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Technical Note TN 34

The solution of wave propagation Problems using an iterative operator

by

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August 1963

SUMMARY

Studies of tidal and open channel hydraulics lead to the basic problem of solving systems of hyperbolic partial differential equations under complicated initial and boundary conditions. In practice these problems are often so involved that they can only be solved with the aid of a digital computer, and for this purpose the equation systems are expressed as numerical operators. Most existing operators, however, utilise an iterative process to obtain more representative data, especially when non-linear equations are concerned, and this suggests that the operator itself may as well be of an iterative type. Such an "Iterative operator" forms the subject of the present paper.

The Iterative operator uses known conditions at one time to determine approximately the conditions at a later time, these approximate conditions then being averaged with the original conditions to determine a better approximation. Such an operator has a complicated explicit form, but may be generated from a very simple programme in the computer.

The paper falls into two parts. In the first part, a linear stability analysis is employed to show that the Iterative operator stability depends upon the extent to which forward time differences are utilised in the iteration. If only these differences are utilised the operator may be stable, while it they are averaged, half and half, with the backward time differences, the operator will be unstable. The equivalent operator for two space dimensions is shown to have the save properties.

In the second part of the paper, an analytically unstable operator for one-dimensional propagation is shown, experimentally, to behave in an apparently stable manner in two widely separated problems. The experimental procedure is described in some detail. The more evidently unstable behaviour in two dimensions is also discussed, with special reference to the North Sea problem. The growth of instability is illustrated.

In conclusion the reasons for this apparent stability, or weak instability, are discussed and related to the possibility of transforming such unstable systems into-stable ones by "smoothing" or "damping" operators, such as properly correspond to convergence acceleration techniques. Using these techniques the iterative operator may be employed very extensively.

INTRODUCTION

The wave propagation problems treated in the present paper relate specifically to "long" water waves in which the motion of the fluid is mainly horizontal. Problems of this type arise in studies of tidal waves and storm surges, as well as in the design of canals for navigation and water supply. In most of these practical problems it is then found that the boundary conditions take forms which cannot be represented by simple functions, so that the process of classical hydrodynamics, whereby the governing hyperbolic equation systems are reduced to elliptic systems by the introduction of a condition of periodicity, cannot be followed. In these problems, therefore, the hyperbolic equation systems have to be retained. These equations may be written for one dimensional propagation (1), and two dimensional propagation (2), neglecting resistance, Coriolis and surface stress terms, as

MOTION $$\frac{\partial u}{\partial t} + \frac{u\partial u}{\partial x} + g\frac{\partial \eta}{\partial x} = 0$$

(1)

CONTINUITY $$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + (h+\eta)\frac{\partial u}{\partial x} = 0$$

MOTION $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial \eta}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} + u\frac{\partial v}{\partial x} + g\frac{\partial \eta}{\partial y} = 0 \quad (2)$$

CONTINUITY $$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} + (h+\eta)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

In these expressions, x and y are Cartesian space coordinates in the horizontal plane, t is the time coordinate, u and v are velocities in the x and y directions respectively, η is the elevation of the water level above its equilibrium level and h is the depth of water below the equilibrium level. The notation and the expressions themselves follow directly from Lamb (Ref. 4).

In most practical cases the equation systems are solved using a digital computer. The instantaneous state of the fluid system is then supposed to be described by a representative set of numbers, associated with a set of equally grid points, while the transition from one instantaneous state to another is supposed to be simulated by a transformation from one representative set to another, as effected by the intercession of an operator (Ref. 2).

Now when a numerical operator is used in practice, the state of the system at one time, t, is usually first advanced tentatively to a later time, $t + \Delta t$, the tentative state at $t + \Delta t$ then being used only to provide more representative data such as convective velocities and friction coefficients for a second computation from t. This, however, introduces an iterative process into many practical wave computations, especially when the convective velocities are significant, and suggests that the operator itself might then also be of an iterative type.

