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MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFDELING TOEGEPASTE WISKUNDE

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On certain Padé polynomials in connection with the block function

by

R.P. van de Riet

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Introduction

In the theory of electronics the block-function $f(t)$, defined by

$$f(t)=\begin{cases} 1 \text{ for } 0 \leqq t \leqq 1 \\ 0 \text{ elsewhere} \end{cases}$$

plays an important role.

Functions $f_n(t)$ approximating this block-function may be found by approximating the Laplace transform

$$(1) \qquad L[f(t)] = F(s) = \frac{1 - e^{-s}}{s} .$$

The function $F(s)$ can be approximated by the functions

$$(2) \qquad F_n(s) = \frac{P_n(s) - P_n(-s)}{sP_n(s)} ,$$

where $P_n(s)$ is the (n,n) Padé-polynomial belonging to the exponential function e^s. In section 1 the definition of the Padé polynomials and their connection with continued fractions will be given.

In this note we prove the following theorem:

The zeros of the (n,n) Padé-polynomials $P_n(z)$ belonging to the exponential function e^z all lie in the left complex half-plane. *)

By aid of this theorem it can easily be proved that the functions

$$(3) \qquad f_n(t) = L^{-1}[F_n(s)]$$

are bounded and that we have for all n the formula:

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*) In a forthcoming report another shorter proof of this theorem will be given, by aid of Laplace-transformation theory.

(4) $$\int_0^\infty \{f_n(t)\}^2 \, dt = 1,$$

and hence the electronic system, based upon these functions $f_n(t)$, is stable.

In order to prove the theorem we need the following lemma: If

(5) $$P_n(ix) = F_n(x) + iG_n(x),$$

where $F_n(x)$ and $G_n(x)$ are polynomials with real coefficients, then the zeros of $F_n(x)$ and $G_n(x)$ are real and separate one another.

This lemma will be proved in section 2.
The theorem will be proved in section 3, and finally we show the relation (4) in section 4.

1. Padé approximation and continued fractions

Let some function $F(x)$ be approximated by a quotient of polynomials $P_{n,m}(x)$ and $Q_{n,m}(x)$ of degree m and n respectively, in such a way that

$$F(x) \cdot Q_{n,m}(x) - P_{n,m}(x) = O(x^{n+m+1}),$$

$$P_{n,m}(0) = F(0) \text{ and } Q_{n,m}(0) = 1,$$

then $P_{n,m}(x)$ and $Q_{n,m}(x)$ are called the (n,m) Padé polynomials belonging to $F(x)$ (see [1] p. 420)

The (n,m) Padé polynomials belonging to the exponential function e^x are

$$(6)\qquad P_{n,m}(x) = 1 + \frac{m}{n+m}\, x + \frac{m(m-1)}{(n+m)(n+m-1)} \frac{x^2}{2!} + \dots + \frac{m!}{(m+n)(m+n-1)\dots(n+1)} \cdot \frac{x^m}{m!}$$

and $Q_{n,m}(x) = P_{m,n}(-x)$.
In this note we are only interested in the (n,n) Padé polynomials, and we write:

$$P_n(x) \overset{\text{def}}{=} P_{n,n}(x) = Q_{n,n}(-x).$$

In the following we need some results from continued fraction theory.

Let a continued fraction C_n be given

$$C_n = b_o + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \dots + \frac{a_n|}{|b_n},$$ which is a shorter notation for

$$b_o + \cfrac{a_1}{b_1 + \cfrac{a_2}{\ddots\; b_{n-1} + \cfrac{a_n}{b_n}}}$$

We define the series $\{A_m\}$ and $\{B_m\}$ by the recurrence relations:

$$(7)\qquad \left.\begin{aligned} A_m &= b_m\, A_{m-1} + a_m\, A_{m-2} \\ B_m &= b_m\, B_{m-1} + a_m\, B_{m-2} \end{aligned}\right\} m \geq 2$$

with

$$A_o = b_o, \quad B_o = 1,$$

$$A_1 = b_o b_1 + a_1 \text{ and } B_1 = b_1.$$

The A_m and B_m are called the $m^{\underline{th}}$ convergents or $m^{\underline{th}}$ approximants of the continued fraction C_n.

