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REDUCTION OF TENSOR PRODUCTS OF SOME REPRESENTATIONS OF THE CONFORMAL GROUP

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Reduction of tensor products of some representations of the conformal group *)

by

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ABSTRACT

Unitary irreducible ray representations of the conformal group $SO_0(4,2)/\mathbb{Z}_2$ with positive energy possess a unique lowest weight. Starting from this fact, the construction of such representations as well as the reduction of their tensor product is not too hard because methods known from the representation theory of the rotation group may be used. In physics, the conformal group is often encountered as a symmetry group of space-time and it is appropriate to construct its representations as induced representations on Minkowski space. Then the reduction of tensor products is much more involved but remains straightforward. It leads to Koornwinder's orthogonal polynomials in two dimensions.

KEY WORDS & PHRASES: *conformal group, Koornwinder's polynomials, representations, tensor products.*

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1. INTRODUCTION

This introductory section gives some background information about the conformal group of space-time in general.

1.1. Space-time

Space-time or *Minkowski space* is the four-dimensional vector space which is used to describe the special theory of relativity. Its points are called events $x = (x^0, \vec{x})$, where $x^0 = t$ and $\vec{x} = (x,y,z)$ or $\vec{x} = (x^1, x^2, x^3)$, and a curve in space-time is called a world line. Distance between events is measured by the line element

$$(1.1) \qquad -d\tau^2 = (dx)^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = -(dx^0)^2 + |d\vec{x}|^2.$$

If $x(t)$ is the world line of a moving observer, $d\tau$ measures proper time, that is, the time measured by that same observer.

1.2. The conformal group

A *conformal transformation of space-time* is a coordinate transformation

$$x' = x'(x)$$

which leads to the new line element

$$(1.2) \qquad d\tau'^2 = \frac{1}{\sigma(x)} d\tau^2 ,$$

that is, it multiplies $d\tau^2$ by a factor $1/\sigma(x)$. From this definition, some subgroups of the conformal group may be guessed immediately. For $\sigma(x) = 1$ one gets the symmetry transformations of space-time, namely, the Poincaré group. The Poincaré group is generated by the

(1.3) *translations* $x'^{\mu} = x^{\mu} + a^{\mu}$,

whose infinitesimal operators P^{μ} are called the four-momentum, and by the

(1.4) *Lorentz transformations* $x'^{\mu} = L^{\mu}{}_{\nu}x^{\nu}$ with $x'^2 = x^2$,

whose infinitesimal operators $M^{\mu}{}_{\nu}$ are called angular momentum tensor. Let me recall that there is a special name for the zero component P^0 of the four-momentum. It is called energy . A second subgroup of the conformal transformations can be guessed, the

(1.5) *dilations* $x'^{\mu} = \lambda x^{\mu}$, $\lambda \neq 0$, $\sigma(x) = \frac{1}{\lambda^2}$.

Their infinitesimal operator will be called D. Finally, there is one more subgroup which together with the subgroups mentioned already generates the full group of conformal transformations with 15 parameters. It is a four-parameter, abelian subgroup of nonlinear transformations, the

(1.6) *special conformal transformations* $x'^{\mu} = \frac{1}{\sigma(x)}(x^{\mu} - x^2 c^{\mu})$,

$$\sigma(x) = 1 - 2c.x + x^2,$$

with infinitesimal operators K^{μ}. They map certain points of Minkowski space to infinity, therefore a compactification of Minkowski space is necessary.

1.3. The invariance of Maxwell's equations under the conformal group

The conformal group entered physics in 1910 when CUNNINGHAM [1] and BATEMAN [2] proved that Maxwell's equations are invariant under conformal transformations. Let me give an argument in support of this statement.

The effect of time dilation in special relativity says that time is slower for a moving object than for an object at rest. For instance, there is the "twin paradox" where one of a pair of twins enters a space ship and travels very fast. After return to earth he is younger than his brother, time has been slower for him.

