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On boundary value problems for Laplace's equation.

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1962

On Boundary Value Problems for Laplace's Equation

R. T. Seeley

Introduction. The object of this paper is to formulate certain boundary value problems for Laplace's equation in a way suitable for the representation of the boundary condition by a singular integral equation. This formulation is modeled to some extent on the Abel summation of Fourier series. The application considered is to show that, for a first order boundary operator that "covers" the Laplacian, the index of the problem is zero.

Section 1 describes the regions considered, and develops the results used to justify the representation of the boundary conditions by singular integral equations. Some lemmas used in this are proved in section 3. Section 2 discusses the question of the index.

The methods used here may prove applicable to wide classes of elliptic boundary problems.

1. On the connection between u and $\partial u/\partial n$ for a harmonic function u.

Let k be $\geqq 2$, and G be a bounded region in $k+1$ dimensional Euclidean space E^{k+1}, with boundary Γ. We assume that Γ is connected, and is almost a C_2 manifold, of dimension k. Specifically

i) Γ is a C_1 manifold; the unit outer normal at the point $\underline{x}$ in Γ is denoted by $\underline{\nu}(\underline{x})$;

ii) there is a number $\tau > 0$ such that for every $\underline{x}$ in Γ the circle of radius τ about $\underline{x} + \tau\,\underline{\nu}(\underline{x})$ intersects Γ only at $\underline{x}$;

iii) for each t in $0 \leqq |t| \leqq \tau$, the points $\{\underline{x} + t\,\underline{\nu}(\underline{x});\ \underline{x} \text{ in } \Gamma\}$ are all distinct and form a C_1 manifold Γ_t, bounding the region G_t; and

iv) if we introduce in E^{k+1} orthonormal coordinates $\xi_1, \cdots, \xi_k, \zeta$ with origin at the point $\underline{x}$ in Γ in such a way that the tangent plane to Γ at $\underline{x}$ is $\zeta = 0$, then in a neighborhood of $\underline{x}$, Γ can be represented by $\zeta = h(\xi_1, \cdots, \xi_k)$, with h a C_1 function.

These conditions are not independent, of course.

It follows from these conditions that Γ_t is of the same type as Γ, with τ replaced by $\tau - |t|$. Since the point $\underline{x}'$ on Γ_t nearest to the point $\underline{x}$ on Γ is $\underline{x}' = \underline{x} + t\,\underline{y}(\underline{x})$, $\underline{y}(\underline{x})$ is the normal to Γ_t at $\underline{x} + t\,\underline{y}(\underline{x})$.

It will be convenient to write simply $\underline{y}$ instead of $\underline{y}(\underline{x})$ in most contexts.

Let $d\sigma$ be the natural surface measure on Γ, and $|\Gamma| = \int_\Gamma d\sigma$. For a function f in $L^1(\Gamma)$, and $\underline{x}$ in E^{k+1}, let $Pf(\underline{x}) = c\int_\Gamma f(\underline{y})\,|\underline{x}-\underline{y}|^{1-k}d\sigma_y$, where $c = \frac{1}{2}\Gamma(\frac{k-1}{2})\Pi^{-\frac{k+1}{2}}$, and $|\underline{x}-\underline{y}|^2 = \sum_1^{k+1}(x_i-y_i)^2$. $Pf(\underline{x})$ is harmonic when $\underline{x}$ is not on Γ; and since locally on Γ, P is dominated by convolution (in local coordinates) with a function in L^1, $Pf(\underline{x})$ is defined almost everywhere on Γ. For each t in $0 < |t| \leqq \tau$ and $\underline{x}$ in Γ we define $P_t f(\underline{x})$ by $P_t f(\underline{x}) = Pf(\underline{x} + t\underline{y})$. Lemma 1 proves that $P_t f \longrightarrow P_o f$ in $L^1(\Gamma)$, and in $L^p(\Gamma)$ if f is in $L^p(\Gamma)$ $(1 \leqq p \leqq \infty)$; here $P_o f$ is the restriction to Γ of Pf.

An operator N, the normal derivative, can be defined on the range of P_o by $NP_o f(\underline{x}) = \lim_{t\to 0-} D_t P_t f(\underline{x})$, where D_t denotes differentiation with respect to t. Lemma 2 shows that there is an operator K_o (whose kernel is the adjoint of that for the double layer potential) such that $D_t P_t f \longrightarrow f + K_o f$ in $L^p(\Gamma)$ as $t \to 0-$; the convergence is uniform if f is continuous. Thus $NP_o f = I + K_o$. The lemma shows moreover that if f is bounded, then $K_o f$ is continuous; and that the kernel of K_o is uniformly $O(|x-y|^{1-k})$.

