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BRANCHING PROCESSES WITH CONTINUOUS STATE SPACE ALLOWING IMMIGRATION

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Branching processes with continuous state space allowing immigration *)

by

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ABSTRACT

In addition to some recent papers on Galton-Watson processes allowing immigration we study asymptotic properties of branching processes with continuous state space allowing immigration. In particular we derive results for the class of processes which do not occur if we consider only processes for which the state space is integer-valued. The methods we use are however very similar to the ones used to obtain results for Galton-Watson processes.

KEY WORDS & PHRASES: *Galton-Watson process; branching process with continuous state space; immigration; subordinator; slow variation; regular variation*

*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In several recent papers ([1],[6],[7],[8]), the asymptotic behaviour of Galton-Watson processes allowing immigration has been studied. Such processes describe the number of individuals of a population, behaving like a Galton-Watson process, in which moreover a random number of immigrants enter the nth generation. These immigrants then act as the other individuals of the population. More formally, we can say, let $\{X_{ij};\ i = 1,2,\dots;\ j = 1,2,\dots\}$ be a sequence of independent random variables, all distributed according to the so-called offspring distribution $\{p_j;\ j = 0,1,2,\dots\}$, and let $\{Y_1, Y_2, Y_3,\dots\}$ be a sequence of i.i.d. integer valued random variables, then the Galton-Watson process allowing immigration is defined as

$$(1.1)\qquad X_0 = 0;\quad X_{n+1} = \sum_{j=1}^{X_n} X_{n+1,j} + Y_{n+1},\qquad n = 0,1,2,\dots .$$

In this paper we study the analog of these processes for branching processes with continuous state space (see e.g. [5]). We use the following definition:

<u>DEFINITION 1.1</u>. Let $\{\{W_n(t);\ t \in [0,\infty)\};\ n = 0,1,2,\dots\}$ be a sequence of i.i.d. subordinators, and $\{Y_1,Y_2,Y_3,\dots\}$ a sequence of i.i.d. random variables, also independent of the subordinators and such that $P(Y_1=0) < 1 = P(Y_1 \geq 0)$. Let $\{X_n;\ n = 0,1,2,\dots\}$ be defined by

$$(1.2)\qquad X_0 = 0;\quad X_{n+1} = W_n(X_n) + Y_{n+1},\qquad n = 0,1,2,\dots .$$

Then the process $\{X_n;\ n = 0,1,2,\dots\}$ is called a branching process with continuous state space allowing immigration.

Writing $h^*(s)$ for the cumulant generating function (c.g.f.) of the offspring distribution of the process without immigration, that is $h^*(s) = -\log E\exp(-sW_n(1))$, $f(s)$ for the c.g.f. of Y_1 and $h_n(s)$ for the c.g.f. of X_n, it follows immediately from Definition 1.1 that

$$(1.3)\qquad h_0(s) = 0;\quad h_{n+1}(s) = h_n(h^*(s)) + f(s),\qquad n = 0,1,2,\dots .$$

Denoting by $h_j^*(s)$ the j-th iterate of the function $h^*(s)$, or equivalently, the c.g.f. of the size of the j-th generation of the process without immigration starting with one particle, (1.3) can also be expressed in the more convenient form

$$(1.4)\qquad h_0(s) = 0; \quad h_{n+1}(s) = \sum_{j=0}^{n} f(h_j^*(s)), \qquad n = 0,1,2,\ldots .$$

In several papers ([5],[10],[11]) it has been shown that there is a great correspondence between Galton-Watson processes and branching processes with continuous state space, both if the expectation of the offspring distribution, denoted by m, exceeds 1 and if $m \le 1$ and the offspring distribution has an atom at 0. This same correspondence can be proved to exist if we consider processes with immigration. We shall not go into details but only refer to [1] and [8]. In this paper we restrict attention to the remaining cases, that is $m \le 1$ and the offspring distribution has no atom at 0. Notice that such processes can not occur if we consider Galton-Watson processes. The methods we shall use are however very similar to the ones used to obtain results for Galton-Watson processes allowing immigration, as described in [8]. They are in fact based on knowledge of the process without immigration, that is of the c.g.f. $h_j^*(s)$, which is used with the help of (1.4) to obtain results for $h_n(s)$.