An elementary result from the continued fraction theory is that

$$C_n = \frac{A_n}{B_n} \text{ , for all n.}$$

The following theorem gives a connection between continued fractions and Padé polynomials (see [1] p. 452)

Let the continued fraction

$$(8) \quad C_o + \frac{C_1 x}{|1+l_1 x} + \frac{k_2 x^2}{|1+l_2 x} + \frac{k_3 x^2}{|1+l_3 x} + \dots + \frac{k_n x^2}{|1+l_n x} + \dots$$

be associated with the function F(x), of which the (n,n) Padé polynomials are $P_n(x)$ and $Q_n(x)$, then $P_n(x)$ and $Q_n(x)$ are the $n^{\underline{th}}$ approximants of this continued fraction and the following relations hold:

if $Q_n(x) = 1 + q_1^{(n)}x + q_2^{(n)}x^2 + \dots + q_n^{(n)}x^n$

then $l_1 = q_1^{(1)}$, $l_m = q_1^{(m)} - q_1^{(m-1)}$ $(m \geq 2)$

and $k_m = q_2^{(m)} - q_2^{(m-1)} - l_m q_1^{(m-1)}$,

moreover C_o and C_1 are defined by

$$C_o = F(0) \text{ , } C_1 = F'(0) .$$

From this theorem it follows immediately that the Padé polynomials $P_n(x)$ and $Q_n(x)$ belonging to exp x and given by formula (6) are the n^{th} approximants of

$$1 + \frac{x}{|1} + \frac{\frac{x^2}{4.3.1}}{|1} + \dots + \frac{\frac{x^2}{4(2n-1)(2n-3)}}{|1} + \dots$$

associated with the exponential function.

From formula (7) it is easily seen that the following recursive relations are valid for $P_n(x)$:

$$(9) \quad P_n(x) = P_{n-1}(x) + \frac{x^2}{4(2n-1)(2n-3)} P_{n-2}(x) \qquad n \geq 2.$$

Let now

$$P_n(ix) = F_n(x) + iG_n(x) ,$$

where $F_n(x)$ and $G_n(x)$ are polynomials with real coefficients. By virtue of formula (9) we obtain the following recursive relations for $F_n(x)$ and $G_n(x)$:

$$\left.\begin{aligned} F_n(x) &= F_{n-1}(x) - \frac{x^2}{4(2n-1)(2n-3)} F_{n-2}(x), \\ G_n(x) &= G_{n-1}(x) - \frac{x^2}{4(2n-1)(2n-3)} G_{n-2}(x), \end{aligned}\right\} n \geq 2$$

while formula (6) gives:

$$F_o = 1 \ , \ G_o = 0$$
$$F_1 = 1 \ , \ G_1 = \frac{x}{2} \ .$$

Now we try to find a continued fraction of which $G_n(x)$ and $F_n(x)$ are the $n^{\underline{th}}$ approximants.

A simple algebraic calculation shows that the following continued fraction is the desired one:

$$\frac{\frac{x}{2}|}{|1} + \frac{-\frac{x^2}{12}|}{|1} + \ldots + \frac{-\frac{x^2}{4(2n-1)(2n-3)}|}{|1} \ ,$$ which is equal to

$$(10) \quad \frac{1|}{|2x^{-1}} - \frac{1|}{|6x^{-1}} - \ldots - \frac{1|}{|(4n-2)x^{-1}} \ .$$

Let the functions $R_o^{(n)}(x)$ and $R_1^{(n)}(x)$ be defined by

$$(11) \quad R_o^{(n)}(x) = x^n F_n(x^{-1}) \text{ and } R_1^{(n)}(x) = x^n G_n(x^{-1}) \ ,$$

then

$$\frac{R_o^{(n)}(x)}{R_1^{(n)}(x)} = 2\,x - \frac{1|}{|6x} - \dots - \frac{1|}{|(4n-2)x} \;.$$