PART I

THE ITERATIVE OPERATOR

For the purpose of illustrating the formation and behaviour of such an "Iterative operator" we may consider the operator in its linear form, as applied to one dimensional propagation. The operator is then based on the linearised form of (1):

$$\begin{aligned} \frac{\partial u}{\partial t} + g\,\frac{\partial \eta}{\partial x} &= 0 \\ \frac{\partial \eta}{\partial t} + h\,\frac{\partial u}{\partial x} &= 0 \end{aligned} \qquad (3)$$

Now we may use these expressions to derive approximate values of the flow variables at one time level $(n+1)\Delta t$, from values at the preceeding time level, $n\Delta t$, for, if we represent these approximate values by (u_j^{n+1}), we have at once that:

$$(u_j^{n+1}) = u_j^n - g\left(\frac{\partial \eta}{\partial x}\right)_j^n \Delta t$$

$$(\eta_j^{n+1}) = \eta_j^n - h\left(\frac{\partial u}{\partial x}\right)_j^n \Delta t$$

where the expressions are supposed centred on a point $j\Delta x$ and where $\left(\frac{\partial}{\partial x}\right)_j^n$ is the space derivative at that point at time $n\Delta t$.

As a first approximation we may represent the partial space derivatives by first differences to give:

$$(u_j^{n+1})_1 = u_j^n - \frac{g}{2}\left(\frac{\Delta t}{\Delta x}\right)\left(\eta_{j+1}^n - \eta_{j-1}^n\right) \qquad (4)$$

$$(\eta_j^{n+1})_1 = \eta_j^n - \frac{h}{2}\left(\frac{\Delta t}{\Delta x}\right)\left(u_{j+1}^n - u_{j-1}^u\right) \qquad (5)$$

If we now use these trial values at j, n+1 to derive a better approximation for the flow variables at the same point, which variables we shall then call $(u_j^{n+1})_2$, $(\eta_j^{n+1})_2$, we find that:

$$(u_j^{n+1})_2 = u_j^n - \frac{g}{2}\left(\frac{\Delta t}{\Delta x}\right)\left\{\left[\eta_{j+1}^{n+1}\right]_1 - \left[\eta_{j-1}^{n+1}\right]_1\right\}, \tag{6}$$

$$= u_j^n - \frac{g}{2}\left(\frac{\Delta t}{\Delta x}\right)\left\{\left[\eta_{j+1}^n - \frac{h}{2}\left(\frac{\Delta t}{\Delta x}\right)(u_{j+2}^n - u_j^n)\right] - \left[\eta_{j-1}^n - \frac{h}{2}\left(\frac{\Delta t}{\Delta x}\right)(u_j^n - u_{j-2}^n)\right]\right\} \tag{7}$$

$$(\eta_j^{n+1})_2 = \eta_j^n - \frac{h}{2}\left(\frac{\Delta t}{\Delta x}\right)\left\{\left[u_{j+1}^{n+1}\right]_1 - \left[u_{j-1}^{n+1}\right]_1\right\}, \tag{8}$$

$$= \eta_j^n - \frac{h}{2}\left(\frac{\Delta t}{\Delta x}\right)\left\{\left[u_{j+1}^n - \frac{g}{2}\left(\frac{\Delta t}{\Delta x}\right)(\eta_{j+2}^n - \eta_j^n)\right] - \left[u_{j-1}^n - \frac{g}{2}\left(\frac{\Delta t}{\Delta x}\right)(\eta_j^n - \eta_{j-2}^n)\right]\right\} \tag{9}$$

We now take some average of the values of the flow variables over these first two steps, representing these average values by $(u_j^{n+1})_{\overline{1,2}}$ and $(\eta_j^{n+1})_{\overline{1,2}}$, so that

$$\begin{aligned}(u_j^{n+1})_{\overline{1,2}} &= (1-\theta)\,(u_j^{n+1})_1 + \theta(u_j^{n+1})_2\\ (\eta_j^{n+1})_{\overline{1,2}} &= (1-\theta)\,(\eta_j^{n+1})_1 + \theta(\eta_j^{n+1})_2\end{aligned} \tag{10}$$

where θ is a constant such that $1 \geq \theta \geq 0$.

