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THE BERRY-ESSEEN BOUND FOR U-STATISTICS

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1. INTRODUCTION

A more or less satisfactory and reasonably complete theory of Berry-Esseen bounds and Edgeworth expansions is available only for sums of independent random variables and vectors. In recent years, however, Berry-Esseen bounds and Edgeworth expansions were also obtained for several statistics of a different structure, which occur in estimation and hypothesis testing problems. To obtain such results for statistics T_N in cases where no explicit representation for the characteristic function of T_N is known, one may proceed by first obtaining a so-called stochastic expansion for T_N. This means that one approximates T_N sufficiently accurately by a statistic T'_N of a simpler structure and then proves the desired results for T'_N instead of T_N. In many of the cases that were considered, T'_N is a smooth function of a sum of independent random vectors and the problem can be solved by appealing to the classical theory for such sums and translating the result into one for T'_N. Thus the classical theory can be extended beyond its original domain of application.

There are other cases, however, where the natural stochastic expansion T'_N is of a different type. In particular, it appears to be quite common that T'_N is a U - statistic, i.e.

$$(1.1) \qquad T'_N = \sum_{1 \le i_1 < \ldots < i_k \le N} h(X_{i_1}, \ldots, X_{i_k}) ,$$

where $X_1, \ldots, X_N$ are independent and identically distributed (i.i.d.) random variables. Examples of statistics T_N for which this is the case are one-sample linear rank statistics and linear combinations of order statistics. For a more detailed account of what has been said so far and for references to relevant literature, the reader is referred to Van Zwet (1977).

In view of this, it would be of considerable interest to have a theory of Berry-Esseen bounds and Edgeworth expansions for U - statistics, similar to the one for sums of independent random variables and vectors. Such a theory would not only be a meaningful generalization of the classical case from a purely probabilistic point of view, but it could also be extended to statistics possessing stochastic expansions of the type (1.1).

As yet, Edgeworth expansions for U - statistics have only been established under very restrictive assumptions (cf. Janssen (1978) and Callaert, Janssen and Veraverbeke (1980)) and much work remains to be done on this problem. However, results of Berry-Esseen type showing that the distribution function of a standardized U - statistic tends to its normal limit at the rate of $N^{-\frac{1}{2}}$, have been obtained in increasing generality by Bickel (1974), Chan and Wierman (1977) and Callaert and Janssen (1978). These authors discuss U - statistics of order 2, viz.

$$(1.2) \qquad U_N = \sum_{1\le i<j\le N}\sum h(X_i,X_j) \ ,$$

but Chan and Wierman as well as Callaert and Janssen point out that all results carry over to U - statistics of any fixed order and also to the multi-sample case (cf. also Janssen (1978)). The best of these results is that of Callaert and Janssen who assume that $E|h(X_1,X_2)|^3 < \infty$. In the present paper we relax this moment condition even further. The results are applied to Von Mises functionals and an example is given to indicate the improvement over previous work. For simplicity we too restrict attention to U - statistics of order 2.

2. THE BERRY-ESSEEN THEOREM

Let $X_1, X_2, \ldots$ be i.i.d. random variables with common distribution function F. Define the U - statistic U_N by (1.2), where h is a symmetric function of two variables with

$$(2.1) \qquad E\,h(X_1,X_2) = 0 \ , \quad E|h(X_1,X_2)|^p < \infty \ ,$$

for some $p > 5/3$. Let g be given by

$$(2.2)\qquad g(x) = E(h(X_1,X_2)|X_1=x) = \int h(x,y)d\,F(y)$$

and suppose that

$$(2.3)\qquad E\,g^2(X_1) > 0\ ,\quad E|g(X_1)|^3 < \infty\ .$$

If we take

$$\psi(x,y) = h(x,y) - g(x) - g(y)\ ,$$

then the random variable $\psi(X_1,X_2)$ has the property that

$$(2.4)\qquad E(\psi(X_1,X_2)|X_1) = E(\psi(X_1,X_2)|X_2) = 0$$

almost surely (a.s.) and

$$(2.5)\qquad U_N = (N-1)\sum_{i=1}^{N} g(X_i) + \sum_{1\le i<j\le N}\sum \psi(X_i,X_j)\ .$$

Let

$$(2.6)\qquad \tau_N^2 = (N-1)^2 N\,E\,g^2(X_1)$$

denote the variance of the first term on the right in (2.5) and let Φ be the standard normal distribution function.