For a particle with the velocity of light, say, for a photon, this effect is extreme. Time does not flow at all for such a particle, it does not become older, $d\tau = 0$. After a conformal transformation, $d\tau'$ is also zero, that is, the transformed world line is again the world line of a photon. Now, photons describe just the particle aspect of light waves, and therefore one will expect that conformal transformations also transform light waves into light waves, in other words, that Maxwell's equations are invariant under conformal transformations.

1.4. A natural linear representation of the conformal group

25 years after Cunningham and Bateman it has been shown by DIRAC [3] that the conformal group of space-time is locally isomorphic to the group $SO_0(4,2)$. The proof is quite simple. Take the cone

$$(1.7) \qquad -(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 + (y^5)^2 - (y^6)^2 = 0$$

in 6 dimensions without the origin. It is a homogeneous space for $SO_0(4,2)$. Identify all the points

$$\lambda y \quad \text{with } y \text{ fixed and } \lambda \neq 0$$

which gives a four-dimensional projective space. Introduce inhomogeneous coordinates

$$(1.8) \qquad x^\mu = \frac{y^\mu}{y^5+y^6} \quad \text{with} \quad \mu = 0,1,2,3$$

and identify the four-vector x^μ with the four vector x^μ of Minkowski space provided $y^5 + y^6$ is different from zero. Then the actions of $SO_0(4,2)$ on the cone translate into the conformal transformations of Minkowski space, because on the cone

$$(1.9) \qquad -d\tau^2 = (dx)^2 = \frac{1}{(y^5+y^6)^2}\left[-(dy^0)^2+(dy^1)^2+(dy^2)^2+(dy^3)^2+(dy^5)^2-(dy^6)^2\right].$$

By this construction one has also obtained a compactification of Minkowski space. It can be shown by direct construction that all conformal transformations are obtained in this way.

The center of the conformal transformations of Minkowski space is trivial, but the center of $SO_0(4,2)$ is $\mathbb{Z}_2 = \{1,-1\}$. Therefore, the group of conformal transformations of Minkowski space is isomorphic to $SO_0(4,2)/\mathbb{Z}_2$ and $SO_0(4,2)$ itself is a twofold covering group of the conformal group.

1.5. The universal covering group of the conformal group

There are many covering groups for the conformal group, for instance SU(2,2), which is easier to work with than $SO_0(4,2)$ because it deals with 4 by 4 matrices only. All of them can of course be derived from the universal covering group which itself may be obtained by using the Iwasawa decomposition of the conformal group, but I shall not go into the details here. However, I need to explain about the covering space of Minkowski space which is obtained from the universal covering group.

The compactified Minkowski space described above is simply the homogeneous space obtained from the conformal group with respect to the stationary group of the origin, and from the explicit formulae for the subgroups given before one sees that the stationary group is generated by Lorentz transformations, dilations and special conformal transformations.

Starting from the universal covering group instead and constructing the homogeneous space with respect to the stationary group mentioned above one obtains

$$\mathbb{R} \times S^3.$$

Since the universal covering group is much larger than the conformal group itself, $\mathbb{R} \times S^3$ is much larger than Minkowski space. In fact, Minkowski space can be embedded into $\mathbb{R} \times S^3$ as a so-called fundamental domain, and by applying operators of the center of the universal covering group one obtains more fundamental domains which are disjoint and the union of which gives (after completion) the whole of $\mathbb{R} \times S^3$. In other words, together with the infinitely sheeted covering group of the conformal group one gets an infinitely sheeted covering space of Minkowski space. If you like so, you may interpret them as spheres of heaven and circles of hell.

Having identified Minkowski space as a fundamental domain of its covering space, one is tempted to translate back the geometry of $\mathbb{R} \times S^3$ to Minkowski space. Parametrization of $\mathbb{R}$ by a parameter τ and of S^3 by a four-vector $e = (\cos\theta, \sin\theta\ \vec{e})$ gives for the line element of Minkowski space:

$$(1.10)\qquad -d\tau^2 = \frac{1}{(\cos\tau+\cos\theta)^2}\{-d\tau^2+d\theta^2+\sin^2\theta|d\vec{e}|^2\}.$$

Therefore, fixed τ gives a spacelike hypersurface and one may interpret τ as a time variable, say *conformal time*. Then one may fix a point of S^3 and ask for the line in Minkowski space when τ varies freely. What one gets is the world line of a body moving with constant acceleration $\sin\theta$. So one finds a group of motions of Minkowski space describing parallel displacement along these lines or, correspondingly, which describes the parallel displacement of S^3 along $\mathbb{R}$. Its generator is $H_0 = 1/2(P^0+K^0)$, it is called *conformal energy*.

