > 0.

To verify finally that assumption (ii) can be dropped in the case that J is zero on $(0,\alpha)$ and $(\beta,1)$ is easy. Of course we need only to consider the case K = 0. Note that it is immediate from (1) and the Glivenko-Cantelli theorem that we may restrict the region of integration in (1) to a closed interval $[\alpha - \eta, \beta + \eta]$ for some fixed $0 < \eta < \min(\alpha, 1-\beta)$. Since F^{-1} is bounded on any such interval we can simply repeat the argument of the proof for K = 0, provided we replace the interval (0,1) by the interval $[\alpha - \eta, \beta + \eta]$ everywhere in the proof. This shows that

$$(19) \qquad P\left(\lim_{n\to\infty} T_n = \int_{\alpha-\eta}^{\beta+\eta} F^{-1}(s)\, J(s)\, ds\right) = 1.$$

Because $\int_{\alpha-\eta}^{\beta+\eta} F^{-1}(s)\, J(s)\, ds = \int_{\alpha}^{\beta} F^{-1}(s)\, J(s)\, ds$ the proof of the theorem is complete.

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