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THE EFFICIENCY OF THREE TESTS FOR LINEAR HYPOTHESES CONCERNING TWO PROBABILITIES

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The efficiency of three tests for linear hypotheses concerning two probabilities*)

by J.M. Buhrman

SUMMARY

Two experiments have probabilities of success p_1 and p_2 respectively. Three methods are described to test hypotheses with the form $p_1 - cp_2 = d$, and their Pitman-efficiencies are given.

KEY WORDS & PHRASES: *Bernoulli trials, linear hypotheses, comparison of probabilities, Pitman efficiency.*

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*) This report will be submitted for publication elsewhere.

1. DESCRIPTION OF THE TESTS

Two experiments A and B have probabilities of success p_1 and p_2 respectively. We consider the general hypotheses

$$(1.1) \qquad p_1 - cp_2 = d$$

where c and d are given numbers satisfying the conditions $c > 0$ and $-c < d < 1$. Our aim is a comparison of the following three methods for testing hypotheses of type (1.1).

I Fixed numbers n_1 and n_2 of experiments A and B respectively are performed. If $\underline{s}_1$ and $\underline{s}_2$ are the respective numbers of successes, then under the hypotheses $p_1 - p_2 = 0$ the statistic $\underline{s}_1 | \underline{s}_1 + \underline{s}_2 = r$ (i.e. $\underline{s}_1$ under the condition $\underline{s}_1 + \underline{s}_2 = r$) has a hypergeometric-$(n_1+n_2, n_1, r)$ distribution. So $(\underline{s}_1, \underline{s}_2)$ provides a test for testing (1.1) with $c = 1$ and $d = 0$ (Fisher - Yates - Irwin test).

II Random numbers $\underline{n}_1$ and $\underline{n}_2$ experiments A and B respectively are performed. $\underline{n}_1$ is binomial-(n,π) and $\underline{n}_2 \overset{\text{def}}{=\!=} n - \underline{n}_1$. Under the hypothesis $p_1 - cp_2 = 0$ the statistic $\underline{s}_1 | \underline{s}_1 + \underline{s}_2 = r$ has a binomial-$(r, \pi c/(\pi c+1-\pi))$ distribution, so the statistic $(\underline{s}_1, \underline{s}_2)$ provides a test for the hypothesis (1.1) with $c > 0$ and $d = 0$.

III If, in the experimental design of II, one chooses $\pi = (1+c)^{-1}$, then under the hypothesis $p_1 - cp_2 = d$ the statistic $\underline{s}_1 + \underline{f}_2$ ($\underline{f}_1$ and $\underline{f}_2$ denote the respective numbers of failures) has a binomial-$(n, (d+c)/(1+c))$ distribution, so it provides a test for (1.1) with $c > 0$ and $-c < d < 1$.

Method I is well-known and frequently discussed (see e.g. [2], [8]), methods II and III are described in [3], [4] and [5].

2. THE PITMAN-EFFICIENCIES

For each method we shall give the asymptotic power of the one-sided test under a sequence of alternatives which converges with order $O(n^{-\frac{1}{2}})$ to the hypothesis as the total number, n, of experiments tends to infinity. Throughout the remainder of this report Φ will denote the standard normal distribution function, and u_α is defined by $\Phi(u_\alpha) = 1-\alpha$. Further p_1 and p_2 will be the probabilities of success of an experiment of type A and an experiment of type B respectively.

First some general lemmas are given. Let $V_\theta(\underline{z})$ denote the distribution of a random variable $\underline{z}$ with respect to a probability measure indicated by its parameter θ.

