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Technical Note TN 18

Note on the vector equation rot $\vec{v} = \vec{w}$

by

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It is well known, that the vector equation

$$\nabla \times \vec{v} = \vec{w} \qquad (1)$$

can be solved only if the field $\vec{w}$ satisfies the condition

$$\nabla \cdot \vec{w} = 0 , \qquad (2)$$

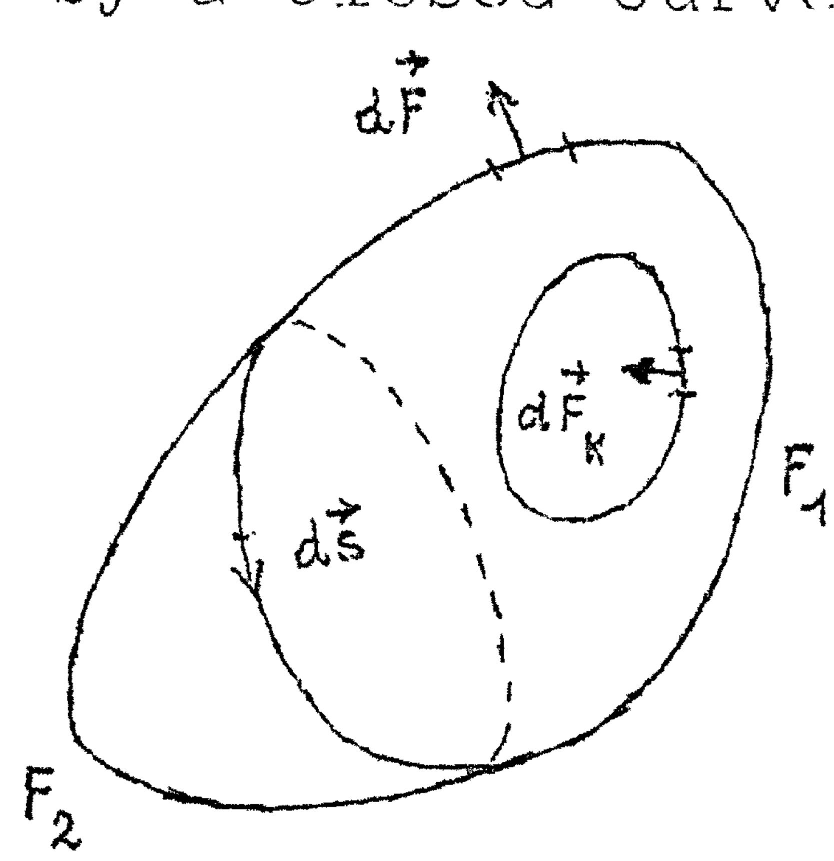
which follows from (1) by taking the divergence of both members.

The condition (2) also appears to be sufficient if the region in which (1) is to be solved, is simply connected and possesses only one closed bounding surface. However, it has been shown by A.F. STEVENSON * that for simply connected regions, characterized by k separate closed bounding surfaces, (2) must be replaced by

$$\oint \vec{w} \cdot d\vec{F}_k = 0 , \qquad (3)$$

where $d\vec{F}_k$ is a surface element of the k^{th} bounding surface. These conditions, again, appear to be both necessary and sufficient. It is the purpose of this note to point out that solutions of (1) may exist for simply connected regions, with several separate bounding surfaces, which satisfy (2), but not the more stringent conditions (3) if $\vec{v}$ is allowed to be singular. We will give an example of such a singular field and apply it to the evaluation of a surface integral.

