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BAYES TESTS FOR LOCALLY ASYMPTOTICALLY NORMAL FAMILIES

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Bayes tests for locally asymptotically normal families *)

by

Donald R. Truax**)

ABSTRACT

This paper generalizes results on the asymptotic behavior of Bayes tests, developed by Johnson and Truax for exponential families, to families of distributions satisfying a sufficiently strong local asymptotic normality condition. The present paper considers only the case of a single parameter and a simple, zero-one, loss function and obtains an approximate form for the Bayes acceptance region in terms of a local sufficient statistic, as well as the asymptotic form of the Bayes risk. Of special interest is the dependence of the risk on the prior distribution.

KEY WORDS & PHRASES: *asymptotic theory, Bayes test, Bayes risk*

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1. INTRODUCTION AND SUMMARY

In a previous paper [3] Johnson and Truax have studied the large sample behaviour of the risk of Bayes tests when the underlying distribution belonged to a multivariate exponential family. The purpose of the present paper is to show that the same results hold if the distributions merely satisfy a sufficiently strong "local asymptotic normality" condition. This condition is satisfied by any exponential family and in a large number of other cases, and is discussed at the end of Section 2.

We suppose that $X_1, X_2 \ldots$ are independent and identically distributed random variables having a common probability density $f(x;\theta)$ with respect to some σ-finite measure μ. The density is assumed to depend on a real valued parameter θ and $\{x : f(x;\theta) > 0\}$ does not depend on θ. The log likelihood function will be denoted by $\ell(x;\theta) = \log f(x;\theta)$. Through this paper we assume that $\ell(x;\theta)$ is a strictly concave function of θ for each x. The derivative of ℓ with respect to θ will be denoted as $\ell'(x;\theta)$, and if this derivative is evaluated at $\theta = 0$ we use the abbreviated notation $\ell'(x)$.

It will be necessary to introduce some notation for the discussion that follows. We will approximate the log likelihood ratio with the log likelihood ratio of a normal family. Let $\underset{\sim}{x} = (x_1, x_2, \ldots)$ and

(1) $$h_n(\underset{\sim}{x};\theta) = \frac{1}{n}\sum_{i=1}^{n} \ell'(x_i)\,\theta - \tfrac{1}{2} J\,\theta^2$$

where

$$J = E_0\{(\ell'(x))^2\} \ .$$

Throughout the paper J is assumed finite and positive. The error of approximation is

(2) $$T_n(\underset{\sim}{x};\theta) = \sum_{i=1}^{n} \{\ell(x_i;\theta) - \ell(x_i) - h_n(x_i;\theta)\}.$$

Define, for each $\varepsilon > 0$

(3) $$B_n(\varepsilon) = \{\underset{\sim}{x} : \sup_{|\theta| \le \frac{\log n}{\sqrt{n}}} |T_n(\underset{\sim}{x};\theta)| \le \varepsilon\}.$$

Our local asymptotic normality property will be expressed in terms of the rate of convergence to zero of $P_0(B_n'(\varepsilon))$, where P_0 is the distribution of $\underset{\sim}{X}$ when $\theta = 0$.

The statistical problem we will be concerned with is the testing of a simple hypothesis against unrestricted alternatives. Without loss of generality we can express the null hypothesis as $\theta = 0$, and the alternative as $\theta \neq 0$. Suppose there is a prior distribution which assigns positive probability γ to the null hypothesis and distributes the remaining probability according to a continuous density $g(\theta)$ $(\int g(\theta) d\theta = 1-\gamma)$. The Bayes test based on $X_1, X_2, \ldots, X_n$ relative to this prior distribution is easily seen to have the acceptance region

(4) $$D_n = \{\underset{\sim}{x} : \int e^{\Sigma_{i=1}^{n}[\ell(x_i,\theta) - \ell(\underset{\sim}{x}_i)]} g(\theta)\, d\theta \le \gamma\}$$

where, for simplicity, we write $\ell(\underset{\sim}{x}_i)$ for $\ell(\underset{\sim}{x}_i; 0)$.

The exact characterization of the acceptance region is complicated by the fact that there is no sufficient statistic as was the case in the exponential family setting. However, in Section 2 we will show that under our local asymptotic normality condition the set D_n can be approximated, in a certain sense, by simpler regions depending on the local sufficient statistic,

(5) $$D_n^{\pm}(\varepsilon) = \left\{\underset{\sim}{x} : \left(\frac{1}{\sqrt{n(\log n)J}} \sum_{i=1}^{n} \ell'(\underset{\sim}{x}_i)\right)^2 \le p + 1 + \frac{c \pm \varepsilon}{\log n} - p\,\frac{\log \log n}{\log n}\right\}$$

where

(6) $$e^{\frac{1}{2}c} = \frac{\gamma J^{(p+1)/2}}{\sqrt{2\pi}\, g_0 (p+1)^{p/2}} .$$

The constant $g_0 > 0$ and $p > 0$ are related to the prior density g by the assumption

(7) $$g(\theta) = g_0 |\theta|^p + o(|\theta|^p) \quad \text{as } |\theta| \to 0 .$$

The risk function for the Bayes procedure is split into two parts. The *type I risk* is the expected loss (here, we consider only the simple zero-one loss function) when H_0 is true

$$\gamma P_0(\underset{\sim}{X} \notin D_n) \ ,$$

and the *type II risk*, which is the expected loss when H_0 fails

$$\int P_0(\underset{\sim}{x} \in D_n) \ g(\theta) d\theta \ .$$

In Section 2 we compute each of these and show that the Bayes risk is asymptotically the type II risk and behaves like

$$C_p \left(\frac{\log n}{n} \right)^{(p+1)/2}$$

where C_p is a constant

These results generalize the previous results of JOHNSON and TRUAX [3] in the case of a single parameter. Analogous generalizations can also be made if the parameter is vector valued. See RUBIN and SETHURAMAN [4] for a somewhat different approach.