LEMMA 2.1. *Let* $V_p(\underline{z}_n) = \text{bin-}(n,p)$ *and let* $\theta_n \to \theta$. *Then*

$$(2.1) \qquad V_{\theta_n}\left(\frac{\underline{z}_n - n\theta_n}{\sqrt{n}}\right) \to N(0,\sqrt{\theta(1-\theta)})$$

PROOF. $\underline{z}_n^* \overset{\text{def}}{=} (\underline{z}_n - n\theta_n)/\sqrt{n\theta_n(1-\theta_n)}$ is asymptotically standard normal, the Lindeberg-Feller condition being satisfied. $\sqrt{\theta_n(1-\theta_n)} \to \sqrt{\theta(1-\theta)}$ proves the lemma. □

LEMMA 2.2. *Let* $V_p(\underline{z}) = \text{bin-}(n,p)$ *and let* k_n *denote the critical value for testing* $p = \theta$ *against* $p > \theta$ *at level* α. *Let* $\theta_n = \theta + b/\sqrt{n} + o(1/\sqrt{n})$, *then*

$$(2.2) \qquad P_{\theta_n}(\underline{z}_n \geq k_n) \to 1 - \Phi(u_\alpha - b/\sqrt{\theta(1-\theta)})$$

PROOF.

$$P_\theta(\underline{z}_n \geq k_n) \to \alpha \quad \Rightarrow \quad \frac{k_n - n\theta}{\sqrt{n\theta(1-\theta)}} \to u_\alpha \qquad \text{(see e.g. [11], Satz 2.11)}$$

$$P_{\theta_n}(\underline{z}_n \geq k_n) = P_{\theta_n}\left(\frac{\underline{z}_n - n\theta_n}{\sqrt{n\theta(1-\theta)}} \geq \frac{k_n - n\theta}{\sqrt{n\theta(1-\theta)}} - \frac{b}{\sqrt{\theta(1-\theta)}} + o(1)\right) \to$$

$$\to 1 - \Phi(u_\alpha - b/\sqrt{\theta(1-\theta)}) \qquad □$$

LEMMA 2.3. *Let* $(\underline{a}_n, \underline{r}_n)$ *be a statistic for testing* H_n *against* K_n *at level* α_n $(n = 1,2,\ldots)$, *with for* $\underline{r}_n = r$ *a conditional critical region of the form* $\{(a,r)\colon a \geq k_{n,r}\}$. *Let*

$$\underline{v}_n \overset{\text{def}}{=} \underline{a}_n - E(\underline{a}_n \mid \underline{r}_n; H_n)$$

$$k^*_{n,r} \overset{\text{def}}{=\!=} \frac{k_{n,r} - E(\underline{a}_n | \underline{r}_n = r; H_n)}{\sigma(\underline{a}_n | \underline{r}_n = r; H_n)}$$

If

(2.3) $\quad k^*_{n,\underline{r}_n} \to u_\alpha$ *in probability under* $\{K_n\}$

(2.4) $\quad \dfrac{\sigma(\underline{a}_n | \underline{r}_n; H_n)}{\sigma(\underline{v}_n | K_n)} \to 1$ *in probability under* $\{K_n\}$

(2.5) $\quad \underline{v}_n$ *is asymptotically normal under* $\{K_n\}$

(2.6) $\quad \eta \overset{\text{def}}{=\!=} \lim\limits_{n\to\infty} \dfrac{E(\underline{v}_n | K_n)}{\sigma(\underline{v}_n | K_n)}$ *exists*

then the power $\beta_n(K_n)$ *satisfies*

(2.7) $\quad \lim\limits_{n\to\infty} \beta_n(K_n) = 1 - \Phi(u_\alpha - \eta)$

PROOF.

$$\beta_n(K_n) = P(\underline{a}_n \geq k_{n,\underline{r}_n} \mid K_n) =$$

$$= P(\underline{v}_n \geq k_{n,\underline{r}_n} - E(\underline{a}_n | \underline{r}_n; H_n) | K_n) =$$

$$= P\left(\frac{\underline{v}_n - E(\underline{v}_n | K_n)}{\sigma(\underline{v}_n | K_n)} \geq k^*_{n,\underline{r}_n} \frac{\sigma(\underline{a}_n | \underline{r}_n; H_n)}{\sigma(\underline{v}_n | K_n)} - \frac{E(\underline{v}_n | K_n)}{\sigma(\underline{v}_n | K_n)} \,|K_n\right) \to 1 - \Phi(u_\alpha - \eta)$$

by application of a well-known theorem (see e.g. [6], theorem 20.6 or [9], Korollar 2.6).