With these remarks I have finished my general introduction to the conformal group of space-time. Many more details are given in the papers by MACK [4], [5], [6].

2. UNITARY RAY REPRESENTATIONS OF THE CONFORMAL GROUP WITH POSITIVE ENERGY: THE METHOD OF LOWEST WEIGHT

There are two main paths towards this goal. The first one, working with lowest weights, seems to be simpler from a mathematical point of view but it does not allow a simple physical interpretation. The second one works with induced representations on compactified Minkowski space. It is certainly the natural approach if one believes that the conformal group has anything to do with Minkowski space. However, one should bear in mind that there are other applications of the conformal group in physics.

In this section I shall describe the method of lowest weights, which is so well known from the representation theory of the rotation group and also of the Lorentz group. The strategy is to construct representations of the Lie algebra first and to show afterwards that they can be integrated to representations of the group. To obtain a representation of the Lie algebra one starts from a Cartan subalgebra. It is possible to choose the same Cartan subalgebra for the Lie algebra of the conformal group $SO_0(4,2)/\mathbb{Z}_2$ and for the Lie algebra of its maximal compact subgroup

$$SO(2) \times SO(4)/\mathbb{Z}_2 = SO(2) \times SO(3) \times SO(3).$$

In other words, one may choose a Cartan subalgebra with basis H_0, H_1, H_2, where H_0 is an infinitesimal operator of SO(2) and H_1 and H_2 are infinitesimal operators of the SO(3) groups. The remaining 4 basis elements of the Lie algebra of the maximal compact subgroup and the still remaining 8 basis elements of the Lie algebra of the conformal group may be chosen as simultaneous eigenvectors of H_0, H_1, and H_2 under the adjoint representation:

$$(2.1) \qquad [H_0, X^{\varepsilon}_{jk}] = \varepsilon X^{\varepsilon}_{jk}, \quad [H_1, X^{\varepsilon}_{jk}] = jX^{\varepsilon}_{jk}, \quad [H_2, X^{\varepsilon}_{jk}] = kX^{\varepsilon}_{jk}.$$

Here ε takes the values 0, $\pm$ 1. It is a bit more complicated to describe the allowed values of j and k. I shall not write them down and likewise I shall not give the commutation relations between the X^{ε}_{jk}.

Now, for a given finite-dimensional representation T of the Lie algebra, one may choose basis vectors in the representation space which are simultaneous eigenvectors $\psi(\lambda)$ of $T(H_0)$, $T(H_1)$ and $T(H_2)$ with corresponding eigenvalues λ_0, λ_1 and λ_2. Then the first of the previous equations gives

$$(2.2) \qquad T(H_0)T(X^{\varepsilon}_{jk})\psi(\lambda) = T(X^{\varepsilon}_{jk})[T(H_0)+\varepsilon]\psi(\lambda) = (\lambda_0+\varepsilon)T(X^{\varepsilon}_{jk})\psi(\lambda),$$

and one sees that $T(X^{1}_{jk})$ acts as a ladder operator which raises the eigenvalue λ_0 by 1, $T(X^{-1}_{jk})$ is a ladder operator which lowers the eigenvalue λ_0 by 1 and that $T(X^{0}_{jk})$ does not change the eigenvalue λ_0. Similarly $T(X^{\varepsilon}_{jk})$ raises the eigenvalue λ_1 by j (j may be positive, negative, or zero) and it raises the eigenvalue λ_2 by k. Finally, one can show that there is a unique vector $\psi(d,-j_1,-j_2)$ with lowest weight $\lambda_0 = d, \lambda_1 = -j_1, \lambda_2 = -j_2$ which is annihilated by $T(X^{\varepsilon}_{jk})$ if just one of the ε, j, k is negative.

