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Technical Note TN 17

Perturbation of orbits of Earth-satellites by atmospheric contact

by

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April 1960

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ZW 1960-005
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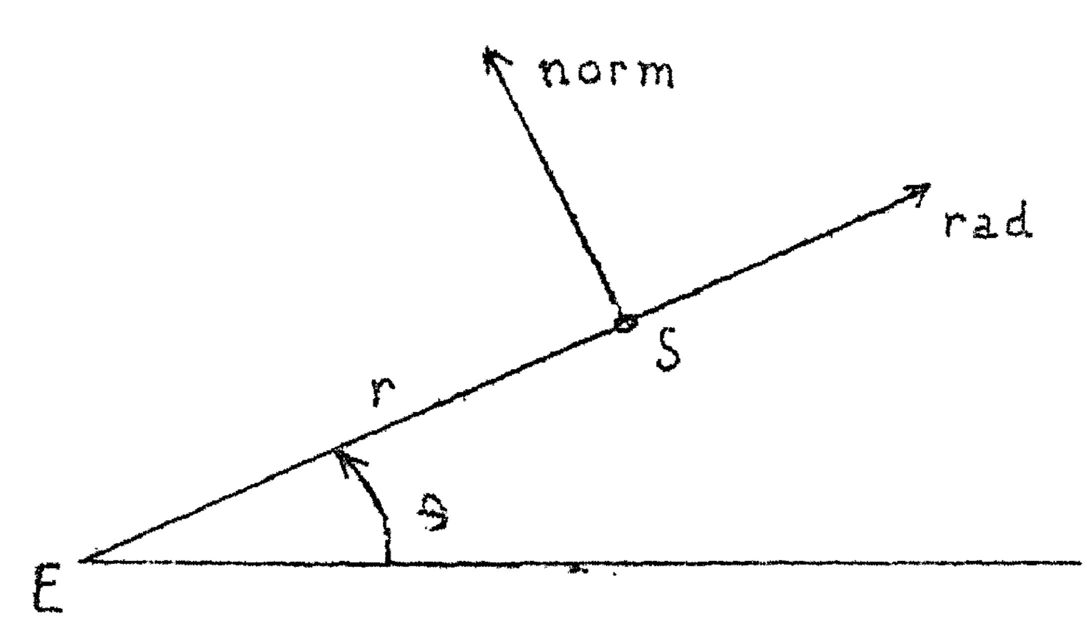
1. Kepler motion

Let $S(r,\theta)$ (satellite) be a point-mass moving in the (r,θ) plane and which is attracted by the origin E (centre of the Earth) with the force μr^{-2}.

Then the components of the velocity and the acceleration of S are

(1.1) $$\begin{cases} v_{rad} = \dot{r} & a_{rad} = \ddot{r} - r\dot{\theta}^2 \\ v_{norm} = r\dot{\theta}, & a_{norm} = 2\dot{r}\dot{\theta} + r\ddot{\theta} \,. \end{cases}$$



The equations of motion are

(1.2) $$\begin{cases} \ddot{r} - r\dot{\theta}^2 = -\dfrac{\mu}{r^2} \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \,. \end{cases}$$

The second equation gives Kepler's second law (conservation of angular momentum)

(1.3) $$r^2\dot{\theta} = M.$$

From (1.2) the following energy relation can be derived

(1.4) $$\tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\mu}{r} = E,$$

i.e. sum of kinetic and potential energy is a constant.

a circular motion

(1.5) $$r = R \qquad \dot{\theta} = \pi/T \,,$$

where T is the semi-period.

From (1.2) we obtain at once Kepler's third law

(1.6) $$T^2 = \pi^2 \mu^{-1} R^3 .$$

Further we note that

(1.7) $$E = -\frac{\mu}{2R} .$$

<u>b</u> elliptic motion

(1.8) $$r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta} ,$$

where ε is the excentricity and 2a the major axis. By using (1.3) we obtain from (1.8) for the components of the velocity

(1.9) $$\dot{r} = \frac{M\varepsilon}{a(1-\varepsilon^2)} \sin\theta \qquad r\dot{\theta} = \frac{M}{a(1-\varepsilon^2)} (1+\varepsilon\cos\theta).$$

If this is substituted in the first equation of (1.2) we find

(1.10) $$M^2 = \mu\, a(1-\varepsilon^2).$$

Substitution in (1.4) gives next

(1.11) $$E = -\frac{\mu}{2a} .$$

The absolute value of the velocity follows easily from (1.9) viz.