The operator P_o on $L^1(\Gamma)$ has trivial null space, as the following argument shows. Since $NP_o = I + K_o$, the null space of P_o is contained in that of $I + K_o$. This latter null space contains only continuous functions; for we can choose n so that the kernel of $K_o{}^n$ is bounded, so that if $f = -K_o f$, then $f = (-1)^n K_o{}^n f$ is bounded, from which it follows by lemma 2 that $f = -K_o f$ is continuous. Thus if $P_o f = 0$ for f in $L^1(\Gamma)$, then f is continuous, and by lemma 1 $P_o f$ is the boundary value in the classical sense of the function Pf in E^{k+1}. It follows from the maximum principle that $Pf(\underline{x}) = 0$ for all $\underline{x}$ in E^{k+1}. Now if $\phi(\underline{x})$ is any function in C_2 on E^{k+1}, with compact support, we have $\int_{\Gamma} \phi(\underline{x}) f(\underline{x}) d\sigma_x$ $= C' \int_{\Gamma} \int_{E^{k+1}} |x-y|^{1-k} \Delta \phi(\underline{y}) f(\underline{x}) d\underline{y}\, d\sigma_x = c'' \int_{E^{k+1}} \Delta \phi(\underline{y}) Pf(\underline{y}) d\underline{y} = 0$. It follows that $f(\underline{x}) \equiv 0$ on Γ.

We can also show that $NP_o f = 0$ only if $P_o f$ is constant. For we have seen that if $NP_o f = 0$, then f is continuous; thus lemmas 1 and 2 yield for $u(\underline{x}) = Pf(\underline{x})$ that $0 = \int_{\Gamma} u\, \partial u/\partial n = \lim_{t \to 0-} \int_{\Gamma_t} u\, \partial u / \partial n = \lim_{t \to 0-} \int_{G_t} |\nabla u|^2 \quad \int_G |\nabla u|^2$. This shows that $u = Pf$ is constant inside G, so that $P_o f$ is constant.

For f in $L^p(\Gamma)$ and g in $L^q(\Gamma)$ we have $(NP_o f, P_o g) = (P_o f, NP_o g)$; this can be verified by approximating f and g with continuous functions and applying the formula $\int_{\Gamma} \overline{(P_o g)}\, \partial P_o f / \partial n = \int_G \overline{(\nabla P_o g)} \cdot (\nabla P_o f) = \int_{\Gamma} P_o f \overline{(\partial P_o g / \partial n)}$, or by using the same formula on Γ_t and letting $t \to 0-$. Part of the same argument, with $f = g$, shows that $(NP_o f, P_o f) \geqq 0$.

If A is the averaging operator, i.e. $Af(\underline{x}) = |\Gamma|^{-1} \int_{\Gamma} f(\underline{y}) d\sigma_y$, then $ANP_o = 0$; for by lemma 2 $\int_{\Gamma} NP_o f = \lim_{t \to 0-} \int_{\Gamma_t} D_t Pf(\underline{x} + t\underline{y}) d\sigma_{x'}$, $(\underline{x}' = \underline{x} + t\,\underline{y})$, and the latter integral is $\int_{\Gamma_t} \partial u / \partial n$ for a function u harmonic in Γ_t and its interior. Now $(A + N)P_o$ has trivial null space. For if f is in $L^1(\Gamma)$ and $(A + N)P_o f = 0$, then $0 = (A^2 + AN)P_o f = AP_o f$, which gives also $NP_o f = 0$. This last equality implies $P_o f$ is constant, and $0 = AP_o f$ shows that this constant is zero. Thus $(A + N)P_o = I + K_o + AP_o = I + L$, where the kernel $L(\underline{x}, \underline{y})$ of L differs from the kernel of K_o by a continuous function of $\underline{y}$; and $I + L$ has trivial null space.

