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On the non-stationary behaviour of a rotating shallow sea under influence of a windfield

by

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§1. Introduction.

The flood disaster of February 1953, when parts of the low lands were flooded by the North Sea, stimulated mathematical research in order to obtain a better insight in the physical background of storm surges in the North Sea.

At the Amsterdam "Mathematisch Centrum" a research group under Prof.Dr D. van Dantzig started an extensive study on the statistical and hydrodynamical aspects of storm surges in the North Sea. This paper and the accompanying paper of G.W. Veltkamp concern some aspects of the work carried out by the applied mathematics division of the "Mathematisch Centrum".

The general problem of the hydrodynamic behaviour of a shallow sea subjected to a storm can be attacked only by means of considerable simplifications: linearisation of the hydrodynamic equations, neglection of vertical motions, etc.

The following equations will be used

$$\begin{aligned} &\frac{\partial u}{\partial t} + \lambda u - \Omega v + g\frac{\partial \zeta}{\partial x} = \frac{U}{\rho h} \\ &\frac{\partial v}{\partial t} + \lambda v + \Omega u + g\frac{\partial \zeta}{\partial y} = \frac{V}{\rho h} \qquad 1.1 \\ &\frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} + \frac{\partial \zeta}{\partial t} = 0. \end{aligned}$$

Here g is the acceleration of gravity, λ a friction coefficient, Ω the coefficient of Coriolis, h the depth, u and v averages over a vertical of the horizontal components of the velocity, ζ the elevation of the sealevel above the undisturbed level, U and V the components of the tangential stress on the surface of the sea due to the wind.

The North Sea is usually considered as a rectangle $|x| < a$, $0<y<b$ where $x=\pm a$, $y=0$ represent coasts and $y=b$ the open end at the ocean. The depth is taken constant. The ocean may be considered infinitely deep and the following boundary conditions hold

$$x = \pm a, \quad u=0 \;; \quad y=0, \; v=0 \;; \quad y=b, \; \zeta=0.$$

The numerical values are approximately

$b=850$ km, $h=65$ m (harmonic average), $\lambda=0.09\ \mathrm{hr}^{-1}$, $\Omega=0.48\ \mathrm{hr}^{-1}$.

Next, if W represents the velocity of the wind at sealevel

$$\frac{\sqrt{U^2+V^2}}{\varrho} = 3.5 \times 10^{-6}\ W^2$$

in corresponding units.

In this paper, however, a simpler model will be discussed. We shall consider a uniformly time-dependent wind upon an infinitely wide "North Sea" i.e. the strip $-\infty < x < \infty$, $0<y<b$. Moreover only wind will be considered the direction of which is perpendicular to the coast i.e. $U=0$, $V=V(t)$. Under these conditions there are no variations in the stream and the elevation in the x-direction and hence the terms in 1.1 with $\frac{\partial}{\partial x}$ disappear. If next the following units are introduced in 1.1

t	$b/2\pi\sqrt{gh}$ hr	; y	$b/2\pi$ km
u,v	$\sqrt{gh}$ km/hr	; ζ	h m

the equations 1.1 become

$$\left(\frac{\partial}{\partial t}+\lambda\right)u - \Omega v = 0$$
$$\left(\frac{\partial}{\partial t}+\lambda\right)v + \Omega u + \frac{\partial\zeta}{\partial y} = V \qquad 1.2$$
$$\frac{\partial v}{\partial y} + \frac{\partial\zeta}{\partial t} = 0$$

with the boundary conditions

$$y=0 \quad v=0 \quad ; \quad y=2\pi \quad \zeta=0. \qquad 1.3$$

In the numerical case the units are as follows

t	1.5 hr	u,v	91 km/hr
y	135 km	ζ	65 m

and $\lambda = 0.14$, $\Omega = 0.71$.

If W is in m/sec, then $|V| = 1.1 \times 10^{-5}\ W^2$.

§2. General solution.