On the other hand, the representation T can be reconstructed from the vector $\psi(d,-j_1,-j_2)$ with lowest weight. Repeated application of the $T(X^{\varepsilon}_{jk})$ gives vectors which span the representation space. It is sufficient here to use only $T(X^{\varepsilon}_{jk})$ with nonnegative indices, because a $T(X^{\varepsilon}_{jk})$ with a negative index can be eliminated with the help of the commutation relations. Furthermore, a (not necessarily positive definite) "scalar product" is defined by

$$(2.3) \qquad (\psi(d,-j_1,-j_2), \psi(d,-j_1,-j_2)) = 1,$$

since the "scalar product" of other vectors can then be calculated from the commutation relations.

Now let us see if this procedure might be useful to construct an infinite-dimensional unitary ray representation of the conformal group, that is, a representation of the universal covering group. At first sight this seems to be improbable. The universal covering group of the maximal compact subgroup $SO(2) \times SO(2) \times SO(3)$ is

$$\mathbb{R} \times SU(2) \times SU(2)$$

(it is no longer compact) and the spectrum of $T(H_0)$ (with H_0 now the infinitesimal operator of $\mathbb{R}$) may contain any real number. It is hard to see how a vector of lowest weight might be in the representation space (which was guaranteed in the construction before by the finiteness of the representation).

At this point physics helps. You may not have noticed that I have used the symbol H_0 before, namely for the conformal energy, the generator of conformal time-translations. In fact, these two objects are indeed the same and you begin to see a chance that a vector with lowest weight might exist for representations with positive energy (that is, for representations, which, when reduced with respect to the Poincaré group, give only representations of the Poincaré group with positve energy). But is conformal energy really the same as energy? Of course it is not. It is much better! This is shown by the following theorems of Mack:

1. *In any unitary, irreducible representation* T *of the universal covering group with positive energy, the infinitesimal operator* $T(H_0)$ *has a discrete spectrum. It contains a lowest eigenvalue* $d \geq 1$ *and all other eigenvalues are of the form* $d + m$, *where* m *is a* positive integer.

2. *A representation* T *with positive energy possesses a unique lowest weight* $(d,-j_1,-j_2)$. *Any two such representations with the same lowest weight are unitarily equivalent.*

Furthermore, Mack has given conditions on d depending on the half-integers j_1 and j_2 which guarantee a positive-definite scalar product, that is, a unitary representation. These conditions are:

3. $d \geq j_1 + j_2 + 2$ *for massive (with respect to the Poincaré subgroup) representations with* $j_1 j_2 \neq 0$,
   $d > j_1 + j_2 + 1$ *for massive representations with* $j_1 j_2 = 0$,
   $d = j_1 + j_2 + 1$ *for mass zero representations.*

The given theorems allow the statement that by now the representation theory for positive energy representations of the conformal group is well established. One might say that the physical requirement of positive energy sorts out the mathematically nice representations, which contain vectors with lowest weight.

Let me finish this chapter with the remark that in this formalism there is a simple procedure for the reduction of tensor products. Seek for a vector with lowest weight in the tensor product and "take away" the corresponding representation. Then seek again for a vector with lowest weight and go on. By this method the abstract results of the following calculations may be checked quite easily.

3. POSITIVE ENERGY REPRESENTATIONS AS INDUCED REPRESENTATIONS

There is a quite different model for the representations described in the last section. They may be constructed as induced representations on compactified Minkowski space.