Now we can construct a right inverse J_o for $A + N$, using P_o as parametrix. Set $J_o = P_o + P_o F$, where F is an integral operator to be determined. $(A + N)J_o = I + L + F + LF$, so we seek F as a solution of $F(I + L) = -L$. F can be determined as an integral operator with kernel $F(\underline{x}, \underline{y})$ given by $F(\underline{x},\underline{y}) + \int_\Gamma L(\underline{x},\underline{z})F(\underline{z},\underline{y})d\sigma_z = -L(\underline{x},\underline{y})$. Since $I + L$ has trivial null space when acting on $L^1(\Gamma)$, this equation has, for each $\underline{y}$ in Γ, a unique solution $F(\underline{x},\underline{y})$ in $L^1(\Gamma)$. The kernel $F(\underline{x},\underline{y})$ constructed in this way has a singularity like that of $L(\underline{x},\underline{y})$, and so defines an operator F on $L^1(\Gamma)$ such that $F + LF = -L$, as desired.

From what we have shown about P_o, any function $P_o f$ in its range is the boundary value of the harmonic function $u = Pf$ in G, with $\partial u/\partial n = NP_o f$ on Γ, both boundary values being assumed in the L^p sense if f is in $L^p(\Gamma)$. Thus any function $J_o f$ in the range of $J_o = P_o(I+F)$ is the boundary value of a harmonic function $u = P(I+F)f$, and on Γ $\partial u/\partial n = NP_o(I+F)f = NJ_o f = f - AJ_o f$. Since $ANP_o = 0$, we find $Af - A^2 J_o f = 0$ or, since $A^2 = A$,

1) $AJ_o f = Af$.

Thus $\partial u/\partial n = f - Af$. The function $J(\underline{x},\underline{y}) = c|\underline{x}-\underline{y}|^{1-k} + c\int_\Gamma |\underline{x}-\underline{z}|^{1-k} F(\underline{z},\underline{y})d\sigma_z$ with $\underline{x}$ in E^{k+1} and $\underline{y}$ in Γ, is then the Neumann function for G.

The results we have just obtained can be formulated as follows.

<u>Theorem 1</u>. If f is in $L^p(\Gamma)$, $1 \leq p < \infty$, then there is a function $u(\underline{x})$ harmonic in G, given by $u = P(I+F)f$, such that $\int_\Gamma |D_t u(\underline{x}+t\,\underline{\nu}) - f(\underline{x}) + Af|^p d\sigma \to 0$, and $\int_\Gamma |u(\underline{x}+t\,\underline{\nu}) - J_o f(\underline{x})|^p d\sigma \to 0$, as $t \to 0-$. If f is bounded, $u(\underline{x}+t\,\underline{\nu}) \to J_o f(\underline{x})$ uniformly. If f is continuous, $D_t u(\underline{x}+t\,\underline{\nu}) \to f(\underline{x}) - Af$ uniformly.

In order to state our next result, the uniqueness result corresponding to theorem 1, in the best way, we need the formula

1a) $J_o Af = Af$,

or what amounts to the same thing, that $J_o \phi = \phi$ if ϕ is constant. Suppose ϕ constant, and let $u = P(I+F)\phi$; then by theorem 1 $\partial u/\partial n = \phi - A\phi = 0$. This boundary value is assumed uniformly, so by classical potential theory u is constant. Hence on Γ, $u = J_o\phi$ is constant. To evaluate the constant, we take averages and use (1) to obtain $J_o\phi = AJ_o\phi = A\phi = \phi$, which establishes (1a).

We can now establish the following converse of theorem 1.

<u>Theorem 2</u>. If u is harmonic in G, and f is a function in $L^p(\Gamma)$, $(1 \leqq p < \infty)$, such that $\int_\Gamma |D_t u(\underline{x}+t\underline{\nu}) - f(\underline{x})|^p d\sigma \to 0$ as $t \to 0-$, then there is a function g in $L^p(\Gamma)$ such that $\int_\Gamma |u(\underline{x}+t\underline{\nu}) - g(\underline{x})|^p d\sigma \to 0$ as $t \to 0-$, and $g = J_o(f + Ag)$.

The theorem is easy to establish if we assume that $\partial u/\partial n$ and u assume uniformly their respective boundary values f and g. For if $D_t u(\underline{x}+t\underline{\nu}) \to f(\underline{x})$ uniformly as $t \to 0-$, and for some g $u(\underline{x}+t\underline{\nu}) \to g(\underline{x})$ uniformly as $t \to 0-$, then the function $v(\underline{x}) = u(\underline{x}) - P(I+F)f(\underline{x})$ satisfies $\partial v/\partial n = 0$ on Γ, in the classical sense, so v is constant. Since f is $\partial u/\partial n$, we have $\int_\Gamma f = 0$, and by (1), $\int_\Gamma P(I+F)f = \int_\Gamma J_o f = \int_\Gamma f = 0$ Then we can evaluate the constant value of v by averaging it on Γ, which leads to $v = Ag$, $u = P(I+F)f + Ag$, and $g = J_o f + Ag = J_o(f + Ag)$.