<u>THEOREM 2.1</u>

Suppose that (2.1) *and* (2.3) *are satisfied for some* $p > 5/3$. *Then there exists a positive number* C_p *depending on* h *and* F *only through* p *and such that for every* $N \ge 2$,

$$(2.7)\qquad \sup_x |P(\tau_N^{-1}U_N \le x) - \Phi(x)| \le C_p N^{-\frac{1}{2}}\left[\frac{E|g(X_1)|^3}{\{Eg^2(X_1)\}^{3/2}} + \frac{E|h(X_1,X_2)|^p}{\{Eg^2(X_1)\}^{\frac{1}{2}p}}\right]\ .$$

In the special case where (2.1) holds for $p = 2$, U_N possesses a finite variance σ_N^2 which is given by

$$(2.8) \qquad \sigma_N^2 = \tau_N^2 + \frac{N(N-1)}{2} E\,\psi^2(X_1,X_2) \ .$$

Obviously

$$1 \le \frac{\sigma_N}{\tau_N} \le 1 + \frac{Eh^2(X_1,X_2)}{2N\ Eg^2(X_1)}$$

and hence σ_N may replace τ_N in (2.7) to yield

<u>COROLLARY 2.1</u>

There exists a universal constant C *such that for every* $N \ge 2$,

$$(2.9) \qquad \sup_x |P(\sigma_N^{-1}U_N \le x) - \Phi(x)| \le CN^{-\frac{1}{2}}\left[\frac{E|g(X_1)|^3}{\{Eg^2(X_1)\}^{3/2}} + \frac{Eh^2(X_1,X_2)}{Eg^2(X_1)}\right] ,$$

provided (2.1) *and* (2.3) *are satisfied for* $p = 2$.

We note that the result of this corollary was announced without proof in Helmers (1978), p.56, and quoted and applied subsequently by Callaert and Veraverbeke (1981) and Müller-Funk and Witting (1981). It also occurs in Borovskikh (1979) but the proof given there is incomplete and ignores a crucial part of the argument.

3. PROOF OF THEOREM 2.1

Assume that (2.1) and (2.3) are satisfied for some $p > 5/3$ so that the quantities

$$a = \frac{E|g(X_1)|^3}{\{Eg^2(X_1)\}^{3/2}} , \quad b_p = \frac{E|h(X_1,X_2)|^p}{\{Eg^2(X_1)\}^{\frac{1}{2}p}}$$

are both finite. If $p \geq 2$, (2.2) ensures that $b_2 \geq 1$ and this implies that for $p > 2$, $b_p \geq b_2^{\frac{1}{2}p} \geq b_2$. It is therefore sufficient to prove the theorem for $5/3 < p \leq 2$ and we shall restrict attention to this case. Define, for $m = 1,\ldots,(N-1)$,

$$\text{(3.1)} \qquad \Delta_N(m) = \sum_{i=1}^{m} \sum_{j=i+1}^{N} \psi(X_i,X_j) \ .$$

We shall need the following bounds for moments.