The necessity of the conditions (3) for non-singular fields may be shown in the following way. Consider a surface F which fully encloses the k-th bounding surface F_k and divide F into two areas F_1 and F_2 by a closed curve s. Using Stokes' Theorem we have



$$\oint_s \vec{v} \cdot d\vec{s} = \int_{F_1} \nabla \times \vec{v} \cdot d\vec{F} = \int_{F_1} \vec{w} \cdot d\vec{F},$$

$$\oint_s \vec{v} \cdot d\vec{s} = -\int_{F_2} \nabla \times \vec{v} \cdot d\vec{F} = -\int_{F_2} \vec{w} \cdot d\vec{F},$$

or

$$\int_{F_1} \vec{w} \cdot d\vec{F} + \int_{F_2} \vec{w} \cdot d\vec{F} = \oint_F \vec{w} \cdot d\vec{F} = 0.$$

Application of Gauss' Theorem to the volume between the surfaces F and F_k gives

$$\oint \vec{w} \cdot d\vec{F}_k = \int \nabla \cdot \vec{w} \, dV - \oint \vec{w} \cdot d\vec{F} = 0$$

which shows the necessity of the conditions (3). For a proof of their

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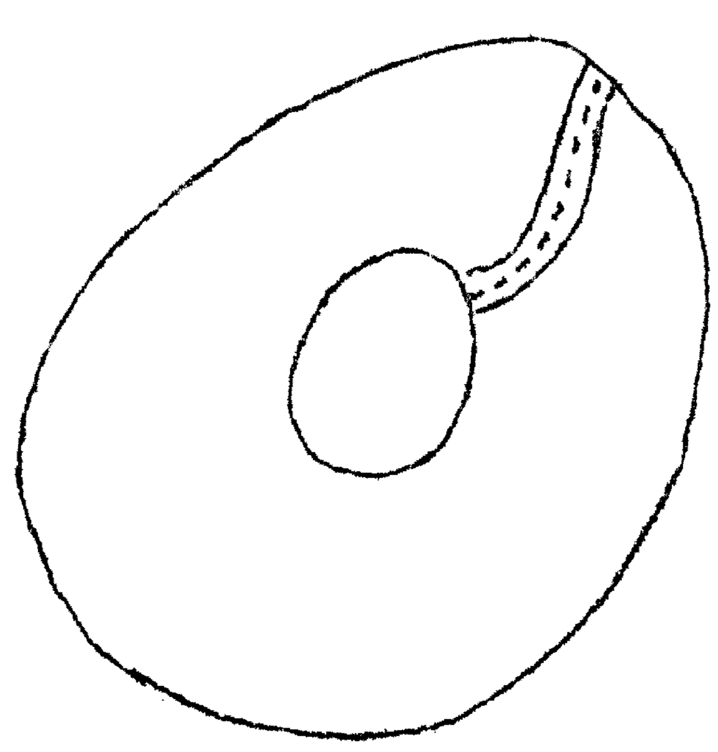
*A.F. Stevenson, Note on the existence and determination of a vector potential. Quat. Appl. Math. 12 (1954), p.194-198.

sufficiency we refer to Stevenson's paper. We note in passing that for a simply connected region with only one bounding surface the conditions (2) and (3) are aequivalent, which follows from Gauss' Theorem.

In the above proof Stokes' Theorem is applicable only if the curve s does not enclose a vortex line, i.e. the proof breaks down if $\vec{v}$ is allowed to contain vortex lines. This means that solutions of (1) may exist which do not satisfy the conditions (3) but contain one or more vortex lines.

The existence of such singular solutions may be elucidated by the following heuristic argument. Consider a simply connected region with an inner and an outer boundary and assume the existence of a



vortex line connecting the two boundaries. We exclude this vortex line by a surrounding vortex tube. The field between the two boundaries and outside the vortex tube now is regular and covers a simply connected region with one bounding surface. For this region the equation (1) can be solved if $\vec{w}$ only satisfies condition (2)

As an illustration consider the equation

$$\nabla \times \vec{v} = \vec{w} = r^{-3}\,\vec{r}\,. \qquad (4)$$

The field $\vec{w}$ is singular at the origin. We consider the region between two spheres with centres in the origin. If F_i is the surface of the inner sphere and F_u the surface of the outer sphere,

$$\oint_{F_i} \vec{w}\cdot d\vec{F} = 4\pi \quad ; \quad \oint_{F_u} \vec{w}\cdot d\vec{F} = -4\pi\,.$$

A regular solution of (4) in the region considered, hence, does not exist. However,

$$\vec{v} = \frac{\vec{r}\times\vec{a}}{r(r-\vec{a}\cdot\vec{r})} \qquad (5)$$

is a solution of (4) which has a vortex line along the half ray in the direction of the unit vector $\vec{a}$ with circulation $\Gamma = -4\pi$.