It then follows from equations 4, 5, 7, 9 and 10 that

$$\begin{aligned}(u_j^{n+1})_{\overline{1,2}} &= u_j^n - \frac{g}{2}\left(\frac{\Delta t}{\Delta x}\right)(\eta_{j+1}^n - \eta_{j-1}^n) + \theta\frac{gh}{4}\left(\frac{\Delta t}{\Delta x}\right)^2(u_{j+2}^n - 2u_j^n + u_{j-2}^n)\\ (\eta_j^{n+1})_{\overline{1,2}} &= \eta_j^n - \frac{h}{2}\left(\frac{\Delta t}{\Delta x}\right)(u_{j+1}^n - u_{j-1}^n) + \theta\frac{gh}{4}\left(\frac{\Delta t}{\Delta x}\right)^2(\eta_{j+2}^n - 2\eta_j^n + \eta_{j-2}^n)\end{aligned} \tag{11}$$

This is the simplest type of iterative operator, being a linear operator with first derivatives replaced by first differences and taken to only one iteration. With further iterations this operator will appear to become more complicated, but this is only so when it is written out explicitly. As in operator in the computer it will always keep the simple form represented by equations (4) and (5) or, what are then effectively the same things, equations (6) and (8). It will be clear from this simple example that the Iterative operator may be easily extended to account for a convective velocity, as well as friction and surface stress terms, with but little increase in complexity. This represents its main advantage over other types of operator currently in use.

The simple operator (11) serves to illustrate the process of forming such an operator in the machine. The situation is very similar if we replace the first space derivatives by second difference approximations (Ref. 3, p. 118) as follows*.

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* While attempting to improve spatial accuracy in this way we in fact maintain lower order time accuracy, at least for $\theta \neq \frac{1}{2}$. With $\theta = \frac{1}{2}$ higher order time accuracy is obtained, but since for stability $\Delta t = 0\ (\Delta x)$, it must seem inconsistent to seek more accurate space differences. These considerations, however, were only pointed out to the author after the experimental work described later was completed, and the description of the second difference form is accordingly retained here as a background for the experimental study

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$$\left(\frac{\partial u}{\partial x}\right)_j^n = \frac{1}{12\Delta x} \left(- u_{j+2}^n + 8u_{j+1}^n - 8u_{j-1}^n + u_{j-2}^n\right)$$

$$\left(\frac{\partial \eta}{\partial x}\right)_j^n = \frac{1}{12\Delta x} \left(- \eta_{j+2}^n + 8\eta_{j+1}^n - 8\eta_{j-1}^n + \eta_{j-2}^n\right)$$

Proceding as before, we then obtain the form of the operator after one iteration as

$$\left(u_j^{n+1}\right)_{\overline{1,2}} = u_j^n - \frac{g}{12}\left(\frac{\Delta t}{\Delta x}\right)\left(-\eta_{j+2}^n + 8\eta_{j+1}^n - 8\eta_{j-1}^n + \eta_{j-2}^n\right)$$

$$+ \frac{\theta g h}{144}\left(\frac{\Delta t}{\Delta x}\right)\left(u_{j+\Lambda}^n - 16u_{j+3}^n + 64u_{j+2}^n + 16u_{j+1}^n\right.$$

$$\left. - 130u_j^n + 16u_{j-1}^n + 64u_{j-2}^n - 16u_{j-3}^n + u_{j-4}^n\right) \qquad (12)$$

$$\left(\eta_j^{n+1}\right)_{\overline{1,2}} = \eta_j^n - \frac{h}{12}\left(\frac{\Delta t}{\Delta x}\right)\left(-u_{j+}^n + 8u_{j+1}^n - 8u_{j-1}^n + u_{j-}^n\right)$$

$$+ \frac{\theta g h}{144}\left(\frac{\Delta t}{\Delta x}\right)\left(\eta_{j+4}^n - 16\eta_{j+3}^n + 64\eta_{j+2}^n + 16\eta_{j+1}^n\right.$$

$$\left. - 130\eta_j^n + 16\eta_{j-1}^n + 64\eta_{j-2}^n - 16\eta_{j-3}^n - \eta_{j-4}^n\right)$$

Again, these take much simpler forms as a programme in a computer.

