STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

TOEGEPASTE WISKUNDE

Technical Note TN 15

A note on the zetafunction of Riemann-Hurwitz

by

H.A. Lauwerier

January 1960

Technical Note TN 15

A note on the zetafunction of Riemann-Hurwitz

by

H.A. Lauwerier

1. Introduction

In this note some old and new results concerning Hurwitz's generalization of the zetafunction of Riemann *)

$$(1.1)\qquad \zeta(s,a) \overset{\text{def}}{=} \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$$

will be discussed.
For simplicity, we shall suppose that $a > 0$. The definition (1.1) holds for Re $s > 1$ but the following analytic continuation can easily be obtained

$$(1.2)\qquad \zeta(s,a) = \frac{\Gamma(1-s)}{2\pi i} \int_{-\infty}^{0^+} \frac{z^{s-1}e^{az}}{1-e^z}\, dz.$$

Hence $\zeta(s,a)$ has a simple pole at $s=1$ with the residue +1.
We note the following simple relation

$$(1.3)\qquad \zeta(s,a) = \zeta(s,a+m) + \sum_{n=0}^{m-1} (n+a)^{-s},$$

where m is a positive integer. This so-called shift-rule can be successfully applied in many numerical applications.

We shall prove the following well-known asymptotic expansion

$$(1.4)\qquad \Gamma(s)\ \zeta(s,a) = \sum_{n=0}^{N} \frac{B_n\Gamma(n+s-1)}{n!(a-1)^{n+s-1}} + R_N,$$

where N is even. The remainder is less in absolute value than the first omitted term. We note that the Bernoullian coefficients B_n in (1.4) are defined by the generating function

$$(1.5)\qquad \frac{t}{e^t-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n .$$

The expansion (1.4) is valid for all s and a with the exception of course of $s=1$. The practical applicability of (1.4) can be greatly enhanced by using the shift rule. We have by combining (1.3) and (1.4)

----

*) Whittaker and Watson. Modern Analysis. 13.11.

(1.6) $$\Gamma(s)\,\zeta(s,a) = \Gamma(s)\sum_{n=0}^{m-1}(n+a)^{-s} + \sum_{n=0}^{\infty}\frac{B_n\Gamma(n+s-1)}{n!\,(m+a-1)^{n+s-1}},$$

where m is an arbitrary integer.

Next we shall prove the convergent expansion of $\zeta(s,a)$ in a faculty series. This result is probably new.

(1.7) $$\Gamma(s)\,\zeta(s,a) = \sum_{n=0}^{\infty} c_n\, B(a,n+s-1),$$

where the coefficients c_n are defined by their generating function

(1.8) $$\left\{-u^{-1}\ln(1-u)\right\}^{s-1} = \sum_{n=0}^{\infty} c_n u^n .$$

The first few coefficients are

(1.9) $$c_0=1 \qquad c_1=\tfrac{1}{2}(s-1) \qquad c_2=\tfrac{1}{24}(s-1)(3s+2) .$$

They can be calculated from the recurrent relation

(1.10) $$nc_n = \sum_{k=0}^{n-1}\frac{n(s-1)-sk}{n-k+1}\,c_k .$$

Their asymptotic behaviour is

(1.11) $$c_n = \frac{s-1}{n+s-1}\left\{\ln(n+s-1)\right\}^{s-2}\left\{1+0(\ln^{-2}n)\right\} .$$

We note that for s=2 the c_n are very simple, viz.

(1.12) $$c_n = (n+1)^{-1}, \qquad s=2.$$

The series (1.7) converges for all values of s, again with $s\neq 1$.

The case with which the generalized zetafunction can be computed by using either (1.6) or (1.7) gives the possibility of a simple and accurate computation of the two functions

(1.13) $$C(x,s) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty}\frac{\cos 2n\pi x}{n^s}, \qquad 0\leq x\leq 1,$$

and

(1.14) $$S(x,s) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty}\frac{\sin 2n\pi x}{n^s}, \qquad 0\leq x\leq 1,$$

if s is a small positive constant.

We shall write

(1.15) $$F(x,s) \stackrel{\text{def}}{=} C(x,s) + i\,S(x,s).$$

Then it follows from a result of Hurwitz *) that

---

*) Whittaker and Watson l.c. 13.15.

$$F(x,s) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \left\{ e^{\frac{1}{2}\pi(1-s)i} \zeta(1-s,x) + e^{-\frac{1}{2}\pi(1-s)i} \zeta(1-s,1-x) \right\} \tag{1.16}$$

valid for $0 < x < 1$.

If e.g. the asymptotic expansion (1.6) is applied upon (1.16) we obtain

$$C(x,s) = (2\pi)^{s-1} \Gamma(1-s) \sin \tfrac{1}{2}\pi s \left\{ \sum_{n=-m}^{n=m-1} |n+x|^{s-1} + \sum_{n=0}^{\infty} \frac{B_n\, \Gamma(n-s)}{n!\, \Gamma(1-s)} \left\{ (m-1+x)^{s-n} + (m-x)^{s-n} \right\} \right\} \tag{1.17}$$

and a similar result for $S(x,s)$.

