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MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
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Technical Note T.N. 13

Asymptotic expansion of Stirling numbers

by

H.A. Lauerier

The following problem was posed by the Statistical Department.

Let Stirling's number C_n^j be defined by

$$x(x+1)(x+2)\dots(x+n-1) = \sum_0^{n-1} C_n^j x^{n-j} \quad (1)$$

then an asymptotic expansion of C_n^j is sought where j is moderate and n very large.

This problem may be solved in the following way. For (1) we may write

$$\frac{\Gamma(x+n)}{x^n \Gamma(x)} = \sum_0^{n-1} C_n^j x^{-j} . \quad (2)$$

If x is very large, $x \gg n$ we have (see e.g. Whittaker and Watson, Modern Analysis 13.6)

$$\log \Gamma(x+n) = (x+n-\frac{1}{2}) \log x - x + \frac{1}{2} \log(2\pi) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} \phi_{m+1}(n)}{m(m+1)x^m} , \quad (3)$$

where $\phi_m(z)$ is the m^{th} Bernoullian polynomial which is defined by

$$\frac{t e^{zt}}{e^t - 1} = \sum_0^{\infty} \phi_m(z) \frac{t^m}{m!} . \quad (4)$$

We note that

$$\phi_0 = 1 \quad \phi_1 = z - \frac{1}{2} \quad \phi_2 = z^2 - z + \frac{1}{6}$$

$$\phi_3 = z^3 - \frac{3}{2}z^2 + \frac{1}{2}z , \quad \text{and generally}$$

$$\phi_m(z) = (B+z)^{(m)} \quad (5)$$

where

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0 \dots$$

From (3) it follows that

$$\log \frac{\Gamma(x+n)}{x^n \Gamma(x)} = \sum_{m=1}^{\infty} \frac{\psi_m(n)}{x^m} , \quad (6)$$

where

$$\psi_m(z) = (-1)^{m-1} \left\{ \frac{\phi_{m+1}(z) - \phi_{m+1}(0)}{m(m+1)} \right\} . \quad (7)$$

We note that

$$\psi_1 = \frac{1}{2}(z^2 - z) = \frac{1}{2}z(z-1)$$

$$\psi_2 = \frac{1}{6}(z^3 - \frac{3}{2}z^2 + \frac{1}{2}z) = -\frac{1}{6}z(z-1)(2z-1)$$

and symbolically

$$\begin{aligned} \psi_m(z) &= (-1)^{m+1} \left\{ \frac{(z+B)^{(m+1)} - B^{(m+1)}}{m(m+1)} \right\} \\ &= (-1)^{m+1} \left\{ \frac{z^{m+1}}{m(m+1)} - \frac{z^m}{2m} + \frac{z^{m-1}}{12} + \dots \right\} \end{aligned} \quad (8)$$

From (2) and (6) we may derive that

$$\begin{aligned} \sum_0^{n-1} c_n^j x^{-j} &= \exp \sum_{m=1} \frac{\psi_m(n)}{x^m} \\ &= \prod_{m=1} \left\{ \exp \frac{\psi_m(n)}{x^m} \right\} \end{aligned} \quad (9)$$

or written out in full changing x in x^{-1}

$$\sum_0^{n-1} c_n^j x^{+j} = \prod_{m=1} \left\{ 1 + \frac{\psi_m x^m}{1!} + \frac{\psi_m^2 x^{2m}}{2!} + \frac{\psi_m^3 x^{3m}}{3!} + \dots \right\}. \quad (10)$$

Comparing the coefficients of x^j on both sides gives

$$c_n^j = \sum_{k_1 k_2 k_3 \dots} \frac{\psi_1^{k_1} \psi_2^{k_2} \psi_3^{k_3} \dots}{k_1! k_2! k_3! \dots} \quad (11)$$

where

$$k_1 + 2k_2 + 3k_3 + \dots = j.$$

This important relation is obviously an identity which of course could have been proved in a more direct way.

The polynomial ψ_m is of the degree $(m+1)$ in n . Therefore the terms on the right-hand side of (11) can be written in the following order

$$c_n^j = \frac{\psi_1^j}{j!} + \frac{\psi_1^{j-2} \psi_2}{(j-2)!} + \left\{ \frac{\psi_1^{j-3} \psi_3}{(j-3)!} + \frac{\psi_1^{j-4} \psi_2^2}{(j-4)! 2!} \right\} + \dots \quad (12)$$

Hence we obtain the asymptotic expression

$$\begin{aligned} c_n^j &\approx \frac{\left\{ \frac{1}{2}n(n-1) \right\}^j}{j!} - \frac{\left\{ \frac{1}{2}n(n-1) \right\}^{j-2} \left\{ -\frac{1}{6}n(n-1)(2n-1) \right\}}{(j-2)!} \\ &\approx \frac{\left\{ \frac{1}{2}n(n-1) \right\}^j}{j!} \left\{ 1 - \frac{2j(j-1)(2n-1)}{3n(n-1)} \right\}. \end{aligned} \quad (13)$$

This indicates that this expression only holds for $j \ll \sqrt{n}$.

For the Stirling numbers of the second species as defined by

$$\frac{x^n}{x(x+1)\dots(x+n-1)} = \sum_0^{\infty} (-1)^j D_n^j x^{-j} \quad (14)$$

we obtain of course instead of (11)

$$D_n^j = (-1)^j \sum_{k_1 k_2 k_3 \dots} (-1)^{k_1+k_2+k_3+\dots} \frac{\psi_1^{k_1} \psi_2^{k_2} \psi_3^{k_3} \dots}{k_1! k_2! k_3!} \quad (15)$$

so that (13) becomes

$$D_n^j \approx \frac{\{\frac{1}{2}n(n-1)\}^j}{j!} \left\{ 1 + \frac{2j(j-1)(2n-1)}{3n(n-1)} \right\} . \quad (16)$$