The governing equations for two-dimensional wave propagation may beexpressed as iterative operators, centred on a point $j\Delta x$, Δy at time $n\Delta t$, in much the same way.

STABILITY ANALYSIS OF THE LINEAR ITERATIVE OPERATOR

For the stability analysis of the operators given above we may employ a procedure described by Richtmeyer (Ref. 5). In this procedure we analyse the Fourier series solutions of the finite difference equations, which solutions we express, for one-dimensional propagation, in the form:

$$u_j^n = A \quad \xi_1^n \; e^{\,imj\Delta x}$$
$$\eta_j^n = A \quad \xi_2^n \; e^{\,imj\Delta x} \qquad (13)$$

After substituting (13) in (11) we obtain, after some simplification that

$$\xi_1^{n+1} = \xi_1^n(1 - \theta gh\alpha^2) + \xi_2^n(- ig\alpha)$$
$$\xi_2^{n+1} = \xi_1^n(- ih\alpha) + \xi_2^n(1 - \theta gh\alpha^2)$$

where $\alpha = (\frac{\Delta t}{\Delta x}) \sin m\Delta x$.

Thus the amplification matrix, H, (Ref. 5), is

$$H = \begin{bmatrix} 1 - \theta gh\alpha^2 & - ig\alpha \\ - ih\alpha & 1 - \theta gh\alpha^2 \end{bmatrix} \qquad (14)$$

The latent roots, λ , of this matrix are then given by

$$\lambda = (1 - \theta A) \pm \sqrt{- A}$$

where $$A = gh\alpha^2 \qquad (15)$$

We now apply the so-called "von Neumann necessary condition for stability", whereby a necessary condition for stability is that

$$|\lambda| \leqslant 1 + O(\Delta t).$$

This yields as a condition for stability that

$$\frac{\Delta t}{\Delta x} \sqrt{gh} \leqslant \frac{\sqrt{2\theta - 1}}{\theta}, \; \theta > \tfrac{1}{2},$$

a result which is depicted graphically for $\theta = \frac{1}{2}$ and $\theta = 1$ in fig. 1. It is then clear that, with $\theta = 1$, the operator may be stable for $1 > A > 0$, i.e. so long as

$$\left(\frac{\Delta x}{\Delta t}\right) \geqslant \sqrt{gh}. \quad \text{i.e. } \Delta t \quad \frac{\Delta x}{\sqrt{gh}}$$

Thus the simple linear operator, with $\theta = 1$, remains stable so long as the domain of dependence of any point in the mesh as given by the difference equations lies within the domain of dependence determined by the differential equation system. $\theta = \frac{1}{2}$, on the other hand, represents the lower limit of stability, so that there is no finite value of $\frac{\Delta x}{\Delta t}$ for which the system will then be stable. In particular, for $\theta = \frac{1}{2}$ it is easily shown that

$$\lambda^2 = 1 + \frac{a^4}{4}$$

so that in this case λ is always greater than unity.

The analysis of the system given in (12) follows the same lines as that given above, and indeed leads to the same result, (15), except that in this case A is given by

$$A = gh\alpha^2$$

with

$$\alpha = \frac{1}{12}\left(\frac{\Delta t}{\Delta x}\right)(8 \sin j\Delta x - \sin 2j\Delta x).$$

Thus, this second difference operator, with $\theta = 1$, will be stable so long as

$$\left(\frac{\Delta x}{\Delta t}\right) \geqslant \frac{8.23}{12} \sqrt{gh}, \text{ approximately.}$$

It may be shown that all these conditions are also sufficient for stability. The stability analysis of the iterative operator for two dimensional propagation may be followed in the same way as described above, to give an amplification matrix, H, of the form:

$$\begin{bmatrix} (1 - \theta gh\alpha^2 - \theta gh\beta^2) & -ih\alpha & -ih\beta \\ -ig\alpha & 1 - \theta gh\alpha^2 & -\theta gh\alpha\beta \\ -ig\beta & -\theta gh\alpha\beta & 1 - \theta gh\beta^2 \end{bmatrix} \quad (18)$$

where $\alpha = (\frac{\Delta t}{\Delta x})$ (sin $j\Delta x$) and $\beta = (\frac{\Delta t}{\Delta y})$ (sin $j\Delta y$)