2. THE SUBCRITICAL CASE

In this section we suppose that $m < 1$. It follows immediately from (1.4) that

$$\lim_{n\to\infty} h_n(s) = \sum_{j=0}^{\infty} f(h_j^*(s))$$

exists, and that this limit is the c.g.f. of a possibly defective distribution. In fact integral comparison yields

$$(2.1)\qquad \sum_{j=0}^{\infty} f(h_j^*(s)) < \infty \text{ for any } s \in (0,\infty) \text{ iff } \int_0^{\infty} f(e^{-x})\,dx < \infty,$$

since $m < 1$ implies that there exist $\delta_1, \delta_2 \in (0,1)$ such that $\delta_1^j \le h_j^*(s) \le$ $\le \delta_2^j$ for sufficiently large j (see Chapter IV of [5]). Furthermore, just as in the discrete case ([2],[4]) we have the following Lemma (see also Theorem 3 in [10]).

LEMMA 2.1. *Equivalent with*(2.1) *is* $E(\log Y_1)^+ < \infty$.

PROOF. Using the representation

$$\frac{1-\exp(-f(s))}{s} = \int_0^\infty e^{-sx}\{1-F(x)\}dx,$$

where F is the distribution function of Y_1, it follows that (2.1) holds iff for any $\varepsilon > 0$

$$\int_0^\varepsilon \frac{1-\exp(-f(s))}{s}\, ds < \infty,$$

that is iff for any $0 < \varepsilon < \infty$

$$\int_\varepsilon^\infty \frac{1-\exp(-x)}{x}\{1-F(x)\}dx < \infty \tag{2.2}$$

by Fubini's theorem. Now (2.2) is equivalent with

$$\int_\varepsilon^\infty \frac{1-F(x)}{x}\, dx = \int_\varepsilon^\infty \left\{\int_{x-}^\infty dF(y)\right\} d\log x$$

$$= \int_{\varepsilon-}^\infty \left\{\int_\varepsilon^y d\log x\right\} dF(y) < \infty \qquad \text{for any } 0 < \varepsilon < \infty,$$

that is with $E(\log Y_1)^+ < \infty$. □

As a consequence of this Lemma we have that X_n converges weakly to a proper distribution iff $E(\log Y_1)^+ < \infty$. Otherwise $X_n \xrightarrow{P} \infty$. We shall now examine this latter case in more detail. Writing $G(x) = f(e^{-x})$, and assuming that $E(\log Y_1)^+ = \infty$, that is that $\int_0^\infty G(x)dx = \infty$, we consider two special types of behaviour of G(x).

LEMMA 2.2. *Suppose there exist a constant* $b \in (0,m]$ *and a positive increasing function* B *such that* $B(0+) = 0$ *and*

$$(2.3) \qquad h^*(s) = B(bB^{-1}(s)),$$

and that $B(s)$ *has a monotone derivative and is regularly varying with exponent* $\delta \in (0,\infty)$ *for* $s \downarrow 0$, *and that furthermore*

$$(2.4) \qquad \lim_{x\to\infty} xG(x) = 0.$$

Then $\Lambda(X_n)/\Lambda(b^{-n})$ *converges weakly to a random variable with distribution function* $A(x) = \min(1,x^{\zeta})$; *here*

$$(2.5) \qquad \Lambda(x) = \begin{cases} \exp\{\int_1^x G(y)\,dy\} & \text{if } x \geq 1 \\ x & \text{if } x < 1 \end{cases},$$

and $\zeta = -1/\log m$.