LEMMA 3.1

If (2.1) *and* (2.3) *hold for some* $5/3 < p \leq 2$, *then*

$$\text{(3.2)} \qquad \frac{E|\psi(X_1,X_2)|}{\{Eg^2(X_1)\}^{\frac{1}{2}}} \leq 2(2a+b_p) \ ,$$

$$\text{(3.3)} \qquad \frac{E|\psi(X_1,X_2)g(X_1)g(X_2)|}{\{Eg^2(X_1)\}^{3/2}} \leq 5a + 2b_p \ ,$$

$$\text{(3.4)} \qquad E\left|\frac{\Delta_N(m)}{\tau_N}\right|^p \leq 4(2a+b_p)m\, N^{-(3/2)p+1}$$

for $m = 1,\ldots,N-1$ *and* $N = 2,3,\ldots$.

Proof

By the c_r - inequality and lemma 1 of Chatterji (1969)

$$E|\psi(X_1,X_2)|^p \leq 2^{p-1}\, E|h(X_1,X_2)|^p + 4\, E|g(X_1)|^p$$
$$\leq 2^{p-1}\, E|h(X_1,X_2)|^p + 4\, \{Eg^2(X_1)\}^{\frac{1}{2}p} \ ,$$

and since $a \geq 1$,

$$\text{(3.5)} \qquad \frac{E|\psi(X_1,X_2)|^p}{\{Eg^2(X_1)\}^{\frac{1}{2}p}} \leq 4\, a + 2^{p-1} b_p \leq 2(2a+b_p) \ .$$

$$\phi_N(t) = E\exp\{it\,\tau_N^{-1}\,U_N\}$$

$$= E\Big[\exp\{it\,\tau_N^{-1}(U_N-\Delta_N(m))\}(1+it\,\tau_N^{-1}\Delta_N(m))\Big] + R_N ,$$

$$|R_N| \le 2|t|^p\,\tau_N^{-p}\,E|\Delta_N(m)|^p \le 8(2a+b_p)mN^{-(3/2)p+1}|t|^p .$$

Let χ denote the characteristic function of $g(X_1)$, thus

$$\chi(t) = E\exp\{it\,g(X_1)\} .$$

Taking $m = (N-1)$ first and using (2.4) as well as the independence of $X_1, X_2, \ldots$, we see that

$$|E\,\tau_N^{-1}\,\Delta_N(N-1)\exp\{it\,\tau_N^{-1}(U_N-\Delta_N(N-1))\}|$$

$$= \Big|\tau_N^{-1}\sum_{1\le r<s\le N}\sum E\,\psi(X_r,X_s)\exp\{it\,\tau_N^{-1}(N-1)\sum_{k=1}^{N}g(X_k)\}\Big|$$

$$\le \frac{N(N-1)}{2\tau_N}|E\,\psi(X_1,X_2)\exp\{it\,\tau_N^{-1}(N-1)(g(X_1)+g(X_2))\}|\cdot|\chi(\tau_N^{-1}(N-1)t)|^{N-2}$$

$$= \frac{N(N-1)}{2\tau_N}|E\,\psi(X_1,X_2)\prod_{k=1}^{2}\Big[\exp\{it\,\tau_N^{-1}(N-1)g(X_k)\}-1\Big]|\cdot|\chi(\tau_N^{-1}(N-1)t)|^{N-2}$$

$$\le \frac{N(N-1)^3t^2}{2\tau_N^3}|\chi(\tau_N^{-1}(N-1)t)|^{N-2}\,E|\psi(X_1,X_2)g(X_1)g(X_2)|$$

$$\le \tfrac{1}{2}(5a+2b_p)N^{-\frac{1}{2}}\,t^2|\chi(\tau_N^{-1}(N-1)t)|^{N-2} ,$$

because of (2.6) and (3.3). Similarly, we find that for $m \le N-2$,

$$|E\,\tau_N^{-1}\,\Delta_N(m)\exp\{it\,\tau_N^{-1}(U_N-\Delta_N(m))\}|$$

$$= |\tau_N^{-1}\sum_{r=1}^{m}\sum_{s=r+1}^{N}E\,\psi(X_r,X_s)\exp\{it\,\tau_N^{-1}[(N-1)\sum_{k=1}^{N}g(X_k)$$