REMARK. Lemma 2.2 is a particular case of lemma 2.3.

THEOREM 2.1.
(a) *Let* $p_2 \in (0,1)$ *and* $b > 0$ *be fixed, and let* $p_{1,n} = p_2 + b/\sqrt{n}$. *If* n_1 *and* n_2, *the respective numbers of experiments* A *and* B, *satisfy* $n_1 + n_2 = n$, $\lim_{n\to\infty} n_1/n = \zeta \in [0,1]$ *and* $n_1, n_2 \to \infty$, *then the unconditional power* $\beta^{(1)}_{\alpha,n}$ *of Fisher's test at level* α *at the alternative* $(p_{1,n}, p_2)$ *satisfies*

$$\lim_{n\to\infty} \beta^{(1)}_{\alpha,n} = 1 - \Phi(u_\alpha - \delta_1 b) \tag{2.8}$$

where

$$\delta_1 = \sqrt{\frac{\zeta(1-\zeta)}{p_2(1-p_2)}} \tag{2.9}$$

(b) *Let* $p_2 \in (0,1)$ *and* $b > 0$ *be fixed, and let* $p_{1,n} = cp_2 + b/\sqrt{n}$. *If method II is applied with a binomially-*(n,π) *distributed number* $\underline{n}_1$ *of experiments* A, *and* $\underline{n}_2 = n - \underline{n}_1$ *experiments* B, *then the unconditional power* $\beta^{(2)}_{\alpha,n}$ *of method II for testing* "$p_1 = cp_2$" *at level* α *at the alternative* $(p_{1,n},p_2)$ *satisfies*

$$\lim_{n\to\infty} \beta^{(2)}_{\alpha,n} = 1 - \Phi(u_\alpha - \delta_2 b) \tag{2.10}$$

where

$$\delta_2 = \sqrt{\frac{\pi(1-\pi)}{(\pi c+1-\pi)cp_2}} \tag{2.11}$$

(c) *Let* $p_2 \in (0,1)$ *and* $b > 0$ *be fixed, and let* $p_{1,n} = cp_2 + d + b/\sqrt{n}$. *The (unconditional) power* $\beta^{(3)}_{\alpha,n}$ *of method III for testing* "$p_1 = cp_2 + d$" *at level* α *at the alternative* $(p_{1,n},p_2)$ *satisfies*

$$\lim_{n\to\infty} \beta^{(3)}_{\alpha,n} = 1 - \Phi(u_\alpha - \delta_3 b) \tag{2.12}$$

where

$$\delta_3 = \sqrt{\frac{1}{(d+c)(1-d)}} \tag{2.13}$$

PROOF. (a) Let $k_{n,r}$ [1)] denote the critical value of the Fisher test at level α. Then (see e.g. [9], Satz 2.11)

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1) Notice, that n_1 and n_2 are functions of n, so $k_{n,r}$ and $k^*_{n,r}$ (see next page) depend on n_1 and n_2 through n only.

$$k^*_{n,r} \stackrel{\text{def}}{=} \frac{k_{n,r} - \frac{n_1 r}{n}}{\sqrt{\frac{n_1 n_2 r(n-r)}{n^2(n-1)}}} \rightarrow u_\alpha$$

for any sequence of hypergeometric-(n,n_1,r) distributions, provided that $r,n-r,n_1,n_2 \rightarrow \infty$, which is a sufficient condition for asymptotic normality (see e.g. [7]). So, if a random variable $\underline{r}$ satisfies $\underline{r}/n \rightarrow c \in (0,1)$ in probability, then $k^*_{n,\underline{r}} \rightarrow u_\alpha$ in probability. Now let

$$H \stackrel{\text{def}}{\Longleftrightarrow} p_1 = p_2$$

$$K_n \stackrel{\text{def}}{\Longleftrightarrow} p_1 = p_2 + b/\sqrt{n}$$

$$\underline{v}_n \stackrel{\text{def}}{=} \underline{s}_1 - E(\underline{s}_1 \mid \underline{s}_1 + \underline{s}_2; H) = (n_2\underline{s}_1 - n_1\underline{s}_2)/n$$