The form of the approximate regions (5) suggests that we might term tests with acceptance regions of the form

$$\{\underset{\sim}{x} : \left(\frac{1}{\sqrt{n(\log n) J}} \sum_{i=1}^{n} \ell'(x_i) \right)^2 \leq c^2 \}$$

"almost Bayes" tests. As in [3], some rather surprising results are obtained when we compare the risks of such tests with the risk of the optimal Bayes test for various values of the constant c.

Finally, in section 3, we discuss the sometimes disasterous consequences of wrongly guessing the prior distribution. The behaviour of the risk depends strongly on the behaviour of the prior density g near zero. Referring to the relationship (7), the rate p at which the prior tends to zero when θ tends to zero, is very important.

2. THE MAIN RESULTS

The principal results of this section will be the approximation of the Bayes acceptance region by simpler regions and the asymptotic behaviour of the Bayes risk. The dependency of the risk on the prior distribution will be discussed. In order to prove the main theorems of this section a number of technical lemmas will be required, and these have been placed in the appendix.

THEOREM 1. *Given any* $\varepsilon > 0$, *we have for all sufficiently large* n

$$B_n(\varepsilon/4) \cap D_n^-(\varepsilon) \subset B_n(\varepsilon/4) \cap D_n \subset B_n(\varepsilon/4) \cap D_n^+(\varepsilon).$$

By invoking a local asymptotic normality condition expressed in terms of the rate at which $P_0(B_n'(\varepsilon))$ tends to zero, and with a further condition on the distribution of $\ell'(X)$ one gets the asymptotic type I risk.

THEOREM 2. *If* $E_0(e^{t\ell'(X)}) < \infty$ *for all* t *in an open neighborhood of* 0, *and if* $P_0(B_n'(\varepsilon)) = o(n^{-q})$ *for all* $\varepsilon > 0$, where $q > \frac{p+1}{2}$, then

$$\gamma P_0(\underset{\sim}{X} \notin D_n) \sim C_p \frac{(\log n)^{(p-1)/2}}{N^{(p+1)/2}}$$

where

$$C_p = 2g_0(p+1)^{(p-1)/2} J^{-(p+1)/2} .$$

Finally, under our local asymptotic normality condition we get the type II risk.

THEOREM 3. *Under the hypothesis of Theorem* 2

$$\int P_\theta(\underset{\sim}{X} \in D_n)\, g(\theta)\, d\theta \sim C_p\left(\frac{\log n}{n}\right)^{(p+1)/2} .$$

COROLLARY. *Under the hypothesis of Theorem* 2 *the Bayes risk satisfies*

$$R_n = \gamma P_0(\underset{\sim}{X} \notin D_n) + \int P_\theta(\underset{\sim}{X} \in D_n)\, g(\theta)\, d\theta \sim C_p\left(\frac{\log n}{n}\right)^{(p+1)/2} .$$

PROOF OF THEOREM 1. We will first show that $\underset{\sim}{x} \in B_n(\varepsilon/4) \cap D_n$ implies $\underset{\sim}{x} \in D_n^+(\varepsilon)$ if n is sufficiently large. The proof will be by contradiction. Suppose that $\underset{\sim}{x}^{(n)} \in B_n(\varepsilon/4) \cap D_n$ for all n, but for infinitely many n, $\underset{\sim}{x}^{(n)} \notin D_n^+(\varepsilon)$. We then have

$$\gamma \geq \int \exp\{\sum_{i=1}^{n} [\ell(x_i^{(n)};\theta) - \ell(x_i^{(n)})]\}\, g(\theta)\, d\theta$$

$$\geq \int_{|\theta| \leq (\log n)/\sqrt{n}} \exp\{\sum_{i=1}^{n} [\ell(x_i^{(n)};\theta) - \ell(x_i^{(n)})]\}\, g(\theta)\, d\theta .$$

Since $\underset{\sim}{x}^{(n)} \in B_n(\varepsilon/4)$, we have by (3)

$$\gamma \geq e^{-\varepsilon/4} \int_{|\theta| \leq (\log n)/\sqrt{n}} \exp\{\theta\sqrt{Jn\log n}\, v_n - \frac{n}{2}\theta^2 J\}\, g(\theta)\, d\theta$$

where

$$v_n = \frac{1}{\sqrt{n(\log n)J}} \sum_{i=1}^{n} \ell(x_i^{(n)}) .$$

Given $\delta > 0$, we have for sufficiently large n

$$\gamma > \frac{e^{-\varepsilon/4}(1-\delta)g_0 e^{\frac{1}{2}v_n^2(\log n)}}{(Jn)^{(p+1)/2}} \int_{|\theta| \leq \sqrt{J}\log n} e^{-\frac{1}{2}(\theta - v_n\sqrt{\log n})^2} |\theta|^p\, d\theta$$

$$(8) \quad = \frac{e^{-\varepsilon/4}(1-\delta)g_0 e^{\frac{1}{2}v_n^2(\log n)}}{(Jn)^{(p+1)/2}} \int_{|\theta + v_n\sqrt{\log n}| \leq \sqrt{J}\log n} e^{-\frac{1}{2}\theta^2} |\theta + v_n\sqrt{\log n}|^p\, d\theta$$

because of (7)

Since v_n is bounded by Lemma A-2, the region of integration in (8) converges to the whole real line, so for sufficiently large n

(9) $$\gamma > \frac{e^{-\epsilon/4}(1-\delta)^2 g_0 \, e^{\frac{1}{2} v_n^2 (\log n)}}{(Jn)^{(p+1)/2}} |v_n \sqrt{\log n}|^p \sqrt{2\pi}.$$

If $\underset{\sim}{x}^{(n)} \notin D_n^+(\varepsilon)$ we have

$$v_n^2 > (p+1) + \frac{c+\varepsilon}{\log n} - p \frac{\log \log n}{\log n}$$

so

(10) $$| v_n |^p > (1-\delta)(p+1)^{p/2}$$

for n sufficiently large, and from (9) and (10) we have for such n

$$\gamma > \frac{e^{-\varepsilon/4}(1-\delta)^3 g_0 (p+1)^{p/2}}{(Jn)^{(p+1)/2}} e^{\frac{1}{2}v_n^2(\log n)} (\log n)^{p/2} \sqrt{2\pi}$$

$$> \frac{e^{-\varepsilon/4}(1-\delta)^3 g_0 (p+1)^{p/2} \sqrt{2\pi}}{(Jn)^{(p+1)/2}} e^{\frac{1}{2}(p+1)\log n + \frac{c+\varepsilon}{2}}$$

$$= e^{-\varepsilon/4}(1-\delta)^3 g_0 (p+1)^{p/2} \sqrt{2\pi}\, e^{(c+\varepsilon)/2} = e^{\varepsilon/4}(1-\delta)^3 \gamma .$$

This gives a contradiction if we choose δ so small that $e^{\varepsilon/4}(1-\delta)^3 > 1$.