(1.12) $$V = \frac{\mu}{M} (1+2\varepsilon\cos\theta + \varepsilon^2)^{\frac{1}{2}} .$$

Kepler's third law may be derived from

(1.13) $$T = \frac{1}{M} \int_0^{\pi} v^2 d\theta ,$$

where r is given by (1.8). Integration gives the well-known result

(1.14) $$T^2 = \pi^2 \mu^{-1} a^3 .$$

2. Atmospheric perturbation, circular motion

An artificial satellite S experiences a drag which depends on its velocity and also on its shape. Usually the drag D is given by the law

$$D = \tfrac{1}{2}\rho V^2 C_D S, \tag{2.1}$$

where V is the velocity, S is some reference area and C_D the drag coefficient based on this area, and where ρ is the density of the air.

The drag causes a decrease of the absolute velocity according to

$$\Delta V = -\lambda \rho V^2 \Delta t, \tag{2.2}$$

where $$\lambda = \frac{C_D S}{2m}.$$

The density ρ is a certain function of the altitude or of r. An exponential law appears to be a reasonably good model. We shall take

$$\rho = e^{-\beta r}. \tag{2.3}$$

Then the equations of motion are, still in their general form,

$$\left\{\begin{aligned} &\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} = -\lambda e^{-\beta r}\,\dot{r}V \\ &2\dot{r}\dot{\theta} + r\ddot{\theta} = -\lambda e^{-\beta r}\,r\dot{\theta}V. \end{aligned}\right. \tag{2.4}$$

If the atmospheric perturbation is small we may put for a perturbed circular motion

$$\left\{\begin{aligned} &r = R(1-x) \\ &\theta = \frac{\pi}{T}(1+y), \end{aligned}\right. \tag{2.5}$$

where x and y are small.

Substitution of (2.5) in (2.4) gives with neglect of higher powers of x and y

$$\left\{\begin{aligned} &3x - 2y = 0 \\ &2\dot{x} - \dot{y} = \frac{\pi\lambda}{T} e^{-\beta R} R, \end{aligned}\right. \tag{2.6}$$

so that with

$$C = \frac{\pi\lambda R}{T} e^{-\beta R}, \tag{2.7}$$

$$\left\{\begin{aligned} &x = 2Ct \\ &y = 3Ct. \end{aligned}\right. \tag{2.8}$$

After a semi-period t=T the radius has decreased according to

$$\frac{\Delta R}{R} = -2CT, \tag{2.9}$$

or

$$\Delta R = -2\pi\lambda R^2 e^{-\beta R}. \tag{2.10}$$

Therefore the total number of revolutions is given by

$$n = \frac{1}{4\pi\lambda} \int_{R_E}^{R} r^{-2} e^{\beta r}\, dr. \tag{2.11}$$

The period decay is after a semi-period

$$\frac{\Delta T}{T} = -3CT, \tag{2.12}$$

or by applying Kepler's third law

$$\Delta T = -3\lambda(\mu\pi)^{1/3} T^{5/3} \exp(-\beta\mu^{1/2}\pi^{-2/3} T^{2/3}). \tag{2.13}$$

3. Atmospheric perturbation, elliptic motion

In this case we consider the momentum and energy equation

$$(3.1)\qquad \begin{cases} \dfrac{d}{dt}\,(r^2\dot\theta) = -\,\lambda(r^2\dot\theta)\;Ve^{-\beta r} \\[2mm] \dfrac{d}{dt}\left\{\tfrac{1}{2}(\dot r^2+r^2\dot\theta^2) - \dfrac{\mu}{r}\right\} = -\,\lambda V^3 e^{-\beta r}\,. \end{cases}$$

If the unperturbed motion is given by (1.8) we put for the perturbed motion

$$(3.2)\qquad \begin{cases} r^2\dot\theta = M(1-x) \\[1mm] \tfrac{1}{2}(\dot r^2+r^2\dot\theta^2) - \dfrac{\mu}{r} = E(1+y)\,. \end{cases}$$

Then it follows from (3.1) that

$$(3.3)\qquad \begin{cases} \dot x = \lambda\, Ve^{-\beta r} \\[1mm] \dot y = 2\lambda\, a\,\mu^{-1}\,V^3 e^{-\beta r}\,. \end{cases}$$

In view of $\dfrac{d}{dt} = \dot\theta\,\dfrac{d}{d\theta} = \dfrac{M}{r^2}\,\dfrac{d}{d\theta}$ it follows that

$$(3.4)\qquad dx = \frac{\lambda}{M}\, r^2 V e^{-\beta r} d\theta \quad\text{and}\quad dy = \frac{2\lambda a}{\mu M}\, r^2 V^3 e^{-\beta r} d\theta.$$