PROOF. Since B is regularly varying with exponent δ, (2.3) implies that $mB(s) \sim h^*(B(s)) = B(bs) \sim b^{\delta}B(s)$, $s \downarrow 0$, and hence $\delta = (\log m)/(\log b)$. It follows also from (2.3) that $h_n^*(s) = B(b^nB^{-1}(s))$, $n = 0,1,2,\ldots$. Defining for all $t \in (0,\infty)$

$$h_t^*(s) = B(b^tB^{-1}(s)),$$

integral comparison and (1.4) yield

$$(2.6) \qquad h_{n+1}(s)-h_1(s) = \sum_{j=1}^{n} f(h_j^*(s)) \leq \int_0^n f(h_t^*(s))\,dt \leq$$

$$\leq \sum_{j=0}^{n-1} f(h_j^*(s)) = h_n(s).$$

Substituting $t = \dfrac{\log\{B^{-1}(e^{-x})/B^{-1}(s)\}}{\log b}$, and using the fact that the regular variation of B(s) and the monotonicity of its derivative imply that

$$\lim_{s\downarrow 0} \frac{s[B^{-1}(s)]'}{B^{-1}(s)} = 1/\delta = \frac{\log b}{\log m},$$

([9], property 1.5.5 and theorem 2.4), it follows from (2.6) that for any sequence $s_n \downarrow 0$,

$$(2.7) \qquad \exp\{-h_n(s_n)\} \sim \exp\{-(1/\log m) \int_{-\log s_n}^{-\log h_n^*(s_n)} G(x)dx\}.$$

Now let

$$(2.8) \qquad s_n(x) = 1/\Lambda^{-1}(x\Lambda(b^{-n})), \; x \in (0,\infty), \qquad n = 1,2,3,\dots ,$$

then $\lim_{n\to\infty} s_n(x) = 0$ for all x, since $\int_0^\infty G(x)dx = \infty$. Furthermore

$$(2.9) \qquad \int_{-\log s_n(x)}^{-\log h_n^*(s_n(x))} G(y)dy = \log\Lambda(1/h_n^*(s_n(x))) - \log\Lambda(1/s_n(x))$$

$$= \log\Lambda(1/h_n^*(s_n(x))) - \log(x\Lambda(b^{-n}))$$

$$= -\log x + \int_{-n\log b}^{-\log h_n^*(s_n(x))} G(y)dy$$

$$= -\log x + I_n,$$

say. Now

$$(2.10) \qquad |I_n| \le \left|\int_{-n\log m}^{-n\log b} G(y)dy\right| + \left|\int_{-n\log m}^{-\log h_n^*(s_n(x))} G(y)dy\right|.$$

Since $b \le m < 1$, and G is decreasing, it follows that

$$(2.11) \qquad \left|\int_{-n\log m}^{-n\log b} G(y)dy\right| \le -G(-n\log m)\cdot n\log(b/m)$$

$$= G(-n\log m)\cdot(-n\log m)\cdot \frac{\log(b/m)}{\log m} \longrightarrow 0 \quad \text{as } n\to\infty \text{ by (2.4)}.$$

Because $s_n(x) \to 0$ as $n \to \infty$, and $h_n^*(s) \le m^n s$, it follows that

$h_n^*(s_n(x)) \le m^n$ for sufficiently large n, and therefore

$$(2.12)\qquad \left|\int_{-n\log m}^{-\log h_n^*(s_n(x))} G(y)\,dy\right| \le -G(-n\log m)\cdot\log\frac{h_n^*(s_n(x))}{m^n}$$

$$= G(-n\log m)\cdot(-n\log m)\left[\frac{-\log h_n^*(s_n(x))}{-n\log m}-1\right]$$

for sufficiently large n. The regular variation of B and B^{-1} imply that $\log B(s) \sim \delta\log s$ and $\log B^{-1}(s) \sim (1/\delta)\log s$ for $s \downarrow 0$, and so

$$1 \le \limsup_{n\to\infty}\frac{-\log h_n^*(s_n(x))}{-n\log m} = \limsup_{n\to\infty}\frac{-\log B(b^nB^{-1}(s_n(x)))}{-n\log m}$$

$$= \limsup_{n\to\infty}\frac{-\delta\log b^n - \delta\log B^{-1}(s_n(x))}{-n\log m}$$

$$= \frac{\delta\log b}{\log m} + \limsup_{n\to\infty}\frac{-\log s_n(x)}{-n\log m}\quad .$$