For the second part of the Theorem we show that if n is sufficiently large $\underset{\sim}{x} \in B_n(\varepsilon/4) \cap D_n^-(\varepsilon)$ implies

(11) $$\int \exp \{ \sum_{i=1}^{n} [\ell(x_i;\theta) - \ell(x_i)]\} g(\theta) d\theta < \gamma.$$

Write the integral in (11) as the sum

$$\int_{|\theta| \le (\log n)/\sqrt{n}} + \int_{|\theta| > (\log n)/\sqrt{n}} \le e^{-\varepsilon/8}\gamma + Ke^{-d(\log n)^2}$$

for some positive constants K and d by Lemmas A-5 and A-6. Then, if n is large enough $Ke^{-d(\log n)^2} < \gamma(1-e^{-\varepsilon/8})$.

PROOF OF THEOREM 2. The existence of the moment generating function of $\ell'(X)$ in a neighborhood of zero is well known (see e.g.[2;p.549]) to imply that

$$P_0\left(\frac{1}{\sqrt{nJ}}\sum_{i=1}^{n}\ell'(X_i)>a_n\right)\sim 1-\Phi(a_n)$$

where $a_n = O(\sqrt{\log n})$ and Φ is the standard normal distribution function.

Our first step will be to compute the "approximate" type I risk $\gamma P_0(\underset{\sim}{X}\notin D_n^+(t))$.

Let $$b_n = \sqrt{(\log n)(p+1+\frac{c+\varepsilon}{\log n} - p\,\frac{\log\log n}{\log n})}\,.$$

$$\gamma P_0(\underset{\sim}{X}\notin D_n^+(\varepsilon)) = \gamma P_0\left(\frac{1}{\sqrt{nJ}}\sum_{i=1}^{n}\ell'(X_i)>b_n\right)+\gamma P_0\left(\frac{1}{\sqrt{nJ}}\sum_{i=1}^{n}\ell'(X_i)<-b_n\right)$$

$$\sim \gamma(1-\Phi(b_n))+\gamma\Phi(-b_n)\sim \frac{2\gamma e^{-\frac{1}{2}b_n^2}}{\sqrt{2\pi}\;b_n}$$

$$\sim \frac{2\gamma e^{-\varepsilon/2}\,n^{-(p+1)/2}(\log n)^{(p-1)/2}}{\sqrt{2\pi}\;\sqrt{p+1}}\,e^{-c/2} = C_p\,\frac{(\log n)^{(p-1)/2}}{n^{(p+1)/2}}\,e^{-\varepsilon/2}\,.$$

In exactly the same way

$$\gamma P_0(\underset{\sim}{X}\notin D_n^-(\varepsilon))\sim C_p\,\frac{(\log n)^{(p-1)/2}}{n^{(p+1)/2}}\,e^{\varepsilon/2}\,.$$

Now, we can write the type I risk as

$$\gamma P_0(\underset{\sim}{X}\in D_n) = \gamma\,(1-P_0(\underset{\sim}{X}\in D_n\cap B_n(\varepsilon))) - \gamma P_0(\underset{\sim}{X}\in D_n\cap B_n'(\varepsilon)). \tag{12}$$

The last term is $o(n^{-q})$. We will show that $\gamma(1-P_0(\underset{\sim}{X}\in D_n\cap B_n(\varepsilon)))$ is the dominant term. By Theorem 1,

$$\gamma P_0(\underset{\sim}{X}\notin D_n\cap B_n(\varepsilon)) \geq \gamma P_0(\underset{\sim}{X}\notin D_n^+(4\varepsilon)\cap B_n(\varepsilon))$$

$$= \gamma P_0(\underset{\sim}{X}\notin D_n^+(4\varepsilon)) + o(n^{-q})$$

$$= C_p\,\frac{(\log n)^{(p-1)/2}\,e^{-2\varepsilon}}{n^{(p+1)/2}}\,(1+o(1)).$$

Similarly,

$$P_0(\underset{\sim}{X} \notin D \cap B_n(\varepsilon)) \le P_0(\underset{\sim}{X} \in D_n^-(4\varepsilon) \cap B_n(\varepsilon))$$

and this implies

$$\gamma P_0(\underset{\sim}{X} \notin D_n \cap B_n(\varepsilon)) \le C_p \frac{(\log n)^{(p-1)/2}}{n^{(p+1)/2}} e^{2\varepsilon} \quad (1+o(1)).$$

Thus, from (12), for any $\epsilon > 0$

$$C_p \frac{(\log n)^{(p-1)/2}}{n^{(p+1)/2}} e^{-2\varepsilon}(1+o(1)) \le \gamma P_0(\underset{\sim}{X} \notin D_n)$$

$$\le C_p \frac{(\log n)^{(p-1)/2}}{n^{(p+1)/2}} e^{2\varepsilon} \quad (1+o(1))$$

and since ε is arbitrary, Theorem 2 follows.

