STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM
AFDELING TOEGEPASTE WISKUNDE

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A flux theorem.

by

D. J. Hofsommer

A FLUX THEOREM

by D. J. HOFSOMMER

Mathematical Centre, Amsterdam, Netherlands

Summary

In the literature [1] [2] some formulae are known which give the total flux incident on an area originating from a surface source. These formulae are derived again by aid of a flux theorem which transforms the original double surface integral into a double line integral over the boundaries of both areas.

In illuminating engineering and elsewhere, one of the many problems is the calculation of the total flux incident on an area and originating from a surface source. We assume a perfectly diffusing source, i.e. the emitted flux obeys Lambert's law.

Let $\mathbf{dA}_1$ be the oriented surface element of the source and $\mathbf{dA}_2$ the oriented surface element of the illuminated area (fig. 1), then, if $\mathbf{r}$ is the radius vector connecting both elements and if L is the luminosity of the source, the total flux incident on the illuminated area is

$$E = \frac{L}{\pi} \iint r^{-4} \mathbf{r} \cdot \mathbf{dA}_1 \, \mathbf{r} \cdot \mathbf{dA}_2. \tag{1}$$

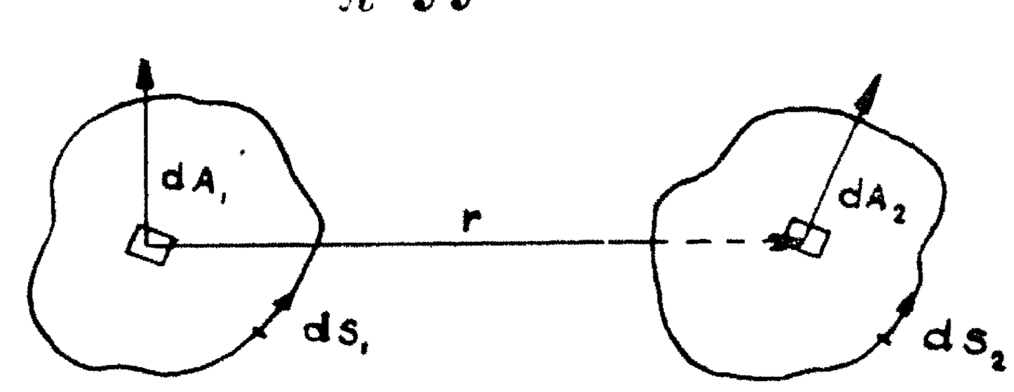

Fig. 1

This double surface integral can be transformed into the double line integral

$$E = -\frac{L}{2\pi} \oint \oint \log r \, \mathbf{ds}_1 \cdot \mathbf{ds}_2, \tag{2}$$

where $\mathbf{ds}_1$ and $\mathbf{ds}_2$ are the oriented line elements of the boundaries of the surface source and the illuminated area. Since $\oint \mathbf{a} \cdot \mathbf{ds} = 0$, we have $\oint\oint \mathbf{ds}_1 \cdot \mathbf{ds}_2 = 0$, and hence $\log r$ in (2) may be replaced by

$\log(r/r_0)$ where r_0 is an arbitrary constant. This means that r in $\log r$ can be regarded as dimensionless.

In the proof we use Stokes' theorem

$$\int \nabla \times \mathbf{v} . \mathrm{d}\mathbf{A} = \oint \mathbf{v} . \mathrm{d}\mathbf{s}.$$

Application of the theorem to the integration with respect to A_2 yields

$$\begin{aligned}\oint\oint \log r \, \mathrm{d}\mathbf{s}_1 . \mathrm{d}\mathbf{s}_2 &= \oint\int (\nabla \times \log r \, \mathrm{d}\mathbf{s}_1) . \mathrm{d}\mathbf{A}_2 \\ &= \oint\int (r^{-2}\mathbf{r} \times \mathrm{d}\mathbf{s}_1) . \mathrm{d}\mathbf{A}_2.\end{aligned}$$

In this integration the elements $\mathrm{d}\mathbf{s}_2$ and $\mathrm{d}\mathbf{A}_2$ have been ,,Aufpunkt''. If we apply Stokes' theorem once again with respect to A_1, the elements $\mathrm{d}\mathbf{s}_1$ and $\mathrm{d}\mathbf{A}_1$ become ,,Aufpunkt''. This means that we have to invert the direction of $\mathbf{r}$. Hence, using the vector formulae

$(\mathbf{a} \times \mathbf{b}) . \mathbf{c} = -(\mathbf{a} \times \mathbf{c}) . \mathbf{b}$, and

$$\begin{aligned}\nabla \times (r^{-2}\mathbf{r} \times \mathbf{a}) &= \mathbf{a} . \nabla r^{-2}\mathbf{r} - \mathbf{a}\nabla . r^{-2}\mathbf{r} \\ &= r^{-2}\mathbf{a} - 2r^{-4}\mathbf{r}.\mathbf{a}\,\mathbf{r} - \mathbf{a}r^{-2} = -2r^{-4}\mathbf{r}.\mathbf{a}\,\mathbf{r},\end{aligned}$$

we find

$$\begin{aligned}\oint\oint \log r \, \mathrm{d}\mathbf{s}_1 . \mathrm{d}\mathbf{s}_2 &= \int\oint (r^{-2}\mathbf{r} \times \mathrm{d}\mathbf{A}_2) . \mathrm{d}\mathbf{s}_1 \\ &= \int\int [\nabla \times (r^{-2}\mathbf{r} \times \mathrm{d}\mathbf{A}_2)] . \mathrm{d}\mathbf{A}_1 \\ &= -2\int\int r^{-4}\mathbf{r} . \mathrm{d}\mathbf{A}_2 \, \mathbf{r} . \mathrm{d}\mathbf{A}_1.\end{aligned}$$

It is an important consequence of (2) that the actual shapes of the surfaces A_1 and A_2 are irrelevant, the flux E depending only on their boundaries.