$$+ \sum_{m<k<\ell\le N}\sum\psi(X_k,X_\ell)]\}| \le \frac{m(N-1)}{\tau_N}E|\psi(X_1,X_2)||\chi(\tau_N^{-1}(N-1)t)|^{m-2}$$

$$\le 2(2a+b_p)m\,N^{-\frac{1}{2}}|\chi(\tau_N^{-1}(N-1)t)|^{m-2}$$

because of (2.6) and (3.2). Finally we note that for $|t| \le T_N = a^{-1}N^{\frac{1}{2}}$,

$$|\chi(\tau_N^{-1}(N-1)t) - 1 + \frac{t^2}{2N}| \le \frac{(N-1)^3|t|^3}{6\tau_N^3} E|g(X_1)|^3$$
$$\le \frac{1}{6} a|t|^3 N^{-3/2} \le \frac{t^2}{6N}$$

and hence

$$|\chi(\tau_N^{-1}(N-1)t)| \le 1 - \frac{t^2}{3N} \le \exp\{-\frac{t^2}{3N}\} .$$

Combining these computations for $m = (N-1)$ and $m \le (N-2)$ respectively we find that for $|t| \le T_N = a^{-1} N^{\frac{1}{2}}$,

$$\text{(3.6)} \quad \begin{aligned} &\phi_N(t) = [\chi(\tau_N^{-1}(N-1)t)]^N + R_N' , \\ &|R_N'| \le 8(2a+b_p)N^{-(3/2)p+2}|t|^p + \tfrac{1}{2}(5a+2b_p)N^{-\frac{1}{2}}|t|^3\exp\{-\frac{(N-2)t^2}{3N}\} \end{aligned}$$

and

$$\text{(3.7)} \quad \begin{aligned} |\phi_N(t)| \le\ & 8(2a+b_p)m\ N^{-(3/2)p+1}|t|^p + \exp\{-\frac{mt^2}{3N}\} \\ & + 2(2a+b_p)m\ N^{-\frac{1}{2}}|t|\exp\{-\frac{(m-2)t^2}{3N}\} \end{aligned}$$

for $m = 1,\ldots,N-2$.

From the proof of the classical Berry-Esseen theorem for sums of i.i.d. random variables we borrow the fact that

$$\text{(3.8)} \quad \int_{-T_N}^{T_N} |t|^{-1}|(\chi(\tau_N^{-1}(N-1)t))^N - e^{-\frac{1}{2}t^2}|dt \le c_1\ a\ N^{-\frac{1}{2}} ,$$

where we shall write $c_1, c_2, \ldots$ to denote universal constants.

Take $\varepsilon = (3p-5)/(2p)$ so that $0 < \varepsilon \le \frac{1}{4}$, and define $T_N' = \min(N^\varepsilon, T_N)$. It follows from (3.6) and (3.8) that

$$\text{(3.9)} \quad \int_{-T_N'}^{T_N'} |t|^{-1}|\phi_N(t) - e^{-\frac{1}{2}t^2}|dt \le c_2(a+b_p)N^{-\frac{1}{2}} .$$

For $T_N' \le |t| \le T_N$ we use (3.7) with $m = [3N\log N/t^2]$. Clearly there exists a natural number N_p depending only on p , such that for $N \ge N_p$ and

$T'_N \le |t| \le T_N$ we do indeed have $1 \le m \le N-2$ for this choice of m and such that (3.7) yields

$$(3.10) \qquad \int_{T'_N \le |t| \le T_N} |t|^{-1} |\phi_N(t) - e^{-\frac{1}{2}t^2}| dt \le c_3(a+b_p)N^{-\frac{1}{2}}$$

for every $N \ge N_p$. It follows that (3.10) will hold for all $N \ge 2$ if we allow c_3 to depend on p and together with (3.9) this implies

$$(3.11) \qquad \int_{|t| \le T_N} |t|^{-1} |\phi_N(t) - e^{-\frac{1}{2}t^2}| dt \le C'_p(a+b_p)N^{-\frac{1}{2}}$$

where C'_p depends only on p. Application of Esseen's smoothing lemma (cf. Feller (1971) p.538) completes the proof of the theorem.