Before proving Theorem 3, it will be helpful to prove three preliminary Lemmas. In order to simplify writing, we denote the type II risk by

$$R_{2,n} = \int P_\theta(\underset{\sim}{X} \in D_n) g(\theta) d\theta \ .$$

<u>LEMMA 1.</u> *If* $P_0(B_n'(\varepsilon)) = o(n^{-q})$, *then*

$$R_{2,n} = \int P_0(\underset{\sim}{X} \in D_n \cap B_n(\varepsilon)) g(\theta) d\theta + o(n^{-q}) \ .$$

<u>PROOF</u>.

$$\int P_0(\underset{\sim}{X} \in D_n \cap B_n'(\varepsilon)) g(\theta) d\theta = \int \int_{D_n \cap B_n'(\varepsilon)} \exp\{\sum_{i=1}^n [\ell(x_i;\theta) - \ell(x_i)]\} dP_0^{(n)}(\underset{\sim}{x}) g(\theta)\, d\theta$$

where $P_0^{(n)}$ denotes the distribution of $X_1, X_2, \ldots, X_n$ when $\theta = 0$. Interchanging the order of integration we can write the integral as

$$\int_{D_n \cap B_n'(\varepsilon)} \int \exp\{\sum_{i=1}^n [\ell(x_i;\theta) - \ell(x_i)]\} g(\theta) d\theta \ dP_0^{(n)}(\underset{\sim}{x})$$

$$\le \gamma P_0(\underset{\sim}{X} \in B_n'(\varepsilon)) = o(n^{-q})$$

LEMMA 2. *If* $P_0(B_n'(\varepsilon)) = o(n^{-q})$, *then*

$$R_{2,n} = \int_{D_n \cap B_n(\varepsilon)} \int_{|\theta| \le (\log n)/\sqrt{n}} \exp\{\sum_{i=1}^{n} [\ell(x_i;\theta) - \ell(x_i)]\} g(\theta) d\theta \, dP_0^{(n)}(\underset{\sim}{x}) + o(n^{-q}).$$

PROOF. The proof is immediate from Lemmas 1 and A-7.

LEMMA 3. *Given* $\epsilon > 0$, *if* $P_0(B_n'(\varepsilon)) = o(n^{-q})$, *then for all sufficiently large* n

$$e^{-3\varepsilon} I_n^-(\varepsilon) + o(n^{-q}) \le R_{2,n} \le e^{3\varepsilon} I_n^+(\varepsilon) + o(n^{-q})$$

where

$$(13) \quad I_n^{\pm}(\varepsilon) = \frac{g_0}{(Jn)^{(p+1)/2}} \int_{D_n^{\pm}(4\varepsilon)} e^{\frac{1}{2} v_n^2 (\log n)} \int e^{-\frac{1}{2}\theta^2} |\theta + \sqrt{\log n}\, v_n|^p d\theta \, dP_0^{(n)}(\underset{\sim}{x})$$

and

$$v_n = \frac{1}{\sqrt{n(\log n)J}} \sum_{i=1}^{n} \ell'(x_i) .$$

PROOF. From Lemma 2, and the relations (3) and (7), we have for sufficiently large n

$$R_{2,n} = \int_{D_n \cap B_n(\varepsilon)} \int_{|\theta| \le (\log n)\sqrt{n}} \exp\{\sum_{i=1}^{n} [\ell(x_i;\theta) - \ell(x_i)]\} g(\theta) d\theta dP_0^{(n)}(\underset{\sim}{x}) + o(n^{-q})$$

$$\le e^{\varepsilon} \int_{D_n \cap B_n(\varepsilon)} \int_{|\theta| \le (\log n)/\sqrt{n}} e^{\theta\sqrt{n \log n} - \frac{n}{2} J \theta^2} g(\theta) d\theta \, dP_0^{(n)}(\underset{\sim}{x}) + o(n^{-q})$$

$$\le \frac{e^{2\varepsilon} \quad g_0}{(Jn)^{(p+1)/2}} \int_{D_n \cap B_n(\varepsilon)} \int_{|\theta| \le \sqrt{J} \log n} |\theta|^p e^{\theta\sqrt{\log n}\, v_n - \frac{1}{2}\theta^2} d\theta \, dP^{(n)}(\underset{\sim}{x}) + o(n^{-q}).$$

Now, $D_n \cap B_n(\varepsilon) \subset D_n^+(4\varepsilon)$ if n is large, and upon completing the square in the exponent of the above integral

$$R_{2,n} \le \frac{e^{2\varepsilon} g_0}{(Jn)^{(p+1)/2}} \int_{D_n^+(4\varepsilon)} e^{\frac{1}{2}v_n^2(\log n)}$$

$$\int_{|\theta+\sqrt{\log n}\, v_n| \le \sqrt{J} \log n} |\theta + \sqrt{\log n}\, v_n|^p e^{-\frac{1}{2}\theta^2} d\theta \, dP_0^{(n)}(\underset{\sim}{x}) + o(n^{-q}) .$$

If $\underset{\sim}{x} \in D_n^+ (4\varepsilon)$, then v_n is bounded so the inner integral above is asymptotically equivalent to the same expression where the region of integration of the inner integral is the whole real line. Thus, if n is sufficiently large

$$R_{2,n} \le e^{3\varepsilon} I_n^+(\varepsilon),$$

giving the required upper bound. For the lower bound, the same kind of arguments give

$$R_{2,n} \ge \frac{e^{-3\varepsilon} g_0}{(Jn)^{(p+1)/2}}$$

$$\int_{D_n^-(4\varepsilon)\cap B_n(\varepsilon)} e^{\frac{1}{2}v_n^2(\log n)} \int | \theta + \sqrt{\log n}\, v_n |^p e^{-\frac{1}{2}\theta^2} d\theta \, dP_0^{(n)}(\underset{\sim}{x}) + o(n^{-q}) .$$

Lemma 3 follows if we can show

$$(14) \qquad \frac{1}{(Jn)^{(p+1)/2}} \int_{D_n^-(4\varepsilon)\cap B_n'(\varepsilon)} e^{\frac{1}{2}v_n^2(\log n)}$$

$$\int | \theta + \sqrt{\log n}\, v_n |^p e^{-\frac{1}{2}\theta^2} d\theta \, dP_0^{(n)}(\underset{\sim}{x}) = o(n^{-q}).$$