Now for any $x \in (0,1)$ it holds that

$$-\log s_n(x) = \log\Lambda^{-1}(x\Lambda(b^{-n})) \le \log\Lambda^{-1}(\Lambda(b^{-n})) = -n\log b,$$

and so for any $x \in (0,1)$

$$1 \le \limsup_{n\to\infty}\frac{-\log h_n^*(s_n(x))}{-n\log m} \le (\delta+1)\,\frac{\log b}{\log m} < \infty.$$

In view of (2.10),(2.11),(2.12) and (2.4) we therefore obtain that $\lim_{n\to\infty}|I_n| = 0$ for all $x \in (0,1)$. This together with (2.7) and (2.9) implies that

$$\lim_{n\to\infty} h_n(s_n(x)) = \frac{\log x}{\log m}\,,\qquad x \in (0,1).$$

Noting that $s_n(x)$ is non-increasing in x, and that $h_n(s) \ge 0$, we see that $\lim_{n\to\infty} h_n(s_n(x)) = \max(0,(\log x)/\log m) = -\log A(x)$. Next we notice that for $x \to \infty$ and $t \in (1,\infty)$

$$1 \le \frac{\Lambda(tx)}{\Lambda(x)} = \exp \int_{\log x}^{\log tx} G(y)\,dy \le \exp \int_{\log x}^{\log tx} \frac{dy}{y} = \frac{\log tx}{\log x} \to 1,$$

that is Λ is slowly varying at ∞. Then we can use Lemma 1 of [3] and the fact that $X_n \xrightarrow{P} \infty$ to obtain the required result. □

LEMMA 2.3. *Suppose that the conditions of Lemma* 2.2. *are satisfied except for* (2.4) *and that*

(2.13) $$\lim_{x\to\infty} xG(x) = \beta \in (0,\infty).$$

Then $(\zeta \log X_n)/n$ *converges weakly to a random variable with distribution function* $C(x) = (1+\delta/x)^{-\zeta\beta}$, *with* $\zeta = -1/\log m$, *and* $\delta = (\log m)/(\log b)$.

PROOF. Just as in the proof of Lemma 2.1 it follows that

(2.14) $$\exp(-h_n(s_n)) \sim \exp\{-1/\log m \int_{-\log s_n}^{-\log h_n^*(s_n)} G(x)\,dx\}$$

for any sequence $s_n \downarrow 0$. Choosing $s_n(x) = b^{xn}$, $x \in (0,\infty)$, $n = 1,2,3,\ldots,$ we obtain by (2.13) that

(2.15) $$\int_{-\log s_n(x)}^{-\log h_n^*(s_n(x))} G(y)\,dy \sim \beta \int_{-\log s_n(x)}^{-\log h_n^*(s_n(x))} \frac{dy}{y}$$

$$= \beta \log \frac{-\log h_n^*(s_n(x))}{-\log s_n(x)} .$$

As in the proof of Lemma 2.1

$$-\log h_n^*(s_n(x)) = -\log B(b^n B^{-1}(s_n(x)) \sim -\delta \log(b^n B^{-1}(s_n(x))),$$

and

$$-\log B^{-1}(s_n(x)) \sim -(1/\delta)\log s_n(x) = -\frac{nx \log b}{\delta} .$$

Therefore

$$\lim_{n\to\infty} \frac{-\log h_n^*(s_n(x))}{-\log s_n(x)} = \lim_{n\to\infty} \frac{-n\delta \log b - nx \log b}{-nx \log b} = 1 + \frac{\delta}{x}.$$

Combining this with (2.14) and (2.15) it follows that

$$\lim_{n\to\infty} h_n(b^{nx}) = \zeta\beta \log(1+\delta/x).$$

We can now use Lemma 9.2.1 in [5], yielding that $(\zeta \log X_n)/n$ converges weakly to a random variable with distribution function $(1+\delta/x)^{-\zeta\beta}$, $x \in (0,\infty)$. □