If upon the equations 1.2 Laplace transformation is applied,

$$\bar{\zeta}(y,p) = \int_{-\infty}^{\infty} e^{-pt}\,\zeta(y,t)\,dt \quad \text{etc.} \qquad 2.1$$

we obtain the equations

$$(p+\lambda)\bar{u} - \Omega\bar{v} = 0$$
$$(p+\lambda)\,\bar{v} + \Omega\,\bar{u} + \frac{\partial\bar{\zeta}}{\partial y} = \bar{V} \qquad 2.2$$
$$\frac{\partial\bar{v}}{\partial y} + p\bar{\zeta} = 0.$$

The boundary conditions are obviously

$$y=0 \quad \bar{v}=0 \quad ; \quad y=2\pi \quad \bar{\zeta}=0.$$

From the system 2.2 we may derive

$$\frac{\partial^2 \bar{\zeta}}{\partial y^2} - q^2 \bar{\zeta} = 0 \qquad 2.3$$

where

$$q^2 = p(p+\lambda) + \Omega^2 \frac{p}{p+\lambda} . \qquad 2.4$$

The boundary conditions of 2.3 are

$$y=0 \qquad \frac{\partial \bar{\zeta}}{\partial y} = \bar{V}$$

$$y=2\pi \qquad \bar{\zeta} = 0.$$

The solution of 2.3 satisfying both boundary conditions is

$$\bar{\zeta}(y,p) = - \bar{V} \frac{\mathrm{sh}(2\pi - y)q}{q \ \mathrm{ch}\ 2\pi q} . \qquad 2.5$$

By means of the inversion theorem of the Laplace transformation we have

$$\zeta(y,t) = - \frac{1}{2\pi i} \int_L e^{pt} \frac{\mathrm{sh}(2\pi - y)q}{q \ \mathrm{ch}\ 2\pi q} \bar{V} \, dp \qquad 2.6$$

where L is a vertical $(\sigma - i\infty , \sigma + i\infty)$ with Re $\sigma > 0$.

In the following sections the following cases will be studied.

a free motions

b a constant wind starting at t=0

$$V=0 \quad t<0 \quad ; \quad V=-1 \quad t>0.$$

c periodic motions of the form

$$\zeta(y,t) = Z(y)e^{i\omega t}$$

d a periodic wind starting at t=0

$$V=0 \quad t<0 \quad ; \quad V= -\sin\omega t \quad t>0.$$

§3. Free motions.

The free motions or the eigenfunctions of the system 2.2 are of the form

$$e^{pt} \cos(\tfrac{k}{2} + \tfrac{1}{4})y \qquad k=0,1,2 \ldots \qquad 3.1$$

where p is determined by

$$q^2 = -(\tfrac{k}{2} + \tfrac{1}{4})^2 \qquad k=0,1,2 \ldots$$

Thus for each k a triplet of eigenvalues are obtained which are the roots of

$$p^3 + 2\lambda p^2 + \left\{ \lambda^2 + \Omega^2 + (\tfrac{k}{2} + \tfrac{1}{4})^2 \right\} p + \lambda (\tfrac{k}{2} + \tfrac{1}{4})^2 = 0 \qquad 3.2$$

In the numerical case $\lambda^2=0.02$, $\Omega^2=0.5$ the first few eigenvalues are

k= 0	-0.0153 ,	-0.134	$\pm$ i 0.749
1	-0.074 ,	-0.104	$\pm$ i 1.028
2	-0.107 ,	-0.088	$\pm$ i 1.434
3	-0.122 ,	-0.080	$\pm$ i 1.886.

The real roots form a decreasing sequence converging to $-\lambda$.
The complex roots are situated left from the line Re $p = -\frac{\lambda}{2}$.
The real parts converge to $-\frac{\lambda}{2}$, the imaginary parts are approximately $\pm(\frac{k}{2}+\frac{1}{4})$.
The lowest real eigenvalue α_o may be approximated by

$$\alpha_o \backsim \frac{-\lambda}{1+16\,\Omega^2} \,. \qquad 3.3$$

§4. The elevation at the coast due to a constant wind.