If $\underset{\sim}{x} \in D_n^-(4\varepsilon)$,

$$\tfrac{1}{2}v_n^2(\log n) \le \tfrac{1}{2}(\log n)(p+1) + \frac{c-4\varepsilon}{2} - \frac{p}{2} \log \log n.$$

The left hand side of (14) is then less than or equal

$$\frac{e^{\frac{1}{2}(c-4\varepsilon)}}{J^{(p+1)/2}(\log n)^{p/2}} \int_{D_n^-(4\varepsilon)\cap B_n'(\varepsilon)}$$

$$\int |\theta + \sqrt{\log n}\, v_n|^p e^{-\frac{1}{2}\theta^2} d\theta\, dP_0^{(n)}(\underset{\sim}{x}) = O(P_0(B_n'(\varepsilon))) = o(n^{-q}) .$$

<u>PROOF OF THEOREM 3</u>. The proof will follow from Lemma 3 if we can show

$$I_n^{\pm}(\varepsilon) \sim C_p\left(\frac{\log n}{n}\right)^{(p+1)/2}$$

for each $\varepsilon > 0$. $I_n^{\pm}(\varepsilon)$ is given by (13). It will be enough to consider $I_n^{+}(\varepsilon)$. Let $P_{0,n}$ be the distribution of the standardized statistic

$$\frac{1}{\sqrt{nJ}} \sum_{i=1}^{n} \ell'(X_i)$$

when $\theta = 0$, and let

$$b_n = \sqrt{(\log n)(p+1) + (c+4\varepsilon) - p \log\log n}.$$

We can then write

$$I_n^{+}(\varepsilon) = \frac{g_0}{(Jn)^{(p+1)/2}} \int_{|u|\le b_n} e^{\frac{1}{2}u^2} \int |\theta + u|^p e^{-\frac{1}{2}\theta^2} d\theta\, dP_{0,n}(u) .$$

By the same argument as in Johnson and Truax [3], one can make use of asymptotic expansion Theorems for $P_{0,n}$ to show

(15) $$I_n^{+}(\varepsilon) \sim \frac{g_0}{(Jn)^{(p+1)/2}} \int_{|u|\le b_n} e^{\frac{1}{2}u^2} \int |\theta+u|^p e^{-\frac{1}{2}\theta^2} d\theta\, d\Phi(u)$$

where Φ is the standard normal distribution.
To evaluate the right hand side of (15), we first let $a_n = (\log n)^{1/4}$ and notice

$$\int |\theta + u|^p e^{-\frac{1}{2}\theta^2} d\theta \sim |u|^p \sqrt{2\pi}$$

uniformly for $|u| \geq a_n$. Secondly,

$$\int_{|u|<a_n} \int |\theta+u|^p e^{-\frac{1}{2}\theta^2} d\theta\, du = a_n \int_{|u|\leq 1} \int |\theta + a_n u|^p e^{-\frac{1}{2}\theta^2} d\theta\, du$$

$$= O(a_n^{p+1}) = o((\log n)^{(p+1)/2}) .$$

Finally,

$$\int_{a_n<|u|\leq b_n} \int |\theta + u|^p e^{-\frac{1}{2}\theta^2} d\theta\, du \sim \sqrt{2\pi} \int_{a_n<|u|\leq b_n} |u|^p du$$

$$= \sqrt{2\pi}(\log n)^{(p+1)/2} \int_{\frac{a_n}{\sqrt{\log n}} < |u| \leq \frac{b_n}{\sqrt{\log n}}} |u|^p du$$

$$\sim \sqrt{2\pi}(\log n)^{(p+1)/2} \int_{|u|\leq\sqrt{p+1}} |u|^p du$$

$$= \sqrt{2\pi}(\log n)^{(p+1)/2}\, 2(p+1)^{(p-1)/2} .$$

This gives

$$I_n^+(\varepsilon) \sim \frac{2g_0(p+1)^{(p-1)/2}}{(Jn)^{(p+1)/2}} (\log n)^{(p+1)/2} = C_p\left(\frac{\log n}{n}\right)^{(p+1)/2} .$$

The corresponding calculation for $I_n^-(\varepsilon)$ is completely analogous.

REMARKS. The local asymptotic normality condition $P_0(B_n'(\varepsilon)) = o(n^{-q})$ for some $q > \frac{(p+1)}{2}$ is alsways satisfied when the underlying distribution belongs to an exponential family since it is easy to show that the set $B_n'(\varepsilon)$ is empty if n is sufficiently large. It also holds in a number of other situations. For example, it can easily be checked for any smooth curved exponential family (for any $q > 0$). If one assumes Cramér's regularity conditions [1; page 500] that ℓ'', ℓ''' also exist for all θ in some interval about 0 and on this interval $|\ell'''(x;\theta)| \leq H(x)$, then a sufficient condition for $P_0(B_n'(\varepsilon) = o(n^{-q})$ for all $\varepsilon > 0$ is that $\ell''(X)$ has a moment generating function in a neighborhood of zero, and H has sufficiently high moments.

The condition that $\ell'(x)$ (or even $\ell''(x)$) have a moment generating function is satisfied in most cases of practical interest. For example, any exponential family, or any curved exponential family satisfies it. If $f(x;\theta) = p(x-\theta)$ where $p(x) > 0$ on R and $p(x)$ is rational or $p(x) = e^{-Q(x)}$ where Q is a polynomial, the condition is also satisfied.

3. ALMOST BAYES TESTS

We will say that a test for $\theta = 0$ is almost Bayes if it has an acceptance region of the form

$$\left| \frac{\sum_{i=1}^{n} \ell'(X_i)}{\sqrt{n(\log n)J}} \right| \leq c \ .$$

Under our assumptions it is easy to compute both the type 1 and type 2 risk functions (as in [3]). The type 1 risk becomes

$$\gamma P_0\left(\left| \sum_{i=1}^{n} \ell'(X_i) \right| > c\sqrt{n(\log n)J}\right) \sim \frac{2\gamma}{\sqrt{2\pi}\, c\sqrt{\log n}\; n^{c^2/2}} \ .$$

Also, the type 2 risk can be shown to be

$$\left(\frac{(\log n)}{n}\right)^{(p+1)/2} \frac{2c^{p+1}}{p+1} \ .$$

Notice that if $c^2 < 1+p$, the type 1 risk is dominant, and the risk of the almost Bayes procedure is much worse than that of the Bayes procedure. For example, if one used a Bayes procedure based on an assumed prior $\tilde{g}(\theta) = \tilde{g}_0 |\theta|^{\tilde{p}} + o(|\theta|^{\tilde{p}})$ where $\tilde{p} < p$ when the actual prior was $g(\theta) = g_0 |\theta|^p + o(|\theta|^p$, the Bayes risk is easily seen to be smaller by a factor approximately $\frac{1}{n^{p-\tilde{p}}}$.