When we define $R_2^{(n)}(x)$ by

$$R_2^{(n)}(x) = R_1^{(n)}(x)\;.\;\left(\frac{1|}{|6x} - \dots - \frac{1|}{|(4n-2)x}\right) \;,$$

then

$$R_o^{(n)}(x) = 2\,x\,R_1^{(n)}(x) - R_2^{(n)}(x) \;.$$

If we define consecutively $R_i^{(n)}(x)$ by

$$(12)\quad R_i^{(n)}(x) = R_{i-1}^{(n)}(x)\;\left(\frac{1|}{|(4i-2)x} - \dots - \frac{1|}{|(4n-2)x}\right), \qquad (n \geqq i \geqq 3),$$

then

$$(13)\quad R_{i-1}^{(n)}(x) = (4i-2)x\,R_i^{(n)}(x) - R_{i+1}^{(n)}(x) \qquad (n-1 \geqq 1 \geqq 2)$$

and $R_{n-1}^{(n)}(x) = (4n-2)\;x\;R_n^{(n)}(x)$.

As is well known $R_n^{(n)}(x)$ is the greatest common divisor of $R_o^{(n)}$ and $R_1^{(n)}$, since $R_o^{(n)}$ has the degree n, it follows that $R_n^{(n)}(x)$ must be constant. The relations (13) will be used in the following section.

2. The zeros of $F_n(x)$ and $G_n(x)$

In this section we prove the following lemma.

The zeros of $F_n(x)$ and $G_n(x)$ are real and separate one another, moreover, $P_n(x)$ has no pure imaginary zeros.

We first give some examples:

$$F_2(x) = 1 - \frac{x^2}{12}, \qquad R_0^{(2)}(x) = x^2 - \frac{1}{12},$$
$$G_2(x) = \frac{x}{2}, \qquad R_1^{(2)}(x) = \frac{x}{2},$$
$$F_3(x) = 1 - \frac{x^2}{10}, \qquad R_0^{(3)}(x) = x^3 - \frac{x}{10},$$
$$G_3(x) = \frac{x}{2} - \frac{x^3}{120}, \qquad R_1^{(3)}(x) = \frac{x^2}{2} - \frac{1}{120}.$$

(In the sequel of this section we omit the upper index of the functions $R_0^{(n)}(x)$ and $R_1^{(n)}(x)$.)
We remark that the degrees of R_0 and R_1 are n and n-1 respectively. When n is even then R_0 and R_1 have the same degrees as F_n and G_n resp., so the zeros of R_0 and R_1, are the reciprocals of the zeros of F_n, resp. G_n, with the exception of the zeros x = 0. When n is odd, R_0 has one zero more (x=0) than F_n and R_1 has one zero less than G_n.
In both cases the following statement is true: if and only if the zeros of R_0 and R_1 are real and separate one another then the same is true for the zeros of F_n and G_n.

Since R_n = G.C.D (R_0,R_1) is a constant, the polynomials R_0 and R_1 have no zeros in common.
In particular it follows that $P_n(x)$ has no pure imaginary zeros, which proves the second statement of the lemma.

Let $R_i(x) = c_i\, x^{n-i}+\ldots$ then it follows from (13) that all c_i are positive.
Let $V\{\lambda_1,\ldots,\lambda_n\}$ be the number of variations of sign in the sequence of the real numbers $\lambda_1,\ldots,\lambda_n$. According to the Sturm theory of zeros of polynomials we study

$$V^*(x) = V\{R_0(x), R_1(x),\ldots, R_n(x)\}.$$

$V^*(x)$ can only change in a zero of $R_i(x)$.