We shall now apply the two preceding lemmas to obtain results in the case which does not occur in Galton-Watson processes, that is we suppose that $P(Z_1=0)$, where $\{Z_n;\ n = 0,1,2,\ldots\}$ is the process without immigration. As is well-known ([5],[11]) we have to distinguish between the cases where $a = \inf\{x; P(Z_1 \le x \mid Z_0 = 1) > 0\} > 0$ and $a = 0$. In the first case the starting point is equation (2.1) in [11]. There it is proved that there exists a sequence c_n such that $c_n Z_n \xrightarrow{a.s.}$ some limit Z, with c.g.f. ϕ satisfying

$$(2.16) \qquad \phi(as) = h^*(\phi(s)).$$

This relation is of the form (2.3), with $B = \phi$ and $b = a$. So we get

<u>THEOREM 2.4</u>. *Suppose that* $a > 0$. *If* $\phi(s)$ *is regularly varying for* $s \downarrow 0$, *then*

a) *if* $\lim_{x\to\infty} xG(x) = 0$, *then* $\Lambda(X_n)/\Lambda(a^{-n})$ *converges weakly to a random variable with distribution function* $A(x)$, *with* Λ *and* A *as in Lemma* 2.2;

b) *if* $\lim_{x\to\infty} xG(x) = \beta \in (0,\infty)$, *then* $(-\log X_n)/(n \log m)$ *converges weakly to a random variable with distribution function* $(1+(\log m)/(x \log a))^{\beta/\log m}$.

<u>PROOF</u>. Since ϕ is a c.g.f. it satisfies the conditions of the Lemmas 2.2 and 2.3. The result now follows from these Lemmas. □

If $a = 0$, we can show just as in Theorem 9.2.12 and the construction preceding it in [5], that there exists a positive, increasing, differentiable function L such that $m^{-n}L(1/Z_n) \xrightarrow{a.s.}$ some limit U, and that

$$h^*(s) = L^{-1}(mL(s)). \tag{2.17}$$

This leads to the following result.

THEOREM 2.5. *Suppose that* $a = 0$, *and that the function* L *is convex or concave;*

a) *if* $\lim_{x\to\infty} xG(x) = 0$, *then* $\Lambda(X_n)/\Lambda(m^{-n})$ *converges weakly to a random variable with distribution function* $A(x)$, *with* Λ *and* A *as in Lemma* 2.2;

b) *if* $\lim_{x\to\infty} xG(x) = \beta \in (0,\infty)$ *then* $(-\log X_n)/(n \log m)$ *converges weakly to a random variable with distribution function* $(1+1/x)^{\beta/\log m}$.

PROOF. In view of the Lemmas 2.2 and 2.3 we only have to prove that L(s) is regularly varying for $s \downarrow 0$. Suppose for instance that L is concave. Since $L(0) = 0$, we then get

$$\frac{L^{-1}(ms)}{ms} \le \frac{L^{-1}(ts)}{ts} \le \frac{L^{-1}(s)}{s}$$

for any $t \in (m,1)$. Furthermore, by (2.17)

$$\frac{L^{-1}(ms)}{ms} \Big/ \frac{L^{-1}(s)}{s} \sim \frac{L^{-1}(mL(s))}{mL(s)} \Big/ \frac{s}{L(s)} = \frac{h^*(s)}{ms} \longrightarrow 1 \quad \text{as } s \downarrow 0$$

and so for any $t \in (m,1)$

$$\lim_{s\downarrow 0} \frac{L^{-1}(ts)}{L^{-1}(s)} = t \lim_{s\downarrow 0} \frac{L^{-1}(ts)}{ts} \Big/ \frac{L^{-1}(s)}{s} = t,$$

implying that L^{-1} and therefore also L, is regularly varying with exponent 1. □

3. THE CRITICAL CASE

In this section we shall pay some attention to critical processes, that is we suppose that $m = 1$. If we want to use a similar method as before, it is not immediately clear which functional equation of the form (2.3) we have to take. We can however proceed in a similar way as in [8]. The starting point there is a functional equation holding for the generating function of the invariant measure of the process without immigration. Making a special assumption on the behaviour of the probability generating function of Z_1, several results for critical processes are deduced. Since the convenient tool in the study of branching processes with continuous state space is a c.g.f., the assumption we make can be stated as follows:

$$(3.1) \qquad h^*(s) = s - s^{1+\alpha}L(s)$$

with $\alpha \in (0,1]$ and $L(s)$ slowly varying for $s \downarrow 0$. Furthermore we can construct a function V which satisfies a functional equation corresponding to the above mentioned equation used in [8]. To this end we define a function V on (0,1) such that

$$(3.2) \qquad V(1) = 0;$$

(3.3) V is decreasing, and continuously differentiable on $(h^*(1),1]$;

$$(3.4) \qquad V'(h^*(1)) = V'(1)/(h^*)'(1).$$

Finally we define

$$(3.5) \qquad V(h_n^*(s)) = V(s) + n, \qquad s \in (h^*(1),1], \qquad n = 1,2,3,\ldots .$$