4. APPENDIX

LEMMA A-1. *If* $\underset{\sim}{x}^{(n)} \in B_n(\varepsilon) \cap D_n$ *for some* $t > 0$

then $\sum_{i=1}^{n} \frac{\ell'(x_i)}{\sqrt{nJ}(\log n)} \to 0$ *as* $n \to \infty$.

PROOF. If this is not the case then there is some positive number δ such that

$$\left| \frac{\sum_{i=1}^{n} \ell'(x_i)}{\sqrt{nJ}(\log n)} \right| \geq \delta$$ for infinitely many n. Without loss we can assume $\sum_{i=1}^{n} \ell'(x_i) \geq \delta \sqrt{nJ}(\log n)$ for some subsequence. Since $\underset{\sim}{x}^{(n)} \in D_n \cap B_n(\varepsilon)$ we have for all n sufficiently large

$$\gamma > \int_{|\theta| \leq \frac{\log n}{\sqrt{n}}} e^{\sum_{i=1}^{n} [\ell(x_i;\theta) - \ell(x_i)]} g(\theta)\, d\theta$$

$$\geq e^{-\varepsilon} \int_{|\theta| \leq \frac{\log n}{\sqrt{n}}} e^{\theta \Sigma_{i=1}^{n} \ell'(x_i) - \frac{n}{2} J \theta^2} g(\theta)\, d\theta$$

$$\geq e^{-2\varepsilon} g_0 \int_{|\theta| \leq \frac{\log n}{\sqrt{n}}} e^{\theta \Sigma_{i=1}^{n} \ell'(x_i) - \frac{n}{2} J \theta^2} |\theta|^p \, d\theta$$

$$\geq g_0\, e^{-2\varepsilon} \int_0^{\log n/\sqrt{n}} e^{\sqrt{nJ}(\log n)\delta\theta - \frac{n}{2} J \theta^2} \theta^p d\theta$$

$$= \frac{g_0 e^{-2\varepsilon}}{(Jn)^{(p+1)/2}} \int_0^{\sqrt{J}\log n} e^{(\log n)\delta\theta - \frac{1}{2}\theta^2} \theta^p d\theta$$

$$= \frac{g_0 e^{-2\varepsilon}}{(Jn)^{(p+1)/2}} \; e^{\frac{1}{2}\delta^2(\log n)^2} \int_0^{\sqrt{J}\log n} e^{\frac{1}{2}(\theta-\delta\log n)^2} \theta^p d\theta$$

$$= \frac{g_0 e^{-2\varepsilon}}{(Jn)^{(p+1)/2}} \; e^{\frac{1}{2}\delta^2(\log n)^2} \int_{-\delta\log n}^{\sqrt{J}\log n-\delta\log n} e^{-\frac{1}{2}\theta^2} |\theta+\delta\log n|^p d\theta \; .$$

By choosing $\delta < \sqrt{J}$ (which is no loss) the integral is asymptotically equivalent to

$$(\delta\log n)^p \int e^{-\frac{1}{2}\theta^2} d\theta = \sqrt{2\pi}(\delta\log n)^p \; .$$

Thus, for large enough n in our subsequence

$$\gamma > \frac{g_0 e^{-3\varepsilon}}{(nJ)^{(p+1)/2}} \; e^{\frac{1}{2}\delta^2(\log n)^2} \sqrt{2\pi}(\delta\log n)^p$$

$$= \frac{g_0 e^{-3\varepsilon}\,\delta^p\sqrt{2\pi}}{(J)^{(p+1)/2}} \; e^{\frac{1}{2}\delta^2(\log n)^2 - \frac{(p+1)}{2}\log n + p\log\log n} \; .$$

Since the right hand side tends to infinity as $n\to\infty$ we arrive at a contradiction.

LEMMA A-2. *For any* $\varepsilon > 0$, *if* $\underset{\sim}{x}^{(n)} \in B_n(\varepsilon) \cap D_n$

then

$$v_n = \frac{1}{\sqrt{n(\log n)J}} \sum_{i=1}^{n} \ell'(x_i^{(n)})$$

is bounded .

PROOF. The proof is again by contradiction. Suppose $|v_n|$ is unbounded. We can assume, without loss, that there is some subsequence n_k such that $v_{n_k} \to \infty$. For convenience we drop the subscript. Since $\underset{\sim}{x}^{(n)} \in D_n(\varepsilon) \cap D_n$ we have

$$\gamma > e^{-\varepsilon} \int_{|\theta|\le(\log n)/\sqrt{n}} e^{\theta\sum_{i=1}^{n} \ell'(x_i^{(n)}) - \frac{1}{2}J\theta^2} g(\theta)d\theta$$

$$> \frac{e^{-2\varepsilon}}{(Jn)^{(p+1)/2}} \int_{|\theta|\le\sqrt{J}\log n} e^{\theta v_n\sqrt{\log n} - \frac{1}{2}\theta^2} |\theta|^p d\theta$$

$$\ge \frac{e^{-2\varepsilon} g_0\, e^{\frac{1}{2}v_n^2(\log n)}}{(Jn)^{(p+1)/2}} \int_0^{\sqrt{J}\log n} e^{\frac{1}{2}(\theta - v_n\sqrt{\log n})^2} |\theta|^p d\theta$$

$$= \frac{e^{-2\varepsilon} g_0\, e^{\frac{1}{2}v_n^2(\log n)}}{(Jn)^{(p+1)/2}} \int_{-v_n\sqrt{\log n}}^{\sqrt{J}\log n - v_n\sqrt{\log n}} e^{-\frac{1}{2}\theta^2} |\theta + v_n\sqrt{\log n}|^p d\theta \,.$$