Let $\tilde{G}$ be the universal covering group of the conformal group acting on the compactified Minkowski space M_c. The stationary group of the origin may be written as $\Gamma_2 MAN$. Here $\Gamma_2 \cong \mathbb{Z}$ is that subgroup of the center which acts nontrivially on the sheets of the covering space of Minkowski space. It has a generating element γ_2. M is the quantum mechanical Lorentz group, A is the group of dilations and N is the group of special conformal transformations. Define a finite-dimensional representation of the stationary group by

$$D^{\lambda}(\gamma man) = |a|^{d-4} e^{i\pi N(d-4)} D^{(j_2,j_1)}(m). \tag{3.1}$$

Here

$$\lambda = (d,-j_1,-j_2) \quad \gamma = \gamma_2^N \in \Gamma_2, \; m \in M, \; a \in A, \; n \in N. \tag{3.2}$$

Since the representation is finite-dimensional it can not be unitary. It should be noted that the special conformal transformations are trivially represented.

Now consider the space E_λ of all infinitely differentiable functions ϕ on G with values in the representation space of $D^{(j_2,j_1)}$ which have the covariance property.

$$\phi(g\gamma man) = D^{\lambda}(\gamma man)^{-1} \phi(g). \tag{3.3}$$

By the covariance property ϕ can be determined for all points of a coset $g\,\Gamma_2 MAN$ if it is known for one point of the coset. Therefore, a representative for each coset may be chosen, and this is done as follows:
Almost every element g of $\tilde{G}$ may be decomposed uniquely as

$$g = x_g\, \gamma man, \quad x_g \in X = \text{group of translations}, \tag{3.4}$$

and one may chose x_g as the representative of the coset which contains g. To see this, one applies the translations $x \in X$ to the coset $\Gamma_2 MAN$ (origin of Minkowski space). One gets all the other cosets (other points of Minkowski space) with the exception of the one which has to do with infinity.

This last coset is obtained as $R\,\Gamma_2 MAN$ where R is a conformal transformation which transforms the origin to infinity. It is defined by

(3.5) $$R(y) = \frac{\theta(y)}{(y)^2} \quad \text{with } \theta(y^0, y) = (-y^0, \vec{y}).$$

It remains to define the action of the group G on the functions $\phi \in \varepsilon_\lambda$. It is given as usual by

(3.6) $$(T(g)\phi)(g') = \phi(g^{-1}g').$$

Having chosen representations for the cosets one can deal with infinitely differentiable functions $\phi(x)$ rather than with infinitely differentiable functions $\phi(g)$. But infinite differentiability is not enough. In addition, one must require a certain asymptotic behaviour of $\phi(x)$ at infinity to guarantee that the corresponding function $\phi(g)$ is infinitely differentiable at points g with the property that they transform the origin to infinity. Consider $g = R$ and find the representative of $R\delta_x$ where δ_x is a small parallel translation. One obtains

(3.7) $$R\delta_x = x\ man$$

with

(3.8) $$x(y) = y + R(\delta_x),$$

$$m(y) = \operatorname{sign}(\delta_x)^2[\theta(y) - 2\theta(y)\ \theta(\delta_x)R(\delta_x)],$$

$$a(y) = \frac{y}{|(\delta_x)^2|},$$

n = special conformal transformation with $c = -R(\delta_x^*)$.

Therefore

$$(3.9)\qquad \phi(R\delta_x) = \phi(x\ \mathrm{man}) = D^{\lambda}(\mathrm{man})^{-1}\ \phi(x) =$$
$$= D^{(j_2 j_1)}(m)\ |(\delta_x)^2|^{d-4}\ \phi(R(\delta_x)) =$$
$$= D^{(j_2 j_1)}(m)\ |(\delta_x)^2|^{d-4}\ \phi\left(\frac{\theta(\delta_x)}{(\delta_x)^2}\right)$$

should behave nicely for $\delta_x \to 0$.

One sees that the function space that one has to deal with is not quite simple. Furthermore, the scalar product turns out to be complicated, due to the fact that the inducing construction has been done with a nonunitary representation of the stationary group.

Let us go to "momentum space", that is, let us introduce Fourier transforms. This has big advantages and serious disadvantages. First of all, to define Fourier transformations one has to confine the functions $\phi(x)$ to Schwartz test functions, which unfortunately are not invariant under the action of the group. It is an open question how the Schwartz test functions are to be extended so that the Fourier transformation is defined for the whole representation space.