We can obtain the general result by applying this special case to the manifolds Γ_t for $t < 0$. To this end, let $\underline{x}'$ and $\underline{y}'$ be on Γ_t, $-\tau \leqq t < 0$, and let L_t, F_t, and J_t be the operators on $L^1(\Gamma_t)$ with kernels

$$L_t(\underline{x}',\underline{y}') = c|\Gamma_t|^{-1}\int_{\Gamma_t} |\underline{x}'-\underline{y}'|^{1-k} d\sigma_{x'}$$

$$+ (1-k)c(\underline{x}'-\underline{y}')\cdot\nu(\underline{x}')|\underline{x}'-\underline{y}'|^{1-k} \tag{2}$$

$$F_t(\underline{x}',\underline{y}') = -L_t(\underline{x}',\underline{y}') - \int_{\Gamma_t} L_t(\underline{x}',\underline{z}')F_t(\underline{z}',\underline{y}')d\sigma_{z'}$$

$$J_t(\underline{x}',\underline{y}') = c|\underline{x}'-\underline{y}'|^{1-k} + c\int_{\Gamma_t} |\underline{x}'-\underline{z}'|^{1-k} F_t(\underline{z}',\underline{y}')d\sigma_{z'};$$

these are for Γ_t what L, F, and J_o are for Γ. Then if $\Delta u = 0$ in G, it follows that u and $\partial u/\partial n$ assume their boundary values on Γ_t uniformly, so that by our reliminary result we have on Γ_t

(3) $u(\underline{x}') - |\Gamma_t|^{-1} \int_{\Gamma_t} u(\underline{x}')d\sigma_{x'} = \int_{\Gamma_t} J_t(\underline{x}',\underline{y}')(\partial u/\partial n)(\underline{y}')d\sigma_{y'}.$

ow let $v_t(x)$ be such that $\int_{\Gamma_t} \phi(\underline{x}')d\sigma_{x'} = \int_{\Gamma} \phi(\underline{x} + t\underline{y})v_t(\underline{x})d\sigma_x$ for every continuous unction ϕ. Then $v_t(\underline{x}) \to 1$ uniformly in $\underline{x}$ as $t \to 0$. Since $\underline{y}(\underline{x})$ is the ormal to Γ_t at $\underline{x} + t\underline{y}(\underline{x})$, $\partial u/\partial n$ on Γ_t is $D_t u(\underline{x} + t\,\underline{y}(\underline{x}))$; thus relation (3) n Γ_t can be written

(4) $u(\underline{x} + t\underline{y}(\underline{x})) - |\Gamma_t|^{-1} \int_{\Gamma} u(\underline{x} + t\underline{y}))v_t(\underline{x})d\sigma$

$$= \int_{\Gamma} J_t(\underline{x} + t\,\underline{y}(\underline{x}),\underline{y} + t\underline{y}(\underline{y}))D_t u(\underline{y} + t\,\underline{y}(\underline{y}))v_t(\underline{y})d\sigma_y.$$

n order to obtain a limiting relation as $t \to 0-$, define the operator J^t on $L^1(\Gamma)$ y the kernel

(5) $J^t(\underline{x},\underline{y}) = J_t(\underline{x} + t\underline{y}(\underline{x}), \underline{y} + t\,\underline{y}(\underline{y}))v_t(\underline{y})$;

efine A^t on $L^1(\Gamma)$ by $A^t f = |\Gamma_t|^{-1}\int_{\Gamma} f v_t d\sigma$; and define u_t on Γ by $u_t(\underline{x}) = u(\underline{x} + t\,\underline{y}(\underline{x}))$. Then according to (4),

(6) $u_t - A^t u_t = J^t D_t u_t$ for $-\tau \leqq t < 0$.

Now by lemma 3, $\|J^t - J_o\| \to 0$ as $t \to 0-$, and the hypothesis of theorem 2 is that $\|D_t u_t - f\| \to 0$, where $\|\cdot\|$ denotes the norm of an operator on, or a function in, $L^P(\Gamma)$. It follows that $J^t D_t u_t \to J_o f$ in $L^1(\Gamma)$, and that $u_t - A^t u_t$ does the same.

