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A note on a paper by H. Levine

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Summary

The diffusion problem considered by H. Levine in a previous paper is solved here in a different way by making use of the known solution of the corresponding problem of Green.

Introduction

In a recent paper H. Levine ¹⁾ considers the problem of solving a non-homogeneous Helmholtz equation in an angle. With a slight change of notation the problem can be formulated as follows. To find a solution $\psi(r, \varphi)$ in the angle $0 < \varphi < \theta$, $0 < r < \infty$ of

$$(\Delta - 1)\psi = -1 \quad (1.1)$$

for which

$$\psi = 0 \quad \text{at} \quad \varphi = 0 \quad \text{and} \quad \varphi = \theta \quad . \quad (1.2)$$

The corresponding problem of Green where $G(r, \varphi, r_0, \varphi_0)$ satisfies

$$(\Delta - 1)G = -r_0^{-1} \delta(r - r_0) \delta(\varphi - \varphi_0) \quad (1.3)$$

and

$$G = 0 \quad \text{at} \quad \varphi = 0 \quad \text{and} \quad \varphi = \theta \quad (1.4)$$

has been solved in Lauwerier ²⁾. Then the solution of (1.1) and (1.2) is simply

$$\psi(r, \theta) = \int_0^\infty r_0 dr_0 \int_0^\theta G(r, \varphi, r_0, \varphi_0) d\varphi_0. \quad (1.5)$$

It will be shown that from (1.5) the solution as given by Levine can easily be derived. We have explicitly

$$G(r, \varphi, r_0, \varphi_0) = \begin{cases} 2\theta^{-1} \sum_{m=1}^{\infty} K_{m\nu}(r) I_{m\nu}(r_0) \sin m\nu\varphi \sin m\nu\varphi_0 & \text{for } r > r_0 \\ 2\theta^{-1} \sum_{m=1}^{\infty} I_{m\nu}(r) K_{m\nu}(r_0) \sin m\nu\varphi \sin m\nu\varphi_0 & \text{for } r < r_0, \end{cases} \quad (1.6)$$

with $\nu = \pi/\theta$, so that by means of (1.5)

$$\psi(r, \varphi) = \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{\sin m\nu\varphi}{m} L_{m\nu}(r), \quad (1.7)$$

where the auxiliary function $L_{\mu}(x)$ is defined by

$$L_{\mu}(x) = I_{\mu}(x) \int_x^{\infty} t K_{\mu}(t) dt + K_{\mu}(x) \int_0^x t I_{\mu}(t) dt, \quad (1.8)$$

and where \sum_1 indicates summation over odd numbers $m=1,3,5\dots$

A few properties of $L_{\mu}(x)$ which may be considered as a modified Lommel function (cf. Watson ³) will be given in the next section. We shall prove that

$$L_{\mu}(x) = 1 - \mu \int_0^{\infty} e^{-\mu u} \cos(x \operatorname{sh} u) du. \quad (1.9)$$

If this is substituted in (1.7) it follows that

$$\psi(r, \varphi) = 1 - 4\theta^{-1} \sum_1 \sin m\nu\varphi \int_0^{\infty} e^{-m\nu u} \cos(r \operatorname{sh} u) du$$

and hence

$$\psi(r, \varphi) = 1 - 2\theta^{-1} \operatorname{Im} \int_0^{\infty} \frac{\cos(r \operatorname{sh} u)}{\operatorname{sh} \nu(u+i\varphi)} du, \quad (1.10)$$

which is equivalent to the result obtained by Levine (l.c. formula (2.18)).

§ 2. Properties of $L_{\mu}(x)$

It can easily be verified that $L_{\mu}(x)$ is a solution of the non-homogeneous Bessel equation

$$L_{\mu}'' + \frac{1}{x} L_{\mu}' - \left(1 + \frac{\mu^2}{x^2}\right) L_{\mu} = -1 \quad (2.1)$$

and that for $\mu > 0$

$$L_{\mu}(0) = 0, \quad L_{\mu}(\infty) = 1. \quad (2.2)$$

By applying cosine transformation in the form

$$L_{\mu}(x) = 1 - \int_0^{\infty} \cos xt f(t) dt \quad (2.3)$$

the equation (2.1) changes into

$$(1+t^2)f'' + 3tf' + (1-\mu^2)f = 0 \quad (2.4)$$

with $f'(0) = -\mu^2$.

By making the substitution $t=\operatorname{sh} u$ the equation (2.4) becomes

$$\frac{d^2}{du^2} (\operatorname{ch} u f) = \mu^2 (\operatorname{ch} u f) \quad (2.5)$$

with $f'(0) = -\mu^2$.

This has the solution vanishing at infinity

$$f(u) = \mu e^{-\mu u} / \text{chu} . \quad (2.6)$$

Therefore (2.3) becomes

$$L_{\mu}(x) = 1 - \mu \int_0^{\infty} e^{-\mu u} \cos(xshu) du. \quad (2.7)$$

A series valid for small values of x can easily be derived. If $\mu \neq 2, 4, 6, \dots$ we obtain (cf. Watson l.c.)

$$L_{\mu}(x) = - \sum_{m=1}^{\infty} \{ (4 - \mu^2)(16 - \mu^2) \dots (4m^2 - \mu^2) \}^{-1} x^{2m}.$$

If μ is a positive even integer logarithmic terms are obtained. The first term is, however, always $(\mu^2 - 4)^{-1} x^2$ except for $\mu = 2$ when it is $-\frac{1}{4} x^2 \ln x$.

The asymptotic expansion for large values of x is (cf. Watson l.c.)

$$L_{\mu}(x) \approx 1 - \sum_{m=0}^{\infty} \mu^2 (4 - \mu^2) \dots (4m^2 - \mu^2) x^{-2m-2}.$$

If $\mu = 2, 4, 6, \dots$ the series on the right-hand side breaks off but then we have the exact relation

$$L_{2n}(x) = \sum_{m=0}^n (-1)^m n^2 (n^2 - 1) \dots (n^2 - (m-1)^2) \left(\frac{1}{2}x\right)^{-2m} - (-1)^n {}_2nK_{2n}(x).$$

References

- 1) Levine, H., Appl. Sci. Res. B 8 (1960) 105-127.
- 2) Lauwerier, H.A., Proc.Kon.Ned.Ak.v.Wet. Amsterdam. Proc.A, 62, no 5, 475-488.
- 3) Watson, G.N., Theory of Bessel functions. Cambridge (1944) 10.7.