If at t=0 the sea is at rest and a sudden constant wind V=-1 starts we obtain from 2.6 for the elevation

$$\zeta(y,t) = \frac{1}{2\pi i}\int_L e^{pt}\,\frac{sh(2\pi-y)q}{p\,q\,ch\,2\pi q}\,dp \qquad 4.1$$

The right-hand side may be evaluated by means of the calculus of residues. The pole at p=0 gives the stationary solution

$$\zeta(y,\infty) = 2\pi - y \qquad 4.2$$

In the numerical case we obtain

$$\zeta(y,t) = (2\pi-y) - 4.56\,e^{-0.0153t}\cos\tfrac{1}{4}y - 0.28\,e^{-0.074t}\cos\tfrac{3}{4}y\ldots \qquad 4.3$$

However, for small t we may use the expansion in rising powers of t. In particular for y=0 we have

$$\zeta(0,t) = e^{-0.14t}\,(t + 0.106t^2 - 0.035t^3 - \ldots) \qquad 4.4$$

From 4.3 and 4.4 the following table is obtained (figure 1)

t	$\zeta(0,t)$	t	$\zeta(0,t)$
t=0	$\zeta(0,t)$=0	t=20	$\zeta(0,t)$= 2.88
1	0.93	30	3.38
2	1.61	40	3.79
5	2.01	50	4.15
10	2.17	∞	2π

Thus the elevation tends very slowly towards its stationary value. At t=30, after about two days, it is still at 54% of its ultimate value. A wind of 30m/sec causes an ultimate elevation at the coast of 4 meters.

§5. Periodic motions.

If $V=-\sin \omega t$ for all t there is a solution of the form

$$\zeta(y,t) = \mathrm{Im}\{Z(y)e^{i\omega t}\} \qquad 5.1$$

According to 2.6 we have

$$Z(y) = \frac{\mathrm{sh}(2\pi-y)q}{q\ \mathrm{ch}\ 2\pi q}$$

where

$$q^2 = \omega i(\lambda+\omega i) + \Omega^2 \frac{\omega i}{\lambda+\omega i}$$

The maximum elevation at the coast is given by

$$M(\omega) = \left|\frac{\mathrm{th}\ 2\pi q}{q}\right|$$

In the numerical case $\lambda^2=0.02$, $\Omega^2=0.5$ we have computed $Z(0)$, $|Z(0)|$ and $\arg Z(0)$. (figures 2,3,4)

ω	$Z(0)$	$\lvert Z(0)\rvert$	$\arg Z(0)$
0	2π	2π	0
0.04	2.21 - 1.67 i	2.78	-37°
0.1	1.62 - 0.90 i	1.85	-29°
0.2	1.49 - 0.57 i	1.64	-20°
0.5	1.72 - 0.68 i	1.85	-22°
0.7	1.56 - 1.72 i	2.32	-48°
0.8	0.67 - 1.67 i	1.80	-68°
0.9	0.70 - 1.30 i	1.48	-62°
1.0	0.52 - 1.77 i	1.85	-74°

$M(\omega)$ appears to have an absolute maximum 2π at $\omega=0$ and a secondary maximum at $\omega=0.69$ where it is only 2.34. This represents the resonance with the first eigenfunction which has the period 0.75 and the Coriolis effect for which $\Omega=0.71$. In the case of the North Sea the dangerous storms extend over a period of about two days, so that ω should be of the order 0.1. Since $M(\omega)$ is some measure of the effect of the storm upon the coast we see that storms of longer duration are more dangerous.

A typical case is given by $\omega=0.1$. In this case we have (figure 5)

$$\zeta(y,t) = M(y)\sin\{0.1t + \theta(y)\}$$

with

y	M	θ
0	1.85	-29°
$1/3\,\pi$	1.13	-45°
$2/3\,\pi$	0.69	-60°
π	0.42	-73°
$4/3\,\pi$	0.24	-85°
$5/3\,\pi$	0.11	-89°
2π	0	-90°

§6. The elevation at the coast due to a periodic wind.