If s is a positive integer the result follows by a simple limit operation.

Finally a simple proof will be given of two expansions due to Lindelöf

$$C(x,s) = (2\pi)^{s-1} \Gamma(1-s) \sin\tfrac{1}{2}\pi s \left\{ x^{s-1} + 2 \sum_{j=0}^{\infty} \binom{s-1}{2j} \zeta(2j+1-s) x^{2j} \right\} \tag{1.18}$$

and

$$S(x,s) = (2\pi)^{s-1} \Gamma(1-s) \cos\tfrac{1}{2}\pi s \left\{ x^{s-1} + 2 \sum_{j=0}^{\infty} \binom{s-1}{2j+1} \zeta(2j+2-s) x^{2j+1} \right\} \tag{1.19}$$

which are convergent for $0 < x < 1$.

2. From (1.1) it follows that for Re s > 1

$$(2.1)\qquad \Gamma(s)\,\zeta(s,a) = \int_0^\infty \frac{e^{-at}}{1-e^{-t}}\, t^{s-1}dt.$$

Since

$$(2.2)\qquad \frac{t}{e^t-1} = \sum_0^N \frac{B_n}{n!}\, t^n + \theta\, \frac{B_{N+2}}{(N+2)!}\, t^{N+2}$$

for even N and $|\theta| < 1$, we obtain

$$\Gamma(s)\,\zeta(s,a) = \sum_0^N \frac{B_n}{n!} \int_0^\infty e^{-at}\, t^{n+s-2}dt + \text{remainder},$$

from which (1.4) easily follows.

If in (2.1) the substitution $t=-\ln(1-u)$ is made we obtain

$$(2.3)\qquad \Gamma(s)\,\zeta(s,a) = \int_0^1 (1-u)^{a-1}\, u^{s-2}\, f(u)du,$$

where

$$(2.4)\qquad f(u) = \left\{-\frac{1}{u}\ln(1-u)\right\}^{s-1}.$$

If f(u) is expanded in the power series

$$(2.5)\qquad f(u) = \sum_{n=0}^\infty c_n u^n ,$$

where

$$c_o = 1 \qquad c_1=\tfrac{1}{2}(s-1) \qquad c_2= \tfrac{1}{24}(s-1)(3s+2) \qquad \text{etc.}$$

we obtain a convergent expansion of the right-hand side of (2.3) in a faculty series viz.

$$(2.6)\qquad \Gamma(s)\,\zeta(s,a) = \sum_{n=0}^\infty c_n\, B(a,n+s-1),$$

or

$$(2.7)\qquad \zeta(s,a) = \frac{\Gamma(a)}{\Gamma(a+s)}\left\{\frac{a+s-1}{s-1} + c_1 + \frac{s}{a+s}\, c_2 + \frac{s(s+1)}{(a+s)(a+s+1)}\, c_3 + \ldots\right\} .$$

The recurrent relation (1.10) can be derived in the following way. Logarithmic differentiation of $u^{s-1}f(u)$ yields

$$\{-\ln(1-u)\}\, \frac{d}{du}\, (u^{s-1}f) = (s-1)(1-u)^{-1}(u^{s-1}f).$$

Substitution of (2.5) gives the identity

$$\sum_{j=1}^\infty j^{-1}u^j \sum_{k=0}^\infty (k+s-1)c_k\, u^{k+s-2} = (s-1)\sum_{j=1}^\infty u^j \sum_{k=0}^\infty c_k\, u^{k+s-2}.$$

Comparison of the coefficient of equal powers on both sides leads at once to the relation (1.10).

The asymptotic behaviour (1.11) of c_n can be found from Cauchy's formula

$$(2.8) \qquad c_n = \frac{1}{2\pi i} \oint z^{-s-n} \{-\ln(1-z)\}^{s-1} dz,$$

where the path of integration is a small circle around the origin. The integrand has a branchpoint at z=1 and the contour can be deformed into a path along the upper and lower sides of the cut from 1 to $+\infty$. This gives

$$(2.9) \qquad \pi c_n = \operatorname{Im} \int_0^\infty e^{-(n+s-1)t} \{\pi i - \ln(e^t-1)\}^{s-1} dt.$$

The asymptotic behaviour of the right-hand side is determined by the nature of $\{\pi i - \ln(e^t-1)\}^{s-1}$ at the origin.
Since

$$\{\pi i - \ln(e^t-1)\}^{s-1} = (\ln \tfrac{1}{t})^{s-1} \left\{1 + \frac{(s-1)\pi i}{\ln 1/t} + O(\ln^{-2} \tfrac{1}{t})\right\}$$

we have asymptotically

$$(2.10) \qquad c_n \backsim \frac{(s-1)\{\ln(n+s-1)\}^{s-2}}{n+s-1}.$$

For s=2 the right-hand side of (2.10) gives also the right expression.