We remark in passing, that Kirchhoff's law, stating that vortex lines are either closed, or end at a boundary (which may be at infinity) of the field, is incomplete. A vortex line may also end in a singularity of the rotation of the field. This happens for the field given by (5) extended over the entire space. The vortex line extends to infinity and ends at the singularity of the field rot $\vec{v} = r^{-3}\vec{r}$, i.e. the origin.

Solutions of (1) may be used for lowering the multiplicity of integrals by means of Stokes' Theorem. Indeed we have

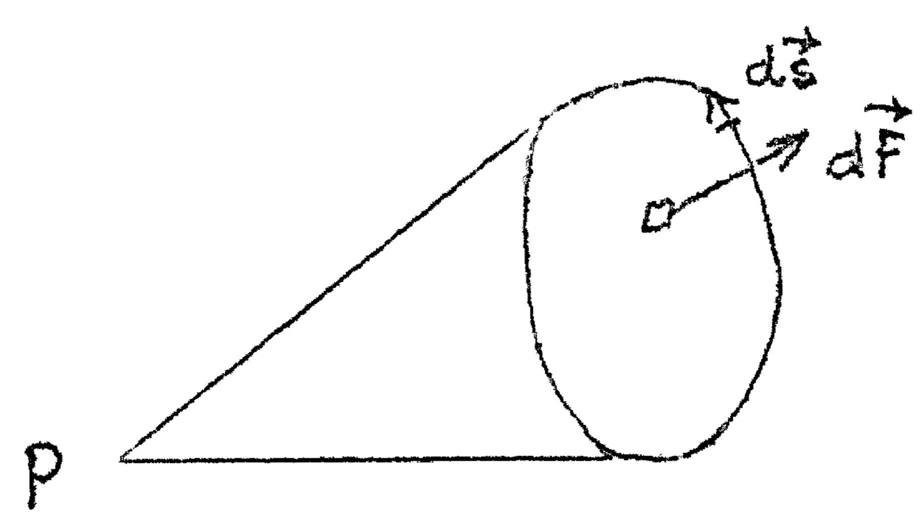
$$\int \vec{w} \cdot d\vec{F} = \int \nabla \times \vec{v} \cdot d\vec{F} = \oint \vec{v} \cdot d\vec{s} \, , \qquad (6)$$

and the surface integral may be evaluated by integration along a closed curve once a particular solution of rot $\vec{v} = \vec{w}$ is known.

The above considerations will be illustrated by two examples.

Example 1.

Consider the solid angle Ω under which a point P sees a surface F.



This solid angle is given by

$$\Omega = \int r^{-3} \vec{r} \cdot d\vec{F} \qquad (7)$$

or, by aid of (5) and (6)

$$\Omega = \oint \frac{z\vec{j} - y\vec{k}}{r(r-x)} \cdot d\vec{s} \qquad (8)$$

if $\vec{a}=\vec{i}$ and the positive X-axis does not intersect the surface F.

Example 2.

The equation

$$\nabla \times \vec{v} = \vec{w} = \frac{\vec{r} \times \vec{a}}{r(r-\vec{a}\cdot\vec{r})} \qquad (9)$$

satisfies condition (2). We consider a region covering the entire space with the exception of the vortex line of $\vec{w}$, i.e. a half ray with direction $\vec{a}$, the origin included. For this region a solution of (9) is given by

$$\vec{v} = \vec{a} \ln(r-\vec{a}\cdot\vec{r}) \, . \qquad (10)$$

This solution, again, is singular on the half ray with direction $\vec{a}$, but the singularity has the character of a line of distributed sources. Combining (4) and (8) it is seen that (10) also is a solution of

$$\nabla \times (\nabla \times \vec{v}) = r^{-3} \vec{r} \, .$$

Furthermore a calculation shows, that the field (10) also satisfies

$$\nabla^2 \vec{v} = 0 \quad \text{and} \quad \nabla \nabla \cdot \vec{v} = r^{-3} \vec{r}$$

in the entire space, with the exception of the half ray.