If initially the sea is at rest and if a sinusoidal wind starts

$$V=0 \quad t<0 \quad , \quad V=-\sin\omega t \quad t>0$$

the elevation may be obtained from

$$\bar{\zeta} = \frac{\omega}{p^2+\omega^2}\,\frac{\mathrm{sh}(2\pi-y)q}{q\,\mathrm{ch}\,2\pi q} \qquad 6.1$$

The poles at $p=\pm i\omega$ give the quasi-stationary solution which has been considered in the preceding section.

In the numerical case $\lambda^2=0.02$, $\Omega^2=0.5$ we have, provided $\omega \gg 0.01$

$$\zeta(y,t) \backsim \mathrm{Im}\left\{Z(y)e^{i\omega t}\right\} + \frac{0.069}{\omega}\,e^{-0.0153t} \qquad 6.2$$

In particular for y=0 and $\omega=0.1$ some results are given below (figure 6)

t=	$\zeta(0,t)=$	dev. =
=0	0	0.90
2π	0.96	0.73
4π	1.89	0.63
6π	2.36	0.54
8π	2.17	0.49
10π	1.33	0.44
12π	0.17	0.39
16π	-1.50	0.32
20π	-0.64	0.26

The column with dev. gives the difference between the actual value of $\zeta(0,t)$ and its quasi-stationary value. The maximum value of $\zeta(0,t)$ is reached at about $t=6.5\pi$ i.e. 1.5π later than the moment of maximum wind intensity. The shape of the actual motion follows very closely that of the quasi-stationary motion.

For a storm with a peak value of 35 m/sec we obtain in this way a maximum elevation of 2.05 m occurring about 7 hours later than the time of maximum wind intensity. The quasi-stationary motion would have given a maximum elevation of 2.55 m.

§7. A half-plane sea.

In order to appreciate the influence of the ocean a comparison should be made with the model of a half-plane sea $y>0$. In this case the windfield is assumed to extend over the whole area.

The equations are as in §2. But here we have the solution

$$\zeta(y,t) = -\bar{V}\,\frac{e^{-qy}}{q} \qquad 7.1$$

with q given by 2.4.

The quasi-stationary motion generated by the windfield $V=-\sin\omega t$ becomes

$$\zeta(0,t) = \mathrm{Im}\,(Z\,e^{i\omega t}) \qquad 7.2$$

with

$$Z = \frac{(\lambda + \omega i)^{\frac{1}{2}}}{(\omega i)^{\frac{1}{2}} \{ (\lambda + \omega i)^2 + \Omega^2 \}^{\frac{1}{2}}} \qquad 7.3$$

For a number of ω values the values of $|Z|$ are given below together with the corresponding values of the ocean case

	$\vert Z\vert$ halfplane	$\vert Z\vert$ strip
ω = 0	2π	2π
0.1	1.84	1.85
0.2	1.59	1.64
0.5	1.85	1.85
0.7	2.26	2.32
0.9	1.62	1.48

We observe here the same resonance at about $\omega = \Omega$.
If ω is of the order 0.1 the results from the two models differ only very slightly. Thus for long storms the influence of the ocean may be expected to be very small.

If the sea is initially at rest and if at t=0 the windfield $-\sin \omega t$ starts we obtain for the particular case ω=0.1

	$\zeta(0,t)$ halfplane	$\zeta(0,t)$ strip
t= 0	0	0
2π	0.96	0.96
4π	1.89	1.89
6π	2.37	2.36
8π	2.18	2.17
10π	1.37	1.33

This confirms the assertion given above.

§8. Final remark.

The results obtained above may have some value in connection with the motion of the North Sea under influence of a homogeneous windfield. This model clearly demonstrates the importance of the Coriolis effect. However, in the case of the North Sea which is represented by a rectangle, the long sides will lessen this effect appreciably. For more details and for related topics the reader is referred to the reports of the "Mathematisch Centrum" of Amsterdam.