Thus it remains only to show that there is a function g in $L^P(\Gamma)$ such that $u_t \to g$; it will then follow that $A^t u_t \to Ag$, and $g - Ag = J_o f$, or $g = J_o(f + Ag)$, as desired. To find g, consider first $I(t) = \int_{\Gamma} u_t$. Then $I'(t) = \int_{\Gamma} D_t u_t \to \int_{\Gamma} f$, so that $I'(t)$ is bounded on $-\tau \leqq t < 0$, and $\lim_{t \to 0-} I(t)$ exists. Now from (6), $\lim_{t \to 0-} |\Gamma|^{-1} \int_{\Gamma} (u_t - A^t u_t) = |\Gamma|^{-1} \int_{\Gamma} J_o f$; we have just seen that $\lim \int_{\Gamma} u_t$ exists, so it follows that $\lim_{t \to 0-} A^t u_t$ exists. Finally, we obtain that $u_t = A^t u_t + J^t D_t u_t$ converges in $L^1(\Gamma)$ to $g = \lim_{t \to 0-} A^t u_t + J_o f$. This concludes the proof of theorem 2.

2. <u>More general boundary value problems.</u>

Theorems 1 and 2 are existence and uniqueness theorems respectively for the L^p formulation of the Neumann problem. In this section we consider more general first order boundary conditions, but require that $\partial u/\partial n$ exist on Γ in the L^p sense, for some $p \geqslant 1$. This is reasonable if the boundary operator is actually of first order at all points of Γ.

Specifically, we look for a function u such that

i) $\Delta u = 0$ in G,

β) ii) $D_t u(\underline{x} + t\,\underline{y})$ converges in $L^p(\Gamma)$ as $t \to 0-$, for some $1 < p < \infty$, and

iii) $Bu = h$ on Γ,

where B is a first order differential operator with continuous coefficients, defined in a neighborhood of Γ; and h is in $L^p(\Gamma)$. In order to apply the theory of [3], we shall assume Γ is of class C_3. Write $Bu = a\partial u/\partial n + Vu + bu$, where a and b are continuous functions on Γ, and V is a continuous vector field on Γ. Suppose u is any solution of (β), and denote the boundary value $\partial u/\partial n$ by f. Then, by theorem 2, there is a g in $L^p(\Gamma)$ such that $u(\underline{x} + t\,\underline{y}) \to g$ as $t \to 0-$, and $g = J_o(f + Ag)$. Writing ϕ for $f + Ag$, we have $\partial u/\partial n = \phi - A\phi$. Thus in the boundary condition $Bu = h$ we can replace u by $J_o\phi$, and obtain the equation $a\partial u/\partial n + Vu + bu = a(\phi - A\phi) + VJ_o\phi + bJ_o\phi = h$. Now it is easy to show (as in [1], section II D), that VP_o is a C_o^∞ singular integral operator on Γ with symbol $i\{V, \xi/|\xi|\}$, where V is thought of as a cross-section of the tangent bundle of Γ, ξ is a vector in the cotangent bundle of Γ, and $\{\ ,\ \}$ is the bilinear form relating the two. Moreover the operators F, bJ_o, and aA are completely continuous. Thus $H = aI + VP_o + (VP_oF + bJ_o - aA)$ is a C_o^∞ singular integral operator with symbol $a + i\{V, \xi/|\xi|\}$; and the solutions u of the problem (β) are isomorphic to the solutions ϕ of the singular integral equation

7) $H\phi = h$

by the isomorphism $u = P(I+F)\phi$.

The difference between the number of orthogonality relations that must be satisfied by h to guarantee a solution of (β), and the dimension of the space of solutions of the associated homogeneous problem: $\Delta u = 0$ in G and $B = 0$ on Γ, is called the index of the problem (β). We obtain the following result.

Theorem 3. If $a + i\{V, \xi/|\xi|\}$ does not vanish at any point on the cotangent bundle of Γ, and the dimension of Γ is $\geqq 2$, then the index of the problem (β) is zero.

Proof. With these assumptions, theorem 3 of [3] asserts that the Fredholm alternative holds for the equation (7), and hence for the original boundary value problem. Q.E.D.

Remark 1. The hypothesis of theorem 3 implies that a is non-vanishing on Γ, i.e. that B is nowhere tangential. If B is real, the condition that $a + i\{V, \xi/|\xi|\}$ vanishes nowhere is equivalent to the condition that B is nowhere tangential.