If we suppose that $\Delta x = \Delta y = \Delta s$. as is usual in such computations, we find that for $\theta = \frac{1}{2}$ the largest latent root of (18) is always greater than zero. With $\theta = 1$ the roots are

$$\lambda = 1$$

$$\lambda = -(2A - 1) \pm i\sqrt{2A}$$

so that, the system being stable so long as

$$|\lambda| < 1,$$

the necessary condition for stability is that

$$\sqrt{A} < \frac{1}{\sqrt{2}}$$

or

$$\Delta t < \frac{\Delta x}{\sqrt{2}c}$$

It may be shown that this condition is also sufficient for stability.

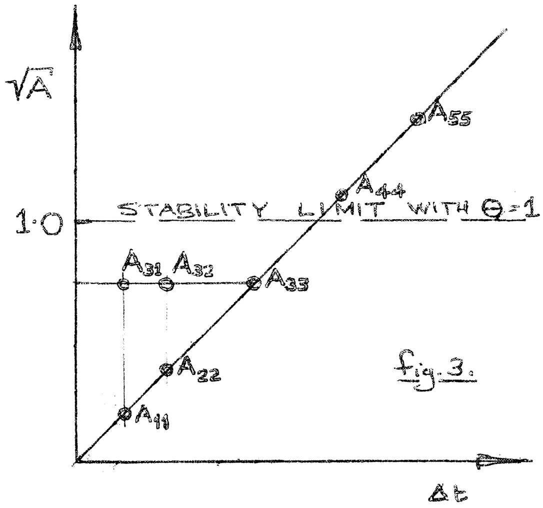
PART II

THE EXPERIMENTAL STABILITY ANALYSIS

The organisation of the experiments carried out in the present work can best be visualised in terms of a "parameter space", R_p, this being the space formed by those parameters which may influence the stability of a computation. Now the study of the linearised computational system suggests that the stability criteria will include the characteristic slopes and the ratios between the space and time increments in the space-time grid. We may accordingly choose, as three components of R_p, the characteristic slope, c, the space increment Δx and the time increment Δt. As a check on the consistency of the stability, the time range should also be introduced as a parameter, this implicitly introducing, with Δt, the number of computational steps.

It would also appear that further parameters appear in the initial conditions and in the conditions in the region of integration, including the boundary conditions, especially when non-linear equations are concerned. These accessory condition parameters may be introduced into the study by carrying out two sets of experiments at widely different accessory conditions, that is to say, roughly speaking, at "end points" in the appropriate R_p, and noting the influence of these conditions upon the overall stability conditions. If there is then no noticeable difference, these parameters may be neglected.

It will be found convenient to define the base plane in R_p by the parameters Δt and ϕ, the latter being a dimensionless number given by $\phi = \frac{c\,\Delta t}{\Delta x}$. This number is seen to be related to the spectral radius of the amplification matrix. Fig. 3 depicts this base plane and shows the stability limit as determined for the linearised operator with $\theta = 1$. (See Appendix 11)



The simplest operator for the investigation of the parameter space is the first difference type (11). It is found in practice, however, that when this simple form is used with $\theta = \frac{1}{2}$, oscillations occur in the solutions, a result which may be anticipated from the preceding analysis. Although these oscillations do not appear to build up (i.e. the operator may appear to remain stable) the errors thereby introduced into the computation are such as to cause some concern. The second difference operator of equation (12), on the other hand, gives much smoother results.

A rather more surprising result, however, concerns the "unstable" operator with $\theta = \frac{1}{2}$. This operator was used in the present work following an error in the linear analysis, the results then being so stable and consistent that the error remained unnoticed until the experimental work described here was completed. In view of the possible interest of this result, the experiments leading to it will be described in a little detail, this description, at the same time, serving to illustrate the experimental procedure.

