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Technical Note TN 22

A note on the summation of some series of Bessel functions

by

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1. Rutgers [1] has derived explicit expressions for the sum of series of the kind

$$S_k(r)=\sum_{n=0}^{\infty}(\nu+2n)^k\, I_{\nu+2n}(r)\ , \tag{1.1}$$

where k is a non-negative integer. In a recent paper Carlitz [2] considered these series from a different point of view. Here it will be shown that such series may be summed in a much more simple and elementary way. From Bessel's differential equation it follows at once that

$$\left\{\left(r\frac{\partial}{\partial r}\right)^2 - r^2\right\} I_\mu(r) = \mu^2\, I_\mu(r)\ . \tag{1.2}$$

Hence the following recurrence relation exists

$$S_{k+2}(r) = \left\{\left(r\frac{\partial}{\partial r}\right)^2 - r^2\right\} S_k(r)\ . \tag{1.3}$$

Therefore we need only consider the cases k=0 and k=1.

In the special case $\nu=0$ we may start from the well-known Fourier expansion

$$e^{r\cos\varphi} = \sum_{n=0}^{\infty}\varepsilon_n \cos n\varphi\, I_n(r)\ . \tag{1.4}$$

Expanding both in rising powers of φ it follows at once that the sum of the series

$$\sum_{n=0}^{\infty}\varepsilon_n\, n^{2k}\, I_n(r) \tag{1.5}$$

is determined by the coefficient of φ^{2k} in the expansion of $\exp(r\cos\varphi)=\exp\left\{r(1-\frac{1}{2}\varphi^2+\dots)\right\}$ which is obviously of the form $\exp(r.\varphi_k(r))$ where $\varphi_k(r)$ is a polynomial of degree k. A recurrence relation between successive $\varphi_k(r)$ could be obtained by noting that both sides of (1.4) satisfy the Helmholtz equation $(\Delta-1)f=0$ where Δ is the Laplacian in polar coordinates (r,φ). However, it is simpler to use (1.4) for $\varphi=0$ which gives the well-known result

$$\sum_{n=0}^{\infty}\varepsilon_n\, I_n(r) = e^r\ , \tag{1.6}$$

and then to apply the recurrence relation (1.3).

This gives

(1.7) $$\sum_{n=0}^{\infty} \varepsilon_n \; n^{2k} \, I_n(r) = \left\{ \left(r \frac{\partial}{\partial r}\right)^2 - r^2 \right\}^k . e^r \quad .$$

Since

(1.8) $$\left\{ \left(r \frac{\partial}{\partial r}\right)^2 - r^2 \right\} . e^r \varphi(r) =$$
$$= e^r \left\{ r^2 \frac{\partial^2}{\partial r^2} + (2r^2+r)\frac{\partial}{\partial r} + r \right\} . \varphi(r),$$

we may also write*)

(1.9) $$\sum_{n=0}^{\infty} \varepsilon_n \; n^{2k} \, I_n(r) = e^r \left\{ r^2 \frac{\partial^2}{\partial r^2} + (2r^2+r)\frac{\partial}{\partial r} + r \right\}^k . \, 1 \quad .$$

By making the substitution $\varphi = \frac{1}{2}\pi$ in (1.4) we obtain

(1.10) $$\sum_{n=0}^{\infty} (-1)^n \, \varepsilon_{2n} \, I_{2n}(r) = 1,$$

and next**)

(1.11) $$\sum_{n=0}^{\infty} (-1)^n \, \varepsilon_{2n} \, (2n)^{2k} \, I_{2n}(r) = \left\{ \left(r \frac{\partial}{\partial r}\right)^2 - r^2 \right\}^k .1 \; .$$

If (1.4) is differentiated with respect to φ we obtain when substituting $\varphi = \frac{1}{2}\pi$

(1.12) $$\sum_{n=0}^{\infty} (-1)^n \, \varepsilon_{2n+1} (2n+1) \, I_{2n+1}(r) = r \quad ,$$

and next ***)

(1.13) $$\sum_{n=0}^{\infty} (-1)^n \, \varepsilon_{2n+1} \, (2n+1)^{2k+1} \, I_{2n+1}(r) = \left\{ \left(r \frac{\partial}{\partial r}\right)^2 - r^2 \right\}^k .r.$$

If r is replaced by ir we obtain expansions containing Bessel functions of the first kind. E.g. from (1.11) may derive

(1.14) $$\sum_{n=0}^{\infty} \varepsilon_{2n} \, (2n)^{2k} \, J_{2n}(r) = \left\{ \left(r \frac{\partial}{\partial r}\right)^2 + r^2 \right\}^k .1.$$

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*) Cf. Carlitz l.c. formula (3.4).