Remark 2. The condition that $a + i\{V, \xi/|\xi|\}$ vanishes nowhere is equivalent to the condition that the boundary operator B covers the Laplacian, in the sense of Schechter (see [1]).

3) The lemmas.

In this section $|||T|||$ denotes the supremum for $1 \leqq p \leqq \infty$ of the norm of the operator T on $L^p(\Gamma)$, $1 \leqq p \leqq \infty$. This norm makes the set of operators T for which $|||T|||$ is finite a Banach algebra. The operators P_o and P_t are described early in section 1.

Lemma 1. $|||P_o - P_t||| \to 0$ as $t \to 0$. If f is bounded, $P_t f \to P_o f$ uniformly, and $P_o f$ is continuous.

Proof. The kernel of $P_o - P_t$ is $c|\underline{x}-\underline{y}|^{1-k} - c|\underline{x} + t\,\underline{\nu}(\underline{x}) - \underline{y}|^{1-k} = cQ_t(\underline{x},\underline{y})$, where $\underline{x}$ and $\underline{y}$ are points on Γ, and $|\underline{x}-\underline{y}|$ is the distance from $\underline{x}$ to $\underline{y}$ in E^{k+1}. Let $\rho(\underline{x},\underline{y}) = (\underline{x}-\underline{y})/|\underline{x} + t\underline{\nu} - \underline{y}|$ $(\underline{\nu} = \underline{\nu}(\underline{x}))$. Then a simple argument, based on the fact that $\underline{y}$ lies outside the sphere of radius τ about $x \pm \tau\underline{\nu}$, shows that $0 \leqq \rho \leqq 2$ for $0 \leqq |t| \leqq \tau$. It follows from this that $|1-\rho| \leqq 1$, and also $|1-\rho| \leqq |t|/|\underline{x} + t\underline{\nu} - \underline{y}| \leqq 2|t|/|\underline{x}-\underline{y}|$. Thus for the kernel of $P_o - P_t$ we have $cQ_t(\underline{x},\underline{y})$, and $|Q_t(\underline{x},\underline{y})| = |\underline{x}-\underline{y}|^{1-k}|1-\rho^{k-1}| \leqq 2^{k-1}|\underline{x}-\underline{y}|^{1-k}|1-\rho|$, or

i) $|Q_t(\underline{x},\underline{y})| \leqq 2^{k-1}|\underline{x}-\underline{y}|^{1-k} \min(1, 2|t|/|\underline{x}-\underline{y}|)$.

Now let $\phi(s)$ be the characteristic function of $0 \leqq s \leqq 1$, and Set $R_t(\underline{x},\underline{y}) = \phi(|\underline{x}-\underline{y}||t|^{-1/2k})Q_t(\underline{x},\underline{y})$, and $S_t(\underline{x},\underline{y}) = Q_t(\underline{x},\underline{y}) - R_t(\underline{x},\underline{y})$. Then $S_t(\underline{x},\underline{y})$ vanishes for $|\underline{x}-\underline{y}| \leqq |t|^{1/2}$, so by (i) $|S_t(\underline{x},\underline{y})| \leqq 2^{k-1}|t|\ |\underline{x}-\underline{y}|^{-k} \leqq 2^{k-1}|t|^{1/2}$, and $S_t(\underline{x},\underline{y})$ defines an operator S_t with $|||S_t||| = O(|t|^{1/2})$. Using (i) again, we find that on each coordinate neighborhood $R_t(\underline{x},\underline{y})$ is dominated by a convolution kernel (in local coordinates), and this dominating kernel comes from a function in $L^1(E^k)$ whose norm is $O(|t|^{1/2\ k})$. Thus $|||P_o - P_t||| = O(|t|^{1/2\ k})$ as $t \to 0$.

$P_t f$ is continuous for f in $L^1(\Gamma)$ and $t \neq 0$. If f is in L^∞ as well, the above result shows that $P_t f$ converges uniformly to $P_o f$, and hence that $P_o f$ is continuous. This establishes lemma 1.

Remember that D_t denotes differentiation with respect to t.