Eq. (2) is very suitable for the calculation of the total flux in special cases.

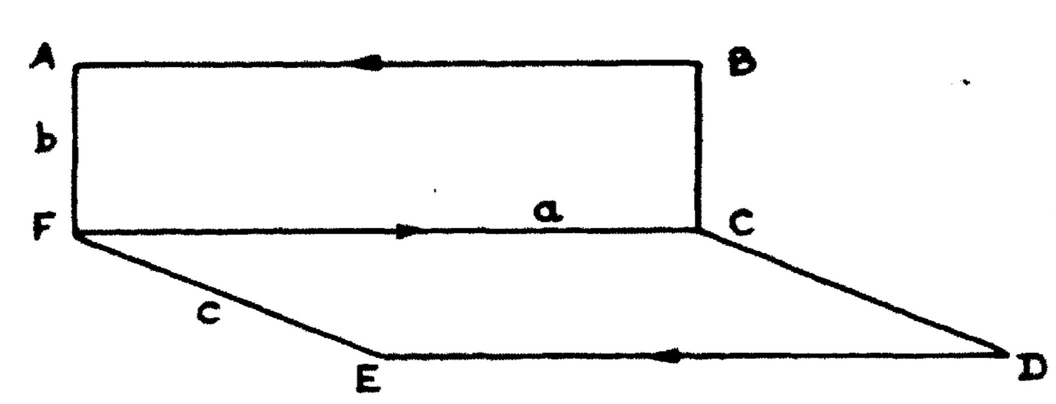

Fig. 2

a) Let ABCF be the surface source and let CDEF be the illuminated area which is perpendicular to the source (fig. 2). Then

$$-\mathrm{d}\mathbf{s}_1 . \mathrm{d}\mathbf{s}_2 = \mathrm{d}s_{AB}\mathrm{d}s_{FC} + \mathrm{d}s_{DE}\mathrm{d}s_{FC} - \mathrm{d}s_{AB}\mathrm{d}s_{FC} - \mathrm{d}s_{FC}\mathrm{d}s_{FC}.$$

If we now define

$$\varphi(p) = \tfrac{1}{2}\int_0^a\int_0^a \log r \, \mathrm{d}s_1\mathrm{d}s_2,$$

where ds_1 and ds_2 are line elements of two parallel lines which are a distance p apart, we find

$$E = \frac{L}{\pi}[\varphi(b) + \varphi(c) - \varphi(\sqrt{b^2 + c^2}) - \varphi(0)],$$

or, with $a^2\phi(p) = \varphi(p) - \varphi(0)$

$$E = \frac{a^2L}{\pi}[\phi(b) + \phi(c) - \phi(\sqrt{b^2 + c^2})].$$

For the function $\phi(p)$ we find

$$\varphi(p) = \tfrac{1}{4}[(a^2 - p^2)\log(a^2 + p^2) - a^2 + p^2 \log p^2] + ap \arctan \frac{a}{p},$$

$$\varphi(0) = \tfrac{1}{4}[a^2 \log a^2 - a^2],$$

$$\phi(p) = \tfrac{1}{2}\left[\left(\frac{p}{a}\right)^2 \log \frac{p}{\sqrt{a^2 + p^2}} - \log \frac{a}{\sqrt{a^2 + p^2}}\right] + \frac{p}{a}\arctan\frac{a}{p},$$

or, putting $a = p \tan \omega$,

$$\phi(p) = \tfrac{1}{2}(\cot^2 \omega \log \cos \omega - \log \sin \omega) + \omega \cot \omega$$

in accordance with Moon [1]).

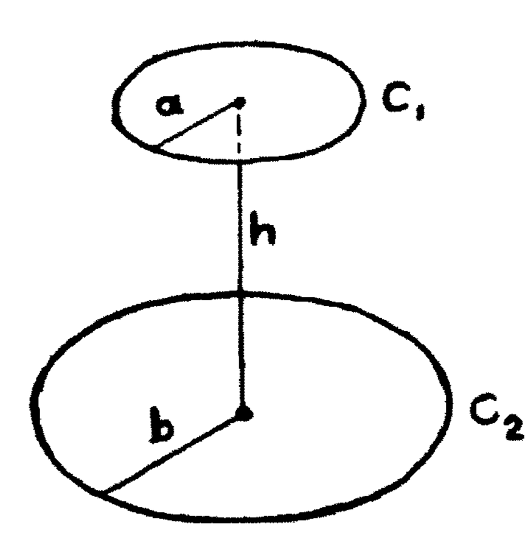

Fig. 3

b) Let C_1 and C_2 be circles in parallel planes (fig. 3) with radii a and b and with centres on a common perpendicular. Then we have, if the distance of the planes is h,

$$E = -\frac{L}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \log[h^2 + a^2 + b^2 - 2ab \cos \varphi]\mathrm{d}\varphi\, \mathrm{d}\psi$$

$$= \frac{h^2L\pi}{4}\left[\sqrt{1 + \left(\frac{a+b}{h}\right)^2} - \sqrt{1 + \left(\frac{a-b}{h}\right)^2}\right]^2,$$

which differs by a factor π with the result given by Moon and is in agreement with Milne's result [2]).

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REFERENCES

1) Moon, P., The scientific basis of illuminating engineering, McGraw Hill, section 10.11.
2) Milne, E. A., Phil. Mag. 7 (1928) 273.