Then

$$E(\underline{v}_n|K_n) = \frac{n_2n_1(p_2+b/\sqrt{n})-n_1n_2p_2}{n} = \frac{n_1n_2b}{n\sqrt{n}}$$

$$\sigma^2(\underline{v}_n|K_n) = \frac{n_2^2}{n^2}\, n_1\left(p_2 + \frac{b}{\sqrt{n}}\right)\left(1 - p_2 - \frac{b}{\sqrt{n}}\right) + \frac{n_1^2}{n^2}\, n_2p_2(1-p_2)$$

Since conditions (2.3) to (2.5) with $\underline{a}_n = \underline{s}_1$ and $\underline{r}_n = \underline{s}_1 + \underline{s}_2$ are obviously fulfilled, lemma 2.3 can be applied:

$$\lim_{n\to\infty} \frac{E(\underline{v}_n|K_n)}{\sigma(\underline{v}_n|K_n)} = \sqrt{\frac{\zeta(1-\zeta)}{p_2(1-p_2)}}\, b \qquad \square$$

<u>REMARK</u>. A stronger result is obtained by ALBERS [1], for the case $0 < \zeta < 1$ (example 2.3.1).

(b) Let k_r denote the conditional critical value for testing $p_1 = cp_2$ against $p_1 > cp_2$ at level α. Then, if $r \rightarrow \infty$, with $\theta \stackrel{\text{def}}{=} \pi c/(\pi c+1-\pi)$

$$k^*_{n,r} \stackrel{\text{def}}{=} \frac{k_r - \theta r}{\sqrt{r\theta(1-\theta)}} \to u_\alpha$$

(see e.g. [9], Satz 2.11).

Now let

$$H \stackrel{\text{def}}{\Longleftrightarrow} p_1 = cp_2$$

$$K_n \stackrel{\text{def}}{\Longleftrightarrow} p_1 = cp_2 + b_2/\sqrt{n}$$

$$\underline{v}_n = \underline{s}_1 - E(\underline{s}_1 \mid \underline{s}_1 + \underline{s}_2; H) =$$

$$= \underline{s}_1 - \theta(\underline{s}_1 + \underline{s}_2) = \frac{(1-\pi)\underline{s}_1 - \pi c \underline{s}_2}{\pi c + 1 - \pi}$$

$\underline{s}_1$ and $\underline{s}_2$ are dependent, but $(\underline{s}_1, \underline{s}_2)$ is asymptotically bivariate normal, so $\underline{v}_n$ is asymptotically normal. Further

$$\sigma^2(\underline{s}_1 \mid \underline{s}_1 + \underline{s}_2; H) = (\underline{s}_1 + \underline{s}_2)\theta(1-\theta)$$

$$\sigma^2(\underline{v}_n | K_n) = E\sigma^2(\underline{v}_n \mid \underline{s}_1 + \underline{s}_2; K_n) + \sigma^2 E(\underline{v}_n \mid \underline{s}_1 + \underline{s}_2; K_n) =$$

$$= E(\underline{s}_1 + \underline{s}_2)\theta_n(1-\theta_n) + \sigma^2((\theta_n - \theta)(\underline{s}_1 + \underline{s}_2))$$

$$\text{(with } \theta_n = \frac{\pi p_1}{\pi p_1 + (1-\pi)p_2} \text{ under } K_n)$$

$$= n\nu_n\theta_n(1-\theta_n) + (\theta_n - \theta)^2 n\nu_n(1-\nu_n)$$

$$\text{(with } \nu_n = \pi p_1 + (1-\pi)p_2 \text{ under } K_n)$$

So, with $\underline{a}_n = \underline{s}_1$ and $\underline{r}_n = \underline{s}_1 + \underline{s}_2$, the conditions (2.3) and (2.4) are obviously fulfilled and lemma 2.3 can be applied.