4. EXTENSION AND APPLICATIONS

The gist of theorem 2.1 is that the U - statistic structure (2.4) - (2.5) permits us to ignore the second order term $\tau_N^{-1} \sum\sum \psi(X_i,X_j)$, even though it only possesses moments of a much lower order than the leading term $\tau_N^{-1}(N-1) \sum g(X_i)$. Many variations on this theme are possible and we shall briefly discuss one here. We shall show that if the functions g and ψ contain terms of lower order, then the contributions of such terms can be neglected under even weaker moment conditions than those imposed on g and ψ in theorem 2.1.

Let α, β, r and s be real numbers satisfying

$$(4.1) \qquad \alpha \ge \tfrac{1}{2}, \quad \beta \ge 0, \quad r > \frac{3}{1+2\alpha}, \quad s > \frac{5}{3+2\beta}.$$

Take h, g, ψ, U_N and τ_N as in section 2 and define (cf. (2.5))

$$(4.2) \qquad V_N = U_N + (N-1)N^{-\alpha} \sum_{i=1}^{N} \tilde{g}(X_i) + N^{-\beta} \sum_{1 \le i<j \le N} \sum \tilde{\psi}(X_i,X_j),$$

where $\tilde{\psi}$ is a symmetric function of its two variables.

We assume that

(4.3) $E|\tilde{g}(X_1)|^r < \infty$ and if $r \geq 1$, $E\,\tilde{g}(X_1) = 0$,

(4.4) $E|\tilde{\psi}(X_1,X_2)|^s < \infty$ and if $s \geq 1$, $E(\tilde{\psi}(X_1,X_2)|X_1) = 0$ a.s..

Thus V_N is another U - statistic which is obtained from U_N by replacing g and ψ by $g + N^{-\alpha}\tilde{g}$ and $\psi + N^{-\beta}\tilde{\psi}$ respectively. The assumptions $\alpha \geq \frac{1}{2}$ and $\beta \geq 0$ suggest that $\tau_N^{-1}(V_N - U_N)$ is stochastically of order $N^{-\frac{1}{2}}$, so that one may hope to obtain a Berry-Esseen theorem for $\tau_N^{-1} V_N$ under appropriate moment conditions. These conditions are spelled out in (4.3), (4.4) and the second part of (4.1) and we see that terms of small order do indeed require only mild moment assumptions.

THEOREM 4.1

Suppose that (4.1), (4.3), (4.4) *as well as the assumptions of theorem* 2.1 *are satisfied. Then there exists a positive number* C *depending only on* α, β, r, s *and* p *and such that for every* $N \geq 2$,

$$\sup_x |P(\tau_N^{-1} V_N \leq x) - \Phi(x)| \leq C\, N^{-\frac{1}{2}}\Big[\frac{E|g(X_1)|^3}{\{Eg^2(X_1)\}^{3/2}} + \tag{4.5}$$
$$+ \frac{E|\psi(X_1,X_2)|^p}{\{Eg^2(X_1)\}^{\frac{1}{2}p}} + \frac{E|\tilde{g}(X_1)|^r}{\{Eg^2(X_1)\}^{\frac{1}{2}r}} + \frac{E|\tilde{\psi}(X_1,X_2)|^s}{\{Eg^2(X_1)\}^{\frac{1}{2}s}}\Big] .$$