By Lemma A-1, $\sqrt{J}\log n - v_n\sqrt{\log n} = \log n\,(\sqrt{J} - \frac{v_n}{\sqrt{\log n}}) \to \infty$ so the upper limit of the integral tends to ∞. The integral is then asymptotic equivalent to

$$|\, v_n \sqrt{\log n}\, |^p \sqrt{2\pi}$$

so that for all sufficiently large n in our subsequence

$$\gamma > \frac{e^{-3\varepsilon} g_0 \sqrt{2\pi}}{(J)^{(p+1)/2}}\, e^{\frac{1}{2}v_n^2 \log n - \frac{p+1}{2}\log n + p\log(v_n\sqrt{\log n})}$$

and if $v_n \to \infty$ we get a contradiction since the right hand side tends to infinity.

<u>LEMMA A-3</u>. *If* f *is a strictly concave function on* R *such that* $f(-\delta) < 0$, $f(\delta) < 0$, *and* $f(0) = 0$, *then* f *has its maximum in* $(-\delta,\delta)$.

<u>PROOF</u>.

$$f'(-\delta) > \frac{f(0)-f(-\delta)}{0-(-\delta)} = \frac{-f(-\delta)}{\delta} > 0,$$

$$f'(\delta) < \frac{f(\delta)-f(0)}{\delta-0} = \frac{f(\delta)}{\delta} < 0 ,$$

so f has its maximum in $(-\delta,\delta)$.

LEMMA A-4. *Given* $t > 0$, *we have for all sufficiently large* n $\underset{\sim}{x} \in B_n(\varepsilon/4) \cap D_n^-(\varepsilon)$ *implies* $\hat{\theta}_n(\underset{\sim}{x}) \in (\frac{-\log n}{\sqrt{n}}, \frac{\log n}{\sqrt{n}})$, *where* $\hat{\theta}_n(\underset{\sim}{x})$ *is the maximum likelihood estimator for* θ *based on* $x_1, x_2, \ldots, x_n$.

PROOF. Recall that

$$h_n(\underset{\sim}{x};\theta) = \frac{\theta}{n} \sum_{i=1}^{n} \ell'(x_i) - \tfrac{1}{2} J\theta^2 ,$$

so

$$h_n(\underset{\sim}{x}; \frac{\log n}{\sqrt{n}}) = \frac{\log n}{\sqrt{n}} \frac{1}{n} \sum_{i=1}^{n} \ell(x_i) - \tfrac{1}{2} \frac{(\log n)^2}{n} J$$

$$\leq \frac{\log n}{\sqrt{n}} \left(\sqrt{\frac{\log n}{n}} \sqrt{J(p+1) + \frac{c-\varepsilon}{\log n} J} \right) - \tfrac{1}{2} J \frac{(\log n)^2}{n} .$$

Also,

$$h_n(\underset{\sim}{x}; \frac{-\log n}{\sqrt{n}}) \leq \frac{\log n}{\sqrt{n}} \left(\sqrt{\frac{\log n}{n}} \sqrt{J(p+1) + \frac{c-\varepsilon}{\log n} J} \right) - \tfrac{1}{2} J \frac{(\log n)^2}{n} .$$

Let

$$f_n(\underset{\sim}{x};\theta) = \frac{1}{n} \sum_{i=1}^{n} (\ell(x_i;\theta) - \ell(x_i))$$

so

$$\frac{T_n(\underset{\sim}{x};\theta)}{n} = f_n(\underset{\sim}{x};\theta) - h_n(\underset{\sim}{x};\theta) .$$

$$x \in B_n(\varepsilon/4) \text{ implies } \left|T_n(\underset{\sim}{x}; \pm \frac{\log n}{\sqrt{n}})\right| \leq \frac{\varepsilon}{4n}$$

so

$$f_n(\underset{\sim}{x}; \pm \frac{\log n}{\sqrt{n}}) \leq h_n(\underset{\sim}{x}; \pm \frac{\log n}{\sqrt{n}}) + \frac{\varepsilon}{4n} < 0$$

for all sufficiently large n. Since $f_n(\underset{\sim}{x};0) = 0$ and $f_n(\underset{\sim}{x};\theta)$ is strictly concave in θ, we have shown, by Lemma A-3, that $\frac{1}{n} \sum_{i=1}^{n} \ell(x_i;\theta)$ has its maximum in $(-\frac{\log n}{\sqrt{n}}, \frac{\log n}{\sqrt{n}})$.

LEMMA A-5. *Given* $t > 0$, *there exist positive constants* K *and* d *so that for all sufficiently large* n, *if* $x \in B_n(\varepsilon/4) \cap D_n^-(\varepsilon)$ *then*

$$\int_{|\theta|>\frac{\log n}{\sqrt{n}}} \exp\{\sum_{i=1}^{n}[\ell(x_i;\theta)-\ell(x_i)]\}\, g(\theta)d\theta \le ke^{-d(\log n)^2} .$$