THE EXPERIMENTAL ANALYSIS OF A DISTINCTLY NON-LINEAR SYSTEM

We shall first take as an example a problem where the non-linear effects are very pronounced. Such an example is provided by the wave induced in a shallow canal by an efflux of water from one of its ends. We require independent solutions for the purposes of comparison, and we take these from a computation using the well established graphical method of characteristics. The nature of the problem is completely described by the appropriate physical and hodograph characteristics, as shown in Fig. 4. In Fig. 5 the results following from this graphical solution are compared with those obtained using the iterative operator with $\theta = \frac{1}{2}$ at small $\emptyset$. This then corresponds to establishing the behaviour of the operator at a point such as A_{11} in R_p, (Fig. 3).

We now proceed along a ray passing through A_{11}. We pass successively through A_{22}, A_{33}, A_{44}, and A_{55}, comparing results with the established system at each point. If, now, we take the mean square deviation of our results from the reference results for several successive time intervals, and plot these in the parameter space, we have a stability "picture" for the operator. A section through the space along the line $A_{11} - A_{55}$ is shown in Fig. 6.
From Fig. 5 it has been found that whereas over one part of the line $A_{11} - A_{55}$ the errors accumulate, over another part they merely oscillate with but little indication of accumulation. This situation is indicated for two time levels in Fig. 5.

So far we have only located the apparent stability limit at a discrete point in R_p: it is also necessary, however, to determine the boundary of the stable region along a distinct line or surface in R_p. Accordingly the experiments described above have been repeated with $\emptyset$ held constant, i.e. along a line such as $A_{33} - A_{31}$ (Fig. 2). A comparison of the results thus obtained with those given by the reference system indicates that, once again, the deviations merely oscillate. The actual deviations are indicated by the lighter lines in Fig. 5. Thus the stability seems to be assured along the line $A_{33} - A_{31}$, which is parallel and close to the experimental stability boundary, and thus, by all reasonable inference, stability should be assured throughout the region where ϕ takes a value which is lower that that on $A_{33} - A_{31}$.

The remaining possible parameter making for instability, outside of the accessory conditions, is the time range, T. To a large extent, however, the influence of this parameter can be discounted by the agreement between the experimental results at A_{11} and A_{33}, which, since they involve different values of Δt, also involve considerable differences in the number of computational steps. As an additional check, however, the computation has been run at A_{33}, (i.e. at a value of ϕ rather less than that associated with instability) over a time range $2\frac{1}{2}$ times that normally taken. The results thereby obtained are shown in Fig. 7, where they are seen to be, if anything, rather better than those obtaining at smaller ranges of T. Thus it would seem that the influence of T as a parameter inducing stability can be discounted, at least for the time range associated with this problem, so that the consistency of the stability appears also to be sufficient.

THE EXPERIMENTAL ANALYSIS OF AN ALMOST LINEAR SYSTEM

The above procedure has been repeated for the case of a standing wave in a straight uniform canal closed at one end and connected at the other end to a tidal sea. In this case an analytical procedure is used as the reference system, the actual case taken being that treated by Lamb (Ref. 4 p. 267). The dependent variables are then given by:

$$\eta = \frac{a \cos\left(\frac{\sigma x}{c}\right)}{\cos\left(\frac{\sigma l}{c}\right)} \cdot \cos(\sigma t + \phi) \qquad (22)$$

and

$$u = \frac{ga}{c} \frac{\sin\left(\frac{\sigma x}{c}\right)}{\cos\left(\frac{\sigma l}{c}\right)} \cdot \sin(\sigma t + \phi) \qquad (23)$$

where
- a = amplitude of its external tide
- σ = its frequency
- c = wave celerity
- l = lenght of canal
- ϕ = phase angle of the external tide.
- a = amplitude of external tide.

This system was also computed from the given initial conditions using the second-difference operator with $\theta = \frac{1}{2}$. A typical case within the apparently stable range of the computation is shown in Fig. 8.