The function V defined in this way is continuously differentiable on (0,1). Analogously to a result in [12] we can prove from (3.1) that V(s) is regularly varying with exponent $-\alpha$ for $s \downarrow 0$. Now we can proceed as in section 2, that is we define

$$h_t^*(s) = V^{-1}(V(s)+t), \qquad s \in (0,1], \qquad t \in [0,\infty),$$

and analogous to (2.7) we get that for every sequence $s_n \downarrow 0$

$$(3.6) \qquad \exp\{-h_n(s_n)\} \sim \exp\{-\int_0^n f(V^{-1}(V(s_n)+t))dt\}$$

$$= \exp\{-\int_{V(s_n)}^{n+V(s_n)} G(x)dx\} \qquad n \to \infty,$$

where $G(x) = f(V^{-1}(x))$. This enables us to prove the results corresponding to the Theorems 9-12 in [8]. We shall give one example of this.

<u>THEOREM 3.1</u>. *Suppose that* $\int_0^\infty G(x)dx = \infty$, *and that* $\lim_{x\to\infty} xG(x) = 0$. *If* V *has a monotone derivative, then* $\Lambda(X_n)/\Lambda(a_n)$ *has a limiting uniform distribution on* $[0,1]$; *here*

$$\Lambda(x) = \begin{cases} \exp \int_0^{V(1/x)} G(y)dy & \text{if } x \geq 1 \\ x & \text{if } 0 \leq x \leq 1 \end{cases},$$

and $a_n = 1/V^{-1}(n)$.

<u>PROOF</u>. Choosing $s_n(x) = 1/\Lambda^{-1}(x\Lambda(a_n))$ for $x \in (0,\infty)$, it follows from (3.6) that

$$\exp\{-h_n(s_n(x))\} \sim \exp\{-\int_{V(s_n(x))}^{n+V(s_n(x))} G(y)dy\}$$

$$= \exp\{\log \Lambda(1/s_n(x)) - \log \Lambda(1/h_n^*(s_n(x)))\}$$

$$= x \exp\{-\int_n^{n+V(s_n(x))} G(y)dy\}.$$

Now for $x \in (0,1]$

$$0 \leq \int_n^{n+V(s_n(x))} G(y)dy \leq V(s_n(x))G(n) \leq V(s_n(1)) \cdot G(n)$$

$$= V(1/a_n)G(n) = nG(n) \to 0 \quad \text{as} \quad n \to \infty$$

by assumption. Therefore

$$\lim_{n\to\infty} h_n(s_n(x)) = \begin{cases} -\log x & \text{if } x \in (0,1] \\ 0 & \text{if } x \in [1,\infty) \end{cases} .$$

Furthermore,

$$\lim_{x\to\infty} \frac{x\Lambda'(x)}{\Lambda(x)} = -\lim_{x\to\infty} \frac{xG(V(1/x))V'(1/x)}{x^2} = -\lim_{s\downarrow 0} s\, G(V(s))\cdot V1/(s) .$$

Since V(s) is regularly varying with exponent $-\alpha$, and V has a monotone derivative it follows from Theorem 2.4 in [9] that $sV'(s) \sim -\alpha V(s)$, $s \downarrow 0$. This means that

$$\lim_{x\to\infty} \frac{x\Lambda'(x)}{\Lambda(x)} = \lim_{s\downarrow 0} \alpha G(V(s))\cdot V(s) = 0,$$

implying that $\Lambda(x)$ is slowly varying for $x \to \infty$. Application of Lemma 1 in [3] now yields the required result. □

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