PROOF. According to Lemma A-4, if $|\theta|> \frac{\log n}{\sqrt{n}}$,

$$\begin{aligned}
\sum_{i=1}^{n}[\ell(x_i;\theta)-\ell(x_i)] &\le \max_{+,-}\{\sum_{i=1}^{n}[\ell(x_i;\pm\frac{\log n}{\sqrt{n}})-\ell(x_i)]\} \\
&\le \max_{+,-}\{T_n(\underset{\sim}{x};\pm\frac{\log n}{\sqrt{n}}) + nh_n(\underset{\sim}{x};\frac{\log n}{\sqrt{n}})\} \\
&\le \frac{\varepsilon}{4} + \max_{+,-}\{\pm\frac{\log n}{\sqrt{n}}\sum_{i=1}^{n}\ell'(x_i) - \tfrac{1}{2}J(\log n)^2\} \\
&\le \frac{\varepsilon}{4} + (\log n)\left(\sqrt{\log n}\sqrt{(p+1)J + \frac{c-\varepsilon}{\log n}J}\right) - \tfrac{1}{2}J(\log n)^2 \\
&\le \frac{\varepsilon}{4} - d(\log n)^2
\end{aligned}$$

for all sufficiently large n, where d is some positive constant. Hence, for all such n

$$\begin{aligned}
\int_{|\theta|>\frac{\log n}{\sqrt{n}}} &\exp\{\sum_{i=1}^{n}[\ell(x_i;\theta)-\ell(x_i)]\}\, g(\theta)d\theta \\
&\le e^{\varepsilon/4}\, e^{-d(\log n)^2}\int_{|\theta|>\frac{\log n}{\sqrt{n}}} g(\theta)d\theta \\
&\le e^{\varepsilon/4} e^{-d(\log n)^2} .
\end{aligned}$$

LEMMA A-6. *Given* $\varepsilon > 0$, *we have for all sufficiently large* n *that if* $\underset{\sim}{x} \in B_n(\varepsilon/4)\cap D_n^-(\varepsilon)$, *then*

$$\int_{|\theta|\le \frac{\log n}{\sqrt{n}}} \exp\{\sum_{i=1}^{n} [\ell(x_i;\theta) - \ell(x_i)]\}\, g(\theta)d\theta \; < e^{-\varepsilon/8}\gamma \; .$$

PROOF. Define

$$v_n = \frac{1}{\sqrt{n(\log n)J}} \sum_{i=1}^{n} \ell'(x_i) \; .$$

Since $\underset{\sim}{x} \varepsilon B_n(\varepsilon/4)$ we have, if n is large enough,

$$\int_{|\theta|\le \frac{\log n}{\sqrt{n}}} \exp\{ \sum_{i=1}^{n} [\ell(x_i;\theta)-\ell(x_i)]\}\, g(\theta)d\theta$$

$$\le e^{(5/16)\varepsilon} g_0 \int_{|\theta|\le \frac{\log n}{\sqrt{n}}} |\theta|^p\, e^{\sqrt{J}\theta\sqrt{n(\log n)}\, v_n - \frac{n}{2} J\theta^2} d\theta$$

$$= \frac{e^{(5/16)\varepsilon} g_0}{(Jn)^{(p+1)/2}} \int_{|\theta|\le \sqrt{J}\log n} |\theta|^p\, e^{\theta\sqrt{\log n}\, v_n - \frac{1}{2}\theta^2} d\theta \; .$$

We may as well suppose $\sqrt{\log n}\; v_n \to \infty$. Otherwise, the assertion of the Lemma is obvious since the integral converges to zero. Then,

$$\int_{|\theta|\le \sqrt{J}\log n} |\theta|^p\, e^{\theta\sqrt{\log n}\, v_n - \frac{1}{2}\theta^2} d\theta$$

$$= e^{\frac{1}{2}v_n^2(\log n)} \int_{|\theta + \sqrt{\log n}\, v_n|\le \sqrt{J}\log n} |\theta+\sqrt{\log n}\, v_n|^p\, e^{-\frac{1}{2}\theta^2} d\theta \; .$$

Since $\underset{\sim}{x} \;\varepsilon\; D_n^-(\varepsilon)$, v_n is bounded, so the region of integration converges to the real line. The integral, above, is asymptotically equivalent to $(\log n)^{p/2}\; |v_n|^p \sqrt{2\pi}$, and this is less than or equal to

$$\sqrt{2}\pi\, [(p+1) + \frac{c-\varepsilon}{\log n}]^{p/2} \le e^{\varepsilon/16} \sqrt{2\pi}\, (p+1)^{p/2} .$$

Finally,

$$\int_{|\theta| \le \frac{\log n}{\sqrt{n}}} \exp\{ \sum_{i=1}^{n} [\ell(x_i;\theta) - \ell(x_i)]\} g(\theta) d\theta$$

$$\le \frac{e^{(5/16)\varepsilon} g_0}{(Jn)^{(p+1)/2}}\, e^{\frac{1}{2}(\log)(p+1) + \frac{1}{2}(c-\varepsilon) - \frac{p}{2} \log \log n}$$

$$\times\, e^{(1/16/)\varepsilon} \sqrt{2\pi}\, (p+1)^{p/2}$$

$$= e^{(5/16)\varepsilon - (\frac{1}{2}\varepsilon)t + (1/16)\varepsilon} g_0 \sqrt{2\pi}(p+1)^{p/2}(J)^{-(p+1)/2}\, e^{\frac{1}{2}c}$$

$$= e^{-\varepsilon/8} \gamma .$$

LEMMA A-7. *Given* $\varepsilon > 0$, *there exist positive constants* K *and* d *so that for all sufficiently large* n, *if* $\underset{\sim}{x} \in B_n(\varepsilon) \cap D_n$

$$\int_{|\theta| > \frac{\log n}{\sqrt{n}}} \exp\{ \sum_{i=1}^{n} [\ell(x_i;\theta) - \ell(x_i)]\} g(\theta) d\theta \le K\, e^{-d(\log n)^2} .$$

PROOF. If $x \in B_n(\varepsilon) \cap D_n$, then by Theorem 1, $x \in D_n^+(4\varepsilon)$ for large enough n. Using the same arguments as in Lemma A-5 the result follows.

REFERENCES

[1] CRAMÉR, H. (1946). *Mathematical Methods of Statistics*. Princeton University Press.

[2] FELLER, W. (1971). *An Introduction to Probability Theory and its Applications*, v.2. Wiley, New York.

[3] JOHNSON, B.R. & D.R. TRUAX (1978). *Asymptotic behavior of Bayes procedures for testing simple hypotheses in multiparameter exponential families*. Ann. Statist. 6 , 346-361.

[4] RUBIN, H. & J. SETHURAMAN (1965). *Bayes risk efficiency*. Sankhyà Ser. A , 21 , 341-356.