An amusing particular case of (2.6) is obtained for s=2 and a=m=positive integer. It follows easily that

$$(2.11) \qquad \sum_{n=m}^{\infty} \frac{1}{n^2} = (m-1)! \sum_{k=0}^{\infty} \frac{1}{(k+1)^2(k+2)\dots(k+m)}.$$

3. Quoting Hurwitz's result *)

(3.1) $\zeta(s,a) = 2(2\pi)^{s-1}\,\Gamma(1-s)\left\{ \sin\tfrac{1}{2}\pi s\; C(a,1-s)+\cos\tfrac{1}{2}\pi s\; S(a,1-s)\right\},$

it can easily be deduced that for $0 < x < 1$

(3.2) $C(x,s) = (2\pi)^{s-1}\Gamma(1-s)\sin\tfrac{1}{2}\pi s\left\{ \zeta(1-s,x)+ \zeta(1-s,1-x)\right\},$

and

(3.3) $S(x,s) =(2\pi)^{s-1}\Gamma(1-s)\cos\tfrac{1}{2}\pi s\left\{ \zeta(1-s,x)- \zeta(1-s,1-x)\right\}.$

These two formulae may be combined into the single relation (1.16).

From

(3.4) $F(x,s) = C(x,s) + i\, S(x,s) = \sum_{n=1}^{\infty} n^{-s}\, e^{2n\pi xi}$

it follows that for any positive integer m

(3.5) $\sum_{j=0}^{m-1} F(x+ \frac{j}{m},s) = m^{1-s}F(mx,s).$

Further we have

(3.6) $F(1-x,s) = F^{*}(x,s),$

where F^{*} denotes the complex conjugate of F.
For m=2 it follows from (3.5) by using (3.6) that

(3.7) $F(x,s) = 2^{s-1}\left\{ F(\tfrac{1}{2}x,s) + F^{*}(\tfrac{1}{2}-\tfrac{1}{2}x,s)\right\}.$

This means that it is sufficient to calculate F(x,s) for x in the interval (0,1/3) only.

Next a simple proof will be given of Lindelöf's expansions (1.18) and (1.19). We have for $0 < a < 1$

$$\zeta(s,a) = a^{-s} + \sum_{n=1}^{\infty} n^{-s}(1+ \frac{a}{n})^{-s} =$$

$$= a^{-s} + \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \binom{-s}{j} a^{j} n^{-s-j} =$$

$$= a^{-s} + \sum_{j=0}^{\infty} \binom{-s}{j} a^{j} \sum_{n=1}^{\infty} n^{-s-j}$$

so that

(3.8) $\zeta(s,a) = a^{-s}+ \sum_{j=0}^{\infty} \binom{-s}{j} \zeta(j+s)a^{j}.$

In a similar way we have

$$(3.9)\qquad \zeta(s,1-a) = \sum_{j=0}^{\infty} \binom{-s}{j} \zeta(j+s)(-a)^j .$$

Substitution of (3.8) and (3.9) in (3.2) and (3.3), changing s in 1-s, yields at once the expansions (1.18) and (1.19).

We shall also give an independent proof which makes no use of the relation of Hurwitz. We start from

$$\Gamma(s)\, F(x,s) = \int_0^{\infty} \frac{e^{2\pi xi}}{e^t - e^{2\pi xi}}\, t^{s-1} dt .$$

Since

$$\frac{e^{2\pi xi}}{e^t - e^{2\pi xi}} = \sum_{m=-\infty}^{\infty} \left\{ t-2(m+x)\pi i \right\}^{-1}$$

we have

$$F(x,s) = \sum_{m=-\infty}^{\infty} \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}dt}{t-2(m+x)\pi i} =$$

$$= \Gamma(1-s) \sum_{m=-\infty}^{\infty} \left\{ -2(m+x)\pi i \right\}^{s-1} =$$

$$= \Gamma(1-s) \left\{ (-2\pi xi)^{s-1} + \sum_{m=1}^{\infty} \left\{ i^{s-1}\left(1-\frac{x}{m}\right)^{s-1} + i^{-s+1}\left(1+\frac{x}{m}\right)^{s-1} \right\} (2m\pi)^{s-1} \right\} =$$

$$= \Gamma(1-s)(2\pi)^{s-1} \left\{ (-xi)^{s-1} + 2\sum_{m=1}^{\infty} \sum_{j=0}^{\infty} \binom{s-1}{j} \cos\tfrac{1}{2}\pi(s+j-1)(ix)^j m^{s-j-1} \right\},$$

so that

$$F(x,s) = \Gamma(1-s)(2\pi)^{s-1} \left\{ (-xi)^{s-1} + 2\sum_{j=0}^{\infty} \binom{s-1}{j} \sin\tfrac{1}{2}\pi(s+j)\, \zeta(j+1-s)(ix)^j \right\}.$$

By taking the real and the imaginary part the relations (1.18) and (1.19) follow at once.