However, it turns out in a quite trivial way that the space of Schwartz test functions is invariant under the Lie algebra of G because under differentiation Schwartz test functions remain Schwartz test functions. Therefore, to reduce tensor products one may use Lie algebraic methods, namely Casimir operators. Having done so one may compare the irreducible representations contained in the tensor product of the Lie algebras with the group representations obtained by the method explained earlier and check that no nonintegrable representation has sneaked in.

In the next and last section I shall show how to use this method to reduce the tensor product of spinless, massive representations, that is, of representations with $d > 1$ and $j_1 = j_2 = 0$.

4. REDUCTION OF $D^{(d_1,0,0)} \otimes D^{(d_2,0,0)}$

The infinitesimal operators of a spinless, massive representation $D^{(d,0,0)}$ in a momentum basis have first been constructed by CASTELL [7].

They are

$$(4.1) \qquad P^{\mu} = p^{\mu},$$

$$M_{\mu\nu} = -i(p_{\mu}\partial_{\nu} - p_{\nu}\partial_{\mu}) \qquad (\partial_{\mu} = \frac{\partial}{\partial p^{\mu}}) \ ,$$

$$D = i(p^{\rho}\partial_{\rho} + d),$$

$$K_{\mu} = p_{\mu}\partial^{\rho}\partial_{\rho} - 2(p^{\rho}\partial_{\rho}+2)\partial_{\mu}.$$

Here μ takes the values 0,1,2,3 and d = 2,4,6,... for representations of the Lie algebra of the conformal group and d > 1 for representations of the Lie algebra of the universal covering group. The infinitesimal operators act on the space $\mathcal{S}(M)$ of Schwartz test functions on Minkowski space with scalar product

$$(4.2) \qquad (\phi,\psi) = \int \phi^{*}(p)\ m^{2(d-2)}\ \theta(m^{2})\ \theta(p^{0})\ d^{4}(p).$$

Because of the θ-distributions the integration is actually over the forward light cone

$$(4.3) \qquad L^{+} = \{p \mid (p^{0})^{2} - p^{i}p_{i} > 0,\ p^{0} > 0,\ i = 1,2,3\}$$

and $m = +\sqrt{-p^{\rho}p_{\rho}}$. $\mathcal{S}(M)$ can be completed to a Hilbert space after dividing out zero norm functions, that is, functions vanishing on the forward light cone.

To construct Casimir operators, it is convenient to replace the infinitesimal operators used in (4.1) by the linear combinations

$$(4.4) \qquad M_{\mu\nu}, \quad M_{\mu 5} = -\frac{1}{2}(P_{\mu}+K_{\mu}), \quad M_{\mu 6} = \frac{1}{2}(P_{\mu}-K_{\mu}), \quad M_{56} = -D,$$

because the $M_{\alpha\beta}$ (α,β = 0,1,2,3,5,6) form a canonical basis for so (4,2) with commutation relations

$$(4.5) \qquad [M_{\alpha\beta}, M_{\gamma\delta}] = g_{\alpha\gamma}\ M_{\beta\delta} + g_{\beta\delta}\ M_{\alpha\gamma} - g_{\alpha\delta}\ M_{\beta\gamma} - g_{\beta\gamma}\ M_{\alpha\delta}.$$

Here the $g_{\alpha\beta}$ are the components of the metric tensor diag(-1,1,1,1,1,-1) in 6 dimensions. The Casimir operators, expressed by the infinitesimal operators (4.4) are

$$(4.6)\qquad \begin{aligned} C_{II} &= \frac{1}{2} M_{\alpha\beta} M^{\alpha\beta}, \\ C_{III} &= \frac{1}{48} \varepsilon_{\alpha\beta\gamma\delta\kappa\lambda} M^{\alpha\beta} M^{\gamma\delta} M^{\kappa\lambda} \\ C_{IV} &= \frac{1}{2} C_{\alpha\beta} C^{\alpha\beta}, \qquad C_{\alpha\beta} = \frac{1}{8} \varepsilon_{\alpha\beta\gamma\delta\kappa\lambda} M^{\gamma\delta} M^{\kappa\lambda}. \end{aligned}$$

Here the $\varepsilon_{\alpha\beta\gamma\delta\kappa\lambda}$ are the components of the completely antisymmetric tensor in 6 dimensions, defined by $\varepsilon_{012356} = 1$.