The proof of this theorem is very similar to that of theorem 2.1, though slightly more laborious. The random variable

$$\Delta_N^*(m) = \Delta_N(m) + (N-1)N^{-\alpha}\sum_{i=1}^{m+1}\tilde{g}(X_i) + N^{-\beta}\sum_{i=1}^{m}\sum_{j=i+1}^{N}\tilde{\psi}(X_i,X_j)$$

plays the part of $\Delta_N(m)$ in the earlier proof. When expanding the characteristic function we now use the inequality

$$|\exp\{i(x+y+z)\} - 1 - i(x+y+z)| \leq 4\,(|x|^p + |y|^r + |z|^s)$$

if $p, r, s \in [1,2]$; if r and/or s are in $(0,1)$, then the corresponding linear terms iy and/or iz should be omitted from the expansion of the exponential. Moments of sums of order smaller than 1 are bounded simply by using $|\sum a_i|^s \le \sum |a_i|^s$ for $0 < s < 1$. Apart from these rather obvious changes the proof contains no new elements and will therefore be omitted.

We conclude this section with two applications. The first of these concerns the Von Mises functional

$$
\begin{aligned}
M_N &= \sum_{i=1}^{N} \sum_{j=1}^{N} h(X_i,X_j) = 2 \sum_{1\le i<j\le N} \sum h(X_i,X_j) + \sum_{i=1}^{N} h(X_i,X_i) \\
&= (N-1) \sum_{i=1}^{N} \{2g(X_i) + (N-1)^{-1} h(X_i,X_i)\} + 2 \sum_{1\le i<j\le N} \sum \psi(X_i,X_j) ,
\end{aligned} \tag{4.6}
$$

with h , g and ψ as in section 2. The statistic $\frac{1}{2} M_N$ is of the form (4.2) with $\alpha = 1$, $\tilde{g}(x) = \frac{1}{2} N(N-1)^{-1} h(x,x)$ and $\tilde{\psi} = 0$. It follows that if for some $p > 5/3$ and $r > 1$, (2.1) and (2.3) hold and

$$
E|h(X_1,X_1)|^r < \infty , \tag{4.7}
$$

then we have the Berry-Esseen bound

$$
\sup_x |P(\tfrac{1}{2}\tau_N^{-1} M_N \le x) - \Phi(x)| = O(N^{-\frac{1}{2}}) , \tag{4.8}
$$

where τ_N^2 is given by (2.6). Note that we have omitted the assumption $E\, h(X_1,X_1) = 0$ even though $r > 1$ (cf. (4.3)), because $\tau_N^{-1}\, N\, E\, h(X_1,X_1) = O(N^{-\frac{1}{2}})$ in any case and a non-random contribution of this order leaves (4.8) unaffected. The bound (4.8) was obtained earlier by Boos and Serfling who assume finite moments of orders $p = 3$ and $r = 3/2$ for $|h(X_1,X_2)|$ and $|h(X_1,X_1)|$ respectively (cf. Serfling (1980), p.230).

Berry-Esseen bounds for U - statistics have been used by a number of authors to obtain similar bounds for various Studentized statistics (e.g. Helmers (1978) for Studentized linear combinations of order statistics and Callaert and Veraverbeke (1981) for Studentized U - statistics).

Since the main argument is the same in each case, we may as well apply our results to the simplest of these statistics, which is of course Student's t. Let $X_1, X_2, \ldots$ be i.i.d. random variables with

$$(4.9) \qquad E\,X_1 = 0\ , \quad E\,X_1^2 = 1\ , \quad E|X_1|^r < \infty$$

for some $r > 4$. Define $\bar{X} = N^{-1} \sum_{i=1}^{N} X_i$ and

$$(4.10) \qquad T_N = \left(\frac{N-1}{N}\right)^{\frac{1}{2}} \frac{\sum_{i=1}^{N} X_i}{\{\sum_{i=1}^{N} (X_i - \bar{X})^2\}^{\frac{1}{2}}} .$$

Then

$$(4.11) \qquad \sup_x |P(T_N \le x) - \Phi(x)| = \mathcal{O}(N^{-\frac{1}{2}}) .$$

A proof of this result starts by noting that $\{T_N \le x\} = \{S_N \le x_N\}$, where

$$S_N = \frac{\sum_{i=1}^{N} X_i}{\{\sum_{i=1}^{N} X_i^2\}^{\frac{1}{2}}}\ , \quad x_N = \left(\frac{N}{N-1+x^2}\right)^{\frac{1}{2}} x .$$

It follows that it suffices to prove (4.11) with T_N replaced by S_N.