$$E(\underline{v}_n | K_n) = \frac{(1-\pi)\pi(cp_2 + b_2/\sqrt{n})n - \pi c(1-\pi)p_2 n}{\pi c + 1 - \pi}$$

$$= \frac{(1-\pi)\pi b\sqrt{n}}{\pi c + 1 - \pi}$$

$$\lim_{n\to\infty} \frac{E(\underline{v}_n|K_n)}{\sigma(\underline{v}_n|K_n)} = \frac{(1-\pi)\pi b}{(\pi c+1-\pi)\sqrt{\nu\theta(1-\theta)}}$$

(with $\nu = (\pi c+1-\pi)p_2$)

$$= \sqrt{\frac{\pi(1-\pi)}{(\pi c+1-\pi)cp_2}}\, b \qquad \square$$

(c) The statistic $\underline{s}_1 + \underline{f}_2$ has a bin-(n,θ_n) distribution with

$$\theta_n = \theta = \frac{p_1}{1+c} + \frac{(1-p_2)c}{1+c} = \frac{cp_2+d}{1+c} + \frac{(1-p_2)c}{1+c} = \frac{d+c}{1+c} \quad \text{under the hypothesis}$$

and

$$\theta_n = \frac{p_1}{1+c} + \frac{(1-p_2)c}{1+c} = \frac{d+c}{1+c} + \frac{b}{(1+c)\sqrt{n}} \quad \text{under the alternative}$$

By lemma 2.2 we obtain the asymptotic power:

$$1 - \Phi\left(u_\alpha - \frac{\frac{b}{1+c}}{\sqrt{\frac{d+c}{1+c}\,\frac{1-d}{1+c}}}\right) = 1 - \Phi(u_\alpha - \delta_3 b). \qquad \square$$

The values of δ_i (see (2.9), (2.11) and (2.13)) immediately lead to the Pitman-efficiencies, which are, according to the definition as given by WITTING and NÖLLE [9], equal to δ_i^2/δ_j^2. Therefore we have

(2.14) $$\text{ARE(I,II)} = \frac{\zeta(1-\zeta)}{\pi(1-\pi)(1-p_2)}$$

(2.15) $$\text{ARE(I,III)} = \frac{\zeta(1-\zeta)}{p_2(1-p_2)}$$

(2.16) $$\text{ARE(II,III)} = \frac{\pi(1-\pi)}{(\pi c+1-\pi)p_2}$$

From (2.9) it is clear that for Fisher's test $\zeta = \frac{1}{2}$ is the best choice. From (2.11) it can easily be derived that $\pi = 1/(1+\sqrt{c})$ is asymptotically optimal for method II. With these values for ζ and π the efficiencies are

$$(2.17) \qquad ARE(I,II) = \frac{1}{1-p_2}$$

$$(2.18) \qquad ARE(I,III) = \frac{1}{4p_2(1-p_2)}$$

$$(2.19) \qquad ARE(II,III) = \frac{1}{p_2(1+\sqrt{c})^2}$$

These results are the same as the results that were presumed and confirmed by numerical investigations in [3]. Formulae (2.17) and (2.18) lead to the conclusion that Fisher's test is at least as good as the other methods for testing $p_1 = p_2$. Only if $p_2 \approx \frac{1}{2}$, methods I and II are equally good, but this is of little importance, since usually $p_2 \approx \frac{1}{2}$ cannot be guaranteed in advance. If $p_1 = cp_2$ $(c \neq 1)$ must be tested, method III is better than method II, except if $p_2 < (1+\sqrt{c})^{-2}$. If $p_1 - cp_2 = d$ $(d \neq 0)$ must be tested, only method III can be used.

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