For the irreducible representation (4.1) the eigenvalues of the Casimir operators are

$$(4.7)\qquad c_{II} = (d-2)^2 - 4, \quad c_{III} = c_{IV} = 0.$$

Given two representations $D^{(d_1,0,0)}$ und $D^{(d_2,0,0)}$ with $d_1 d_2 > 0$, one has the infinitesimal operators $P^{(1)\mu}$, $M^{(1)}_{\mu\nu}$, $D^{(1)}$ and $K^{(1)}_{\mu}$, obtained from (4.1) by replacing p^{μ}, d by $p^{(1)\mu}$, d_1, respectively, and one has the infinitesimal operators $P^{(2)\mu}$, $M^{(2)}_{\mu\nu}$, $D^{(2)}$ and $K^{(2)}_{\mu}$, obtained from (4.1) by replacing p^{μ}, d by $p^{(2)\mu}$, d_2, respectively. The infinitesimal operators of the product representation $D^{(d_1,0,0)} \otimes D^{(d_2,0,0)}$ are

$$(4.8)\qquad \begin{aligned} P^{\mu} &= P^{(1)\mu} + P^{(2)\mu}, \\ M_{\mu\nu} &= M^{(1)}_{\mu\nu} + M^{(2)}_{\mu\nu}, \\ D &= D^{(1)} + D^{(2)}, \\ K_{\mu} &= K^{(1)}_{\mu} + K^{(2)}_{\mu}, \end{aligned}$$

acting on the product space $\mathcal{S}(M) \otimes \mathcal{S}(M)$. Inserting (4.8) into (4.6) one obtains the Casimir operators of the product representation.

The simultaneous eigenfunctions of the Casimir operators may be used as basis functions for the invariant subspaces of $\mathcal{S}(M) \otimes \mathcal{S}(M)$. Therefore the most important step in the reduction of the tensor product is to find those simultaneous eigenfunctions. The details of the calculation may be found in reference [8]. Here only the most important steps are given.

The first step is to introduce variables which simplify the Casimir operators. Of course one has to introduce the variables

$$p^{\mu} = p^{(1)\mu} + p^{(2)\mu} , \tag{4.9}$$

which are the components of the "total momentum". The remaining 4 variables may be chosen as the components of "relative momentum"

$$r^{\mu} = p^{(1)\mu} - p^{(2)\mu}. \tag{4.10}$$

Next several transformations of the variables are to be performed until one arrives at variables p^{μ} and R^{μ}. Expressed in terms of these variables, the Casimir operators will be independent of p^{μ} and invariant under rotations in the three-dimensional space parametrized by (R^1,R^2,R^3). Therefore the variables p^{μ} may be thought of as separated off and it will be useful to introduce basis functions

$$B^{(\ell)}_{m'} = p^{(\ell)}(R^0,R)\mathcal{Y}_{\ell m'}(R^1,R^2,R^3) \tag{4.11}$$

in "spin space" which describe "states with spin ℓ". Here $R = \sqrt{(R^1)^2+(R^2)^2+(R^3)^2}$ and the $\mathcal{Y}_{\ell m'} = R^{\ell}\, Y_{\ell m'}$ are the harmonic polynomials.

Now, in general, an irreducible representation of the conformal group will contain a "spin spectrum", that is, several values of ℓ will occur. Only the infinitesimal operators of the special conformal transformations can jump between different values of ℓ (it is well-known that the infinitesimal operators of the Poincaré group do not and D certainly does not). By well-known techniques one can isolate those parts of K_{μ} which are responsible for such transitions and one gets ladder operators

$$E^{+}(\ell) = (\ell+1\|K\|\ell), \qquad E^{-}(\ell) = (\ell-1\|K\|\ell), \tag{4.12}$$

which map from spin ℓ to spin $\ell \pm 1$. The symbols on the right hand side are called *reduced matrix elements*, cf. [9].