Assume $R_i(x_o) = 0 \quad (i \geqslant 1)$, then $R_{i+1}(x_o)$ and $R_{i-1}(x_o)$ have different signs, as follows from (13), provided they are not zero, but this is impossible since $R_n = G.C.D\,(R_i, R_{i-1}) = \text{const.} \neq 0$ for $i = 1, \ldots, n$.
For a sufficiently small ε it holds therefore that $R_{i+1}(x_o \pm \varepsilon)$ and $R_{i-1}(x_o \pm \varepsilon)$ have different signs.
Therefore the eventual change of sign of $R_i(x)$, $(i \geq 1)$ does not effect the value of $V^*(x)$.
Thus the only points where $V^*(x)$ can alter are zeros of $R_o(x)$ and then $V^*(x)$ actually alters (if this zero is simple).
Now $V^*(-\infty) = n$ (all c_i are positive) and $V^*(+\infty) = 0$, therefore $R_o(x)$ has n real different zeros x_k, furthermore, we see that between each zero of $R_o(x)$ there must lie a zero of $R_1(x)$; otherwise $R_1(x)$ should have the same sign in $(x_k - \varepsilon, x_{k+1} + \varepsilon)$, where ε is a sufficiently small positive number, and the change of $V^*(x)$ in x_k is then cancelled by the change in x_{k+1}.
So we have shown that $R_o(x)$ and $R_1(x)$ have only real zeros which separate one another; this proves the lemma stated in the beginning of this section.

3. Proof of the theorem

In this section we shall prove the theorem, mentioned in the introduction, concerning the zeros of the (n,n) Padé polynomial belonging to the exponential function.
The theorem is:

The zeros of $P_n(z)$ all lie in the left complex half-plane $Re(z) < 0$.

In order to prove this theorem we use the following function-theoretic theorem:

Let D be a simply connected domain with boundary C (double-

point free) and $f(z)$ holomorphic in D, continuous in D + C and $\neq 0$ on C.
If the change of argument of $f(z)$ is Δ, when $f(z)$ is continued along C in the positive sense, then the number of zeros of $f(z)$ in D is equal to

$$\frac{\Delta}{2\pi} .$$

This theorem will now be applied to the Padé polynomial $P_n(z)$, with the contour C, which consists of a part of the imaginary axis and a semi-circle with centre in z=0 and radius R as shown in fig.1.

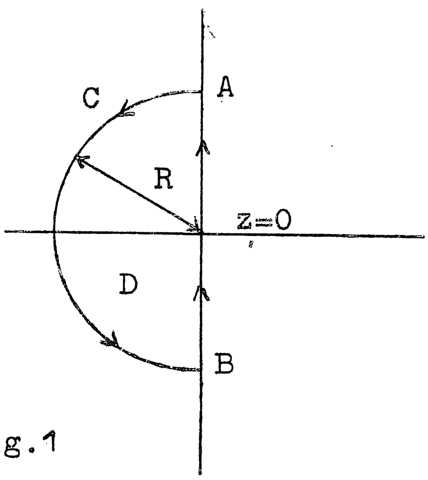

fig.1

The direction of the contour C is positive.
We choose R so large that C encloses all the zeros of $P_n(z)$ which lie in the left half plane. From the lemma proved in section 2 we know already that $P_n(z)$ is $\neq 0$ on the imaginary axis, so that we may apply the above stated function-theoretic theorem.
Moreover we assume R so large that only the main term

$$\frac{z^n}{(2n)\dots(n+1)}$$

of $P_n(z)$ is important.
We now continue $P_n(z)$ along C starting in A with $z= Re^{i\pi/2}$.
The argument of $P_n(z)$ is $\frac{1}{2}n\pi$ in A and after continuing $P_n(z)$ along the semi-circle unto B the argument is $\frac{3}{2}n\pi$, so the increase is $n\pi$.

The path B A along the imaginary axis will now be studied. We decompose $P_n(ix)$ in its real and imaginary parts.

(5) $P_n(ix) = F_n(x) + i\, G_n(x)$,

where $F_n(x)$ and $G_n(x)$ are polynomials with real coefficients. It is clear that the function $Q_n(x)$ defined by

$$Q_n(x) = \frac{G_n(x)}{F_n(x)} ,$$

is the tangent of the argument of $P_n(ix)$.

When n is even F_n has degree n and G_n degree n-1, when n is odd then F_n has degree n-1 and G_n degree n . Thus we see that when n is even $Q_n(x) \downarrow 0$ for $x \to -\infty$, and $Q_n(x) \uparrow 0$ for $x \to +\infty$.