**) Ib. formula (6.2).

***) Ib. formula (6.1).

2. More generally we now consider the series

$$(2.1) \qquad \sum_{n=0}^{\infty} c_n(\nu+n)^k \, I_{\nu+n}(r)$$

where again k is a non-negative integer and where the coefficients c_n are given by the power series expansion

$$(2.2) \qquad f(t) = \sum_{n=0}^{\infty} c_n \, t^n \quad .$$

It is sufficient to consider only the case k=0. Since

$$(2.3) \qquad \mu I_{\mu}(r) = r \, I_{\mu-1}(r) - r \, I'_{\mu}(r)$$

the case k=1 may be reduced to the previous one. For larger values of k we may use the recurrence relation (1.3). We shall write

$$(2.4) \qquad S\,(r,\nu\,,f(t)) = \sum_{n=0}^{\infty} c_n \, I_{\nu+n}\,(r).$$

Using Sommerfeld's integral expression

$$(2.5) \qquad I_{\mu}\,(r) = \frac{1}{2\pi i} \int_{-\infty-\pi i}^{-\infty+\pi i} e^{r \operatorname{ch} w+\mu w} \, dw$$

we obtain without difficulty

$$(2.6) \qquad S(r,\nu,f) = \frac{1}{2\pi i} \int_{-\infty-\pi i}^{-\infty+ i} e^{r \operatorname{ch} w+\nu w} \; f(e^w)dw$$

The right-hand side of (2.6) is obviously reducible to a Bessel function for the following particular choice

$$(2.7) \qquad e^{\nu w} \, f(e^w) = \frac{d}{dw} \, \frac{e^{\nu w}}{1-e^{-2w}} \quad ,$$

i.e. when

$$(2.8) \qquad f(t) = t^{1-\nu}\left(\frac{t^{\nu}}{1-t^2}\right)' = \sum_{n=0}^{\infty} (\nu+2n)t^{2n} \quad .$$

After partial integration it follows that

$$(2.9) \qquad S(r,\nu,f) = \tfrac{1}{2} r \, I_{\nu-1}(r) \; .$$

Applying (1.3) we obtain*)

$$(2.10)\quad 2\sum_{n=0}^{\infty}(\nu+2n)^{2k+1}\, I_{\nu+2n}(r)=\left\{\left(r\frac{\partial}{\partial r}\right)^2-r^2\right\}^k . r\, I_{\nu}(r).$$

The same technique enables us to find simple expressions for series such as

$$(2.11)\qquad \sum_{n=0}^{\infty}\varepsilon_n\, I_{\nu+n}(r)\quad .$$

In this case we have $f(t)=(1+t)/(1-t)$ so that

$$(2.12)\qquad S(r,\nu,\frac{1+t}{1-t})=\frac{-1}{2\pi i}\int_{-\infty-\pi i}^{-\infty+\pi i} e^{r\,\mathrm{ch}\,w+\nu w}\,\mathrm{cth}\,\tfrac{1}{2}w\,dw.$$

A simple calculation shows that

$$(2.13)\qquad \left(\frac{\partial}{\partial r}-1\right)S=\frac{\nu}{r}I_{\nu}(r)$$

so that

$$(2.14)\qquad S=\nu e^{r}\int_{0}^{r}e^{-\varrho}\,\varrho^{-1}\,I_{\nu}(\varrho)d\varrho\quad .$$

The derivation of similar results of this kind may be left to the reader.

<u>References</u>

1. Rutgers, J.G. Extension d'une serie des fonctions de Bessel, I and II. Kon.Ned.Akad.v.Wetensch. Proc.<u>45</u>, 929-936 and 987-993 (1942).

2. Carlitz,L. Summation of some series of Bessel functions. Ibid. A 65, 47-54 (1962).

*) Ib. formula (5.3).