The ladder operator $E^{\pm}(\ell)$ may be used to simplify the Casimir operators:

$$(4.13)\qquad \begin{aligned} C_{II} &= \frac{1}{2\ell+1} E^{+}(\ell-1)E^{-}(\ell) - \frac{1}{2\ell+1} E^{-}(\ell+1)E^{+}(\ell), \\ C_{III} &= 0, \\ C_{IV} &= \frac{1}{2\ell+1} E^{+}(\ell-1)E^{-}(\ell) + \frac{\ell+1}{2\ell+1} E^{-}(\ell+1)E^{+}(\ell). \end{aligned}$$

On the other hand, the $E^{\pm}(\ell)$ are already well-known objects because they are also ladder operators for KOORNWINDER's [10] orthogonal polynomials in two dimensions $P_{m,n}^{(d_1-2,d_2-2,\ell)}(x,y)$, as can be seen from the explicit formulae

$$(4.14)\qquad \begin{aligned} E^{+}(\ell) &= \frac{1}{x-y}\,[(x^2-1)\partial_x^2+(\kappa x+\lambda)\partial_x-(y^2-1)\partial_y^2-(\kappa y+\lambda)\partial_y] \\ E^{-}(\ell) &= \frac{1}{(x-y)^{2\ell}}\,[(x^2-1)\partial_x^2+(\kappa x+\lambda)\partial_x-(y^2-1)\partial_y^2-(\kappa y+\lambda)\partial_y](x-y)^{2\ell+1} \end{aligned}$$

and by comparison with the formulae given in reference [11]. Here $\kappa = d_1 + d_2 - 2$, $\lambda = d_1 - d_2$, and x,y are obtained by a linear transformation of R^{0^2}, R.

One conclues that Koornwinder's polynomials multiplied by $Y_{\ell m}$, are simultaneous eigenfunctions of the Casimir operators. This may be further checked by the observation that the measure of the product space contains a factor

$$(4.15)\qquad \begin{aligned} dm = {} & [(1-x)(1-y)]^{\alpha}[(1+x)(1+y)]^{\beta}(x-y)^{2\ell}\,\cdot \\ & \cdot\,\theta(1-x^2)\theta(1-y^2)\theta(x-y)\,dx\,dy, \end{aligned}$$

which is exactly the measure which makes Koornwinder's polynomials orthogonal.

The action of the ladder operators upon Koornwinder's polynomials is

$$(4.16)\qquad E^{+}(\ell)P_{m,n}^{(d_1-2,d_2-2,\ell)} = (L-1)(\nu-\ell-1)\ P_{m,n-1}^{(d_1-2,d_2-2,\ell+1)},$$

$$E^{-}(\ell)P_{m,n}^{(d_1-2,d_2-2,\ell)} = (L+\ell+1)(\nu+\ell)\ P_{m,n+1}^{(d_1-2,d_2-2,\ell-1)}$$

with

$$(4.17)\qquad L = \ell + n, \qquad \nu = d_1 + d_2 + 2m + \ell + n - 2.$$

From this one calculates not only the eigenvalues of the Casimir operators

$$(4.18)\qquad c_{II} = \nu^2 + L(L+2) - 4, \qquad c_{III} = 0, \qquad c_{IV} = (\nu^2-1)\ L(L+2),$$

but also the action of the infinitesimal operators upon the basis functions. By comparison with the action of the infinitesimal operators of the irreducible representations $D^{(d,j_1,j_2)}$ given in reference [4] one finds that $P_{m,n}^{(d_1-2,d_2-2,\ell)}(x,y)Y_{\ell m'}(R_1,R_2,R_3)$ is a basis function for the representation $D^{(\nu+2,L/2,L/2)}$. One concludes that the tensor product $D^{(d_1,0,0)} \otimes D^{(d_2,0,0)}$ contains all the representations $D^{(d,L/2,L/2)}$ with L a non-negative integer and

$$(4.19)\qquad d = d_1 + d_2 + 2m + L,$$

where again m is a non-negative integer. Each L and m occur exactly once.

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