By Markov's inequality and that of Marcinkiewicz, Zygmund and Chung,

$$P\left(\frac{|\sum(X_i^2-1)|}{N} \ge \frac{N^{-\frac{1}{4}}}{\log N}\right) \le \left(\frac{\log N}{N^{\frac{3}{4}}}\right)^{r/2} E|\sum(X_i^2-1)|^{r/2}$$

$$= \mathcal{O}((N^{-\frac{1}{4}}\log N)^{r/2}) = \mathcal{O}(N^{-\frac{1}{2}})$$

and hence

$$(4.12) \qquad \left(\frac{N}{\sum X_i^2}\right)^{\frac{1}{2}} = \left(1 + \frac{\sum(X_i^2-1)}{N}\right)^{-\frac{1}{2}} = 1 - \frac{1}{2N}\sum(X_i^2-1) + \mathcal{O}\left(\frac{N^{-\frac{1}{2}}}{(\log N)^2}\right)$$

uniformly on a set of probability $1 - O(N^{-\frac{1}{2}})$. In view of the Berry-Esseen theorem for $N^{-\frac{1}{2}} \sum X_i$ we have

$$P(|N^{-\frac{1}{2}}\sum X_i| \geq \log N) = O(N^{-\frac{1}{2}})$$

and as a result

$$S_N = N^{-\frac{1}{2}} \sum X_i(1 - \frac{1}{2N} \sum (X_i^2-1)) + O(N^{-\frac{1}{2}})$$

uniformly on a set of probability $1 - O(N^{-\frac{1}{2}})$. It is therefore sufficient to prove (4.11) with T_N replaced by

$$N^{-\frac{1}{2}} \sum X_i(1 - \frac{1}{2N} \sum (X_i^2-1)) + \tfrac{1}{2}N^{-\frac{1}{2}}E\, X_1^3$$

$$= N^{-\frac{1}{2}} \sum\{X_i - \frac{1}{2N}(X_i^3-X_i-EX_1^3)\} - \tfrac{1}{2}N^{-3/2}\sum_{i\neq j}\sum X_i(X_j^2-1) \ .$$

But this is of the form $\tau_N^{-1} V_N$ studied in theorem 4.1 with $g(x) = x$, $\psi(x,y) = -\frac{1}{2}(N-1)N^{-1}\{x(y^2-1) + y(x^2-1)\}$, $\alpha = 1$, $\tilde{g}(x) = -\frac{1}{2}(x^3-x-EX_1^3)$ and $\tilde{\psi} = 0$. The conditions of theorem 4.1 reduce to (4.9) for some $r > 10/3$. As we have $r > 4$ the theorem may be applied and the proof is complete.

This Berry-Esseen theorem for Student's t provides us with a convenient yardstick to measure the strength of our results. If one recalls that(4.9) for $r = 3$ is needed to establish the classical Berry-Esseen bound for $N^{-\frac{1}{2}} \sum X_i$, then one must admit that the present condition (4.9) for some $r > 4$ is still considerably worse. It is small consolation that the full force of this condition is needed only to bound the remainder term in (4.12); elsewhere any $r > 10/3$ would have been sufficient. However, if one compares with what could previously be obtained, one sees that some progress has certainly been made. A similar analysis based on the best previous result of Callaert and Janssen (1978) would require (4.9) for $r = 6$. If one would use only theorem 2.1 or corollary 2.1 of the present paper, one would need (4.9) for $r = 9/2$ (cf. Helmers (1978) and Callaert and Veraverbeke (1981)).

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