When n is odd $Q_n(x) \downarrow -\infty$ for $x \to -\infty$ and $Q_n(x) \uparrow \infty$ for $x \to \infty$.

We now make use of the lemma shown in section 2, which states that F_n and G_n have only real zeros which separate one another, therefore $Q_n(x)$ takes the form shown in the figures 2 and 3 for n is even and n is odd respectively.

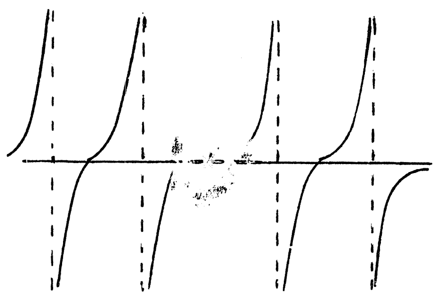

fig.2

n is even

n-1 zeros

n vertical asymptotes

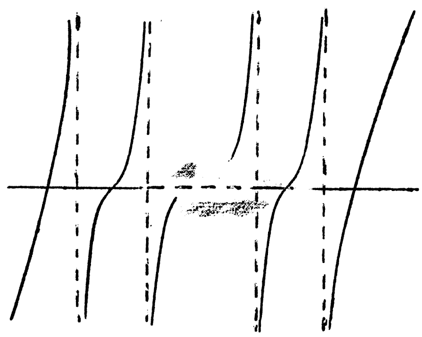

fig.3

n is odd

n zeros

n-1 vertical asymptotes.

From these figures it follows immediately that the argument of $P_n(z)$ is increased with $n\pi$ when $P_n(z)$ is continued along the imaginary axis from B to A. Hence the total increase of the argument is $2n\pi$, thus according to the stated function-theoretic theorem, the number of zeros of $P_n(z)$ which lie in D is n, or all the zeros of $P_n(z)$ lie in the left half of the complex plane, which proves the theorem.

4. Application of the theorem

We now apply the theorem to the functions $F_n(s)$ defined in formula (2)

$$F_n(s) = \frac{P_n(s) - P_n(-s)}{s\, P_n(s)} .$$

Applying the inverse Laplace transformation to the functions $F_n(s)$ we obtain:

$$(14) \quad f_n(t) = L^{-1}\left[F_n(s)\right] = \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{+i\infty+\sigma} e^{st} \frac{P_n(s) - P_n(-s)}{s\, P_n(s)}\, ds,$$

Where σ should be chosen such that all the poles of the integrand in (14) lie at the left hand side of the path of integration.

A consequence of the theorem is that σ can be taken negative ($s=0$ is not a pole!)

We consider now the following integral:

$$I = \int_0^\infty \{f_n(t)\}^2 dt = \frac{1}{2\pi i} \int_0^\infty f_n(t) \int_{-i\infty+\sigma}^{+i\infty+\sigma} e^{st} \frac{P_n(s) - P_n(-s)}{s\, P_n(s)}\, ds\; dt.$$

Interchanging the order of integration, we get

$$I = \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{+i\infty+\sigma} \frac{P_n(s) - P_n(-s)}{s\, P_n(s)} \cdot \frac{P_n(-s) - P_n(s)}{-s\, P_n(-s)}\, ds =$$

$$= \frac{1}{2\pi i} \int_L \frac{P_n(s) - P_n(-s)}{s^2\, P_n(s)}\, ds$$

where L is a contour consisting of L_1 and L_2 as shown in fig. 4;

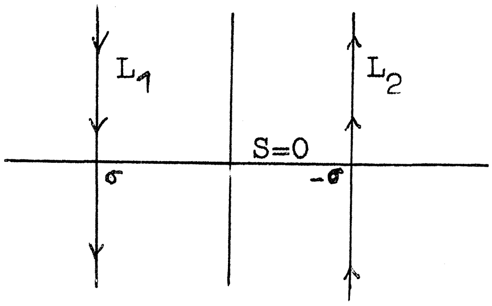

fig.4

The path of integration L encloses only one simple pole s = 0 with residue 1, hence

$$\int_0^\infty \{f_n(t)\}^2 dt = 1.$$

which proves formula (4).

References

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