Hence after a semi-period T we have

$$\begin{cases} \Delta x = \lambda a(1-\varepsilon^2) \displaystyle\int_0^\pi \frac{(1+2\varepsilon\cos\theta+\varepsilon^2)^{\frac{1}{2}}}{(1+\varepsilon\cos\theta)^2}\exp\Big(-\frac{\beta a(1-\varepsilon^2)}{1+\varepsilon\cos\theta}\Big)\,d\theta \\[4mm] \Delta y = 2\lambda a \displaystyle\int_0^\pi \frac{(1+2\varepsilon\cos\theta+\varepsilon^2)^{3/2}}{(1+\varepsilon\cos\theta)^2}\exp\Big(-\frac{\beta a(1-\varepsilon^2)}{1+\varepsilon\cos\theta}\Big)\,d\theta. \end{cases}$$

For orbits with a large excentricity the expressions (3.5) can be estimated by noting that the integrand has its maximum at $\theta=0$ (apogeum). Then

$$(3.6)\qquad \begin{cases} \Delta x \approx \lambda e^{-\beta a(1-\varepsilon)}\sqrt{\dfrac{\pi a(1-\varepsilon^2)}{2\beta\varepsilon}} \\[4mm] \Delta y \approx \lambda e^{-\beta a(1-\varepsilon)}\sqrt{\dfrac{2\pi a(1+\varepsilon)^3}{\beta\,\varepsilon(1-\varepsilon)}}\,. \end{cases}$$

From (1.10) and (1.11) we may derive

$$(3.7)\qquad \Delta a = -a\,\Delta y \quad\text{and}\quad \Delta\varepsilon = -\,\frac{1-\varepsilon^2}{\varepsilon}\,(\tfrac{1}{2}\,\Delta y - \Delta x).$$

Hence the approach of the apogeum follows from

$$(3.8)\qquad \frac{\Delta\{a(1+\varepsilon)\}}{a(1+\varepsilon)} = -\frac{1}{\varepsilon}\left\{\tfrac{1}{2}(1+\varepsilon)\,\Delta y - (1-\varepsilon)\,\Delta x\right\}$$
$$\approx -2\lambda e^{-\beta a(1-\varepsilon)}\sqrt{\frac{\pi a(1+\varepsilon)}{2\beta\,\varepsilon(1-\varepsilon)}}\ .$$

For the perigeum we obtain in a similar way

$$(3.9)\qquad \frac{\Delta\{a(1-\varepsilon)\}}{a(1-\varepsilon)} = -\frac{1}{\varepsilon}\left\{(1+\varepsilon)\,\Delta x - (1-\varepsilon)\tfrac{1}{2}\,\Delta y\right\}$$
$$\approx 0\ ,$$

so that, at least to a first approximation, for orbits of large excentricity the perigeum is stationary.

More precise results can be obtained by expanding the integrals of (3.5) in a series of modified Bessel functions. By taking the new variable of integration u defined by

$$(3.10)\qquad 1+\varepsilon\cos\theta = \frac{1-\varepsilon^2}{1+\varepsilon\cos u}\ ,$$

we find after some elementary calculations

$$(3.11)\qquad \begin{cases} \Delta x = \lambda a e^{-\beta a}\displaystyle\int_0^{\pi} e^{-\varepsilon\beta a\cos u}\,(1-\varepsilon^2\cos^2 u)^{\frac{1}{2}}du \\ \Delta y = 2\lambda a e^{-\beta a}\displaystyle\int_0^{\pi} e^{-\varepsilon\beta a\cos u}\,\frac{(1-\varepsilon\cos u)^{3/2}}{(1+\varepsilon\cos u)^{\frac{1}{2}}}\,du. \end{cases}$$

Noting that

$$(3.12)\qquad I_m(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos u}\cos mu\,du,$$

we have

$$(3.13)\qquad \begin{cases} \Delta x = \pi\lambda a\, e^{-\beta a}\{I_0(\varepsilon\beta a) + O(\varepsilon^2)\} \\ \Delta y = 2\pi\lambda a\, e^{-\beta a}\{I_0(\varepsilon\beta a) + 2\varepsilon I_1(\varepsilon\beta a) + O(\varepsilon^2)\}\ . \end{cases}$$

By using the latter expressions for the approach of the perigeum we obtain

$$(3.14)\qquad \frac{\Delta\{a(1-\varepsilon)\}}{a(1-\varepsilon)} = -2\pi\lambda a e^{-\beta a}\{I_0(\varepsilon\beta a)-(1-\varepsilon)I_1(\varepsilon\beta a)\ldots\}\ .$$

By taking $\varepsilon \to 0$ the result (2.10) is confirmed.

Remarks

More and detailed information about the subject is to be found in the recent volumes of the Journal of the British Interplanetary Society and the Acta Astronautica. We mention in particular two papers by T.R.F. Nonweiler, J.B.I.S. Vol. 16, 368-379 (1958) and ib. Vol. 17, 14-20 (1959).