<u>Lemma 2</u>. If f is in $L^p(\Gamma)$, $1 \leqq p < \infty$, then $D_t P_t f \to K_o f + f$ in L^p norm, as $t \to 0-$, where $K_o f(\underline{x}) = \int_\Gamma K_o(\underline{x},\underline{y}) f(\underline{y})\, d\sigma_y$, and for $\underline{x} \neq \underline{y}$ $K_o(\underline{x},\underline{y}) = D_t c|\underline{x} + t\underline{\nu}(\underline{x}) - \underline{y}|^{1-k}$ evaluated at $t = 0$. $K_o(\underline{x},\underline{y})$ is uniformly $O(|\underline{x}-\underline{y}|^{1-k})$, and on $\Gamma \times \Gamma$ is a continuous function for $x \neq y$. If f is continuous, $D_t P_t f$ converges uniformly to $K_o f + f$. If f is bounded, $K_o f$ is continuous.

Proof. For $t \neq 0$, $D_t P_t f(\underline{x}) = (1-k)c \int_\Gamma (\underline{x} + t\underline{y} - \underline{y}) \cdot \underline{y} |\underline{x}+t\underline{y}-\underline{y}|^{-k-1} f(y) d\sigma_y$, where $\underline{\alpha} \cdot \underline{\beta}$ is the inner product of $\underline{\alpha}$ and $\underline{\beta}$ in E^{k+1}. The kernel here is $K_t(\underline{x},\underline{y}) + B_t(\underline{x},\underline{y})$, with $K_t(\underline{x},\underline{y}) = (1-k)c(\underline{x}-\underline{y}) \cdot \underline{y} |\underline{x}+t\underline{y}-\underline{y}|^{-k-1}$, and $B_t(\underline{x},\underline{y}) = (1-k)ct(\underline{x}+t\underline{y}-\underline{y}|^{-k-1}$. Since Γ lies between the spheres of radius τ with centers at $\underline{x} \pm \tau \underline{y}$, we have for $\underline{x}$ and $\underline{y}$ on Γ and $|\underline{x}-\underline{y}| < \tau$ that $|(\underline{x}-\underline{y}) \cdot \underline{y}| \leq |\underline{x}-\underline{y}|^2/\tau$. Thus $K_t(\underline{x},\underline{y})$ is $O(|\underline{x}-\underline{y}|^{1-k})$ uniformly in $\underline{x}$, $\underline{y}$, and t, and an argument like that of lemma 1 shows that if K_t and K_0 are the operators with kernels $K_t(\underline{x},\underline{y})$ and $K_0(\underline{x},\underline{y})$, then $|||K_t - K_0||| \to 0$ as $t \to 0$. As a consequence, if f is bounded then $K_0 f$ is continuous, as the uniform limit of the continuous functions $K_t f$.

We show next that $B_t(\underline{x},\underline{y})$ satisfies (i): for each $\delta > 0$, $B_t(\underline{x},\underline{y}) \to 0$ uniformly in $\underline{x}$ and $\underline{y}$, for $|\underline{x}-\underline{y}| > \delta$, as $t \to 0$; and (ii): $\int_\Gamma B_t(\underline{x},\underline{y}) d\sigma_y \to 1$ uniformly as $t \to 0-$ (and to -1 as $t \to 0+$). Claim (i) is clear, so we turn to (ii). Introduce orthonormal coordinates $(\xi_1, \cdots, \xi_k, \zeta)$ in E^{k+1}, with origin at the point $\underline{x}$ in Γ, so that $\zeta = 0$ is tangent to Γ at $\underline{x}$; and choose α so that $0 < \alpha < \tau$, and for $|\underline{x}-\underline{y}| < \alpha$ Γ can be represented by $\zeta = h(\xi_1, \cdots, \xi_k) = h(\underline{\xi})$. For simplicity of notation, we let $\underline{\xi}$ stand for the k-tuple such that $(\underline{\xi}, h(\underline{\xi})) = (\xi_1, \cdots, \xi_k, h(\underline{\xi}))$ are the coordinates of the point $\underline{y}$ in $\Gamma \cap \{|\underline{x}-\underline{y}| < \alpha\}$. Then referring again to the two spheres about $x \pm \tau \underline{y}$, we have the rough estimate $(\tau-|\underline{\xi}|)/(\tau+|\underline{\xi}|) \leq |\underline{x} + t\underline{y} - \underline{y}|^2/(|\underline{\xi}|^2 + t^2) \leq (\tau+|\underline{\xi}|)/(\tau-|\underline{\xi}|)$, where $|\underline{\xi}|^2 = \sum_1^k \xi_i^2$. Also, in $\{|x-y| < \alpha\} \cap \Gamma$ we have $d\sigma_y = v_x(\underline{\xi}) d\underline{\xi}$, and $v_x(\underline{\xi}) \to 1$ uniformly in $\underline{x}$, as $\underline{\xi} \to 0$. It is not hard to show that, for $t < 0$, $(1-k)c \int_{E^k} t(|\xi|^2 + t^2)^{-(k+1)/2} d\,\underline{\xi} = 1$, and hence that for any $\delta > 0$ $(1-k)c \int_{|\xi| < \delta} t(|\xi|^2 + t^2)^{-(k+1)/2} d\underline{\xi} \to 1$, as $t \to 0-$. Thus for any positive $\eta < 1$ we can choose a positive $\delta < \alpha$ so that for all $\underline{x}$ and $|\underline{\xi}| < \delta$ we have $\eta^2 < |\underline{x} + t\underline{v} - \underline{y}|^2/(|\underline{\xi}|^2 + t^2) < \eta^{-2}$ and $\eta < v_x(\underline{\xi}) < \eta^{-1}$; and then we can choose $t < 0$ so small that $\eta < (1-k)ct \int_{|\underline{\xi}|<\delta} (|\underline{\xi}|^2 + t^2)^{-(k+1)/2}\, \underline{\xi} \leq 1$. Now let U_x be the neighborhood of $\underline{x}$ in Γ projecting onto $|\xi| < \delta$. Then $\int_{\Gamma - U_x} B_t(\underline{x},\underline{y})\, d\sigma_y \to 0$ uniformly in $\underline{x}$ as $t \to 0$; and with $t < 0$ we have for $\int_{U_x} B_t(\underline{x},\underline{y})\, d\sigma_y = \int_{|\underline{\xi}|<\delta} (1-k)ct|\underline{x}+t\underline{v}-\underline{y}|^{-k-1} v_x(\underline{\xi})\, d\underline{\xi}$ that $\eta^{k+3} < \int_{U_x} B_t(\underline{x},\underline{y})\, d\sigma_y < \eta^{-k-2}$. Thus property (ii) of $B_t(\underline{x},\underline{y})$ is established.

Now $B_t(\underline{x},\underline{y})$ is clearly positive for $t < 0$, hence has the essential properties of an approximate identity. By a standard argument, $B_t f \to f$ uniformly for any continuous function f. Then another standard argument, relying on the fact that continuous functions are dense in $L^p(\Gamma)$ for $1 \leq p < \infty$, shows that for f in L^p, $1 \leq p < \infty$, $B_t f \to f$ in L^p norm. This, together with the results for K_t, establishes lemma 2.

It is easy to see that $\lim_{t \to 0+} D_t P_t f = -f + K_o f$. This with lemma 2 is a formulation of the "jump" relations for the adjoint of the double layer potential.

Lemma 3. If J^t is the operator on $L^p(\Gamma)$ $(1 \leq p \leq \infty)$ defined by the kernel (5), then for each p $\|J^t - J_o\|_p \to 0$ as $t \to 0$.

Proof. With reference to the kernels defined in (2), set $L^t(\underline{x},\underline{y}) = L_t(\underline{x}',\underline{y}')v_t(\underline{y})$, $F^t(\underline{x},\underline{y}) = F_t(\underline{x}',\underline{y}')v_t(y)$, and $P^t(\underline{x},\underline{y}) = c|\underline{x}'-\underline{y}'|^{1-k} v_t(\underline{y})$, where $\underline{x}' = \underline{x} + t\underline{v}(\underline{x})$, $\underline{y}' = \underline{y} + t\underline{v}(\underline{y})$, and $\int_{\Gamma_t} \phi(\underline{y}')\, d\sigma_{y'} = \int_{\Gamma} \phi(\underline{y} + t\underline{v}(\underline{y}))v_t(\underline{y})\, d\sigma_y$ for all continuous ϕ. Then $F^t(\underline{x},\underline{y}) = -L^t(\underline{x},\underline{y}) - \int_{\Gamma} L^t(\underline{x},\underline{z})F^t(\underline{z},\underline{y})\, d\sigma_z$ and $J^t(\underline{x},\underline{y}) = P^t(\underline{x},\underline{y}) + \int_{\Gamma} P^t(\underline{x},\underline{z})F^t(\underline{z},\underline{y})\, d\sigma_z$. We will achieve the desired result by showing in turn that $|||P^t - P_o||| \to 0$, $|||L^t - L||| \to 0$, $\|F^t - F\|_p \to 0$, and finally $\|J^t - J_o\|_p \to 0$. The first