In this set of experiments the deviations from the reference system are much the same as those found for the efflux problem and illustrated in Fig. 4. This then suggests that, for the range of flows between the standing wave and the efflux problem, the influence of the initial and boundary condition parameters is negligible, so that their influence may accordingly be discounted.

The above study indicates that an apparently "unstable" operator may in face be "sufficiently stable" to provide acceptable results in a practical problem.

In fact, such is the difference between the results of the stability analysis and the experimental results that it might be supposed that even the two-dimensional operator might provide acceptable results in practice. Accordingly a similar experimental procedure to that described above was followed using the two dimensional operator on an open sea region (The North Sea). However, the solution quickly showed signs of instability and the experiments had to be abandoned at an early stage. This not withstanding, it was very noticeable in this experiment that the instability was most clearly marked in those parts of the sea which were deepest, (i.e. in the Norwegian deep), while in the shallower parts of the region the tidal conditions could be predicted with a quite acceptable accuracy for some four hours. This, again is at considerable variance with the stability analysis

CONCLUSIONS

By analysing the stability of the iterative operator, we have shown that it may be used to provide stable solutions to wave propagation problems. We have also indicated the advantages of this operator for wave computations when such terms as convective velocity and friction need to be taken into account.

Following convention, we have defined a number θ as the fraction of the forward differences comprehended by the operator, and we have shown that the optimum stability conditions are attained with $\theta = 1$. At the same timewe have shown that, with $\theta = \frac{1}{2}$, the operator is at the lower limit of stability, so that for any finite value of $\frac{\Delta t}{\Delta x}$ it should give unstable results. In practice, however, we have shown that this operator , with $\theta = \frac{1}{2}$, gives results of acceptable accuracy, at least for the problems treated in the text, and we have accordingly described it as "sufficiently stable". We may assume, of course, that this "sufficient" stability is only attainable for the order of time ranges employed, and that if the experiments described above had been pursued further, then, during the course of time, instability would have become apparent. In this case, however, the non-linear computations considered above would probably have broken down anyway, due to the intercession of vertical accelerations with increasing wave steepness. Similarly, in the case of the linear computations, there would not normally be much point in pursuing so many oscillations of the system that the very small closing errors could build up. Thus the operator with $\theta = \frac{1}{2}$, if not actually stable in the analytical sense, would appear to be sufficiently stable for certain practical purposes.

In the two-dimensional problem treated in this study the instability was too strong to justify an unmodified use of the operator, but in a work of Uusitalo(Ref. 7) a very similar operator to that described has been successfully employed. In this latter work instability was encountered, but overcone by use of "smoothing terms". These correspond to well-known convergence acceleration techniques.

The above results, and the existence of acceleration techniques, especially adapted to iterative schemes, suggests that the commonly accepted sharp demarcation between stable and unstable schemes may be too severe, and that in fact there exists a hinterland of considerable potential value. By the further study of such "almost-stable" schemes, and the means to make them entirely stable, this hinterland could pass satisfactorily into engineering computing practice.

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APPENDIX

For the purposes of comparing the different states of a system we suppose that the members of the system's representative subsets (in u and h, or $\sqrt{gh}$) form a Euclidean function space. We take a projection from out of this space and into the space of one of the dependent variables (e.g.h). Then the norm, the difference between any two states of the system, may be measured by a distance ε in this space, given by:

$$\varepsilon = \sqrt{\sum_i (x_i^2 - y_1^2)}$$

where the x_i are the function space coordinates corresponding to one state and the y_i are those corresponding to the other state. We have used this norm in the present work when constructing figure (6).

The concept of a function space serves also to illustrate the difference between the two fluid systems considered in the text. We then suppose that the motion of each system is followed by the motion of its representative point in the function space, so as to generate a trajectory. Then the almost linear system, which describes a "pendulating" flow in which all energy may at some stage be either kinetic or potential, will describe a closed trajectory which will be homeomorphic with the trajectories of all other such systems. The much less linear "propagating" system, on the other hand, will describe an open trajectory which cannot be homeomorphic with the pendulating one. From this point of view, therefore, the differences between the two cases considered reduce to topological differences between their respective hydronamic trajectories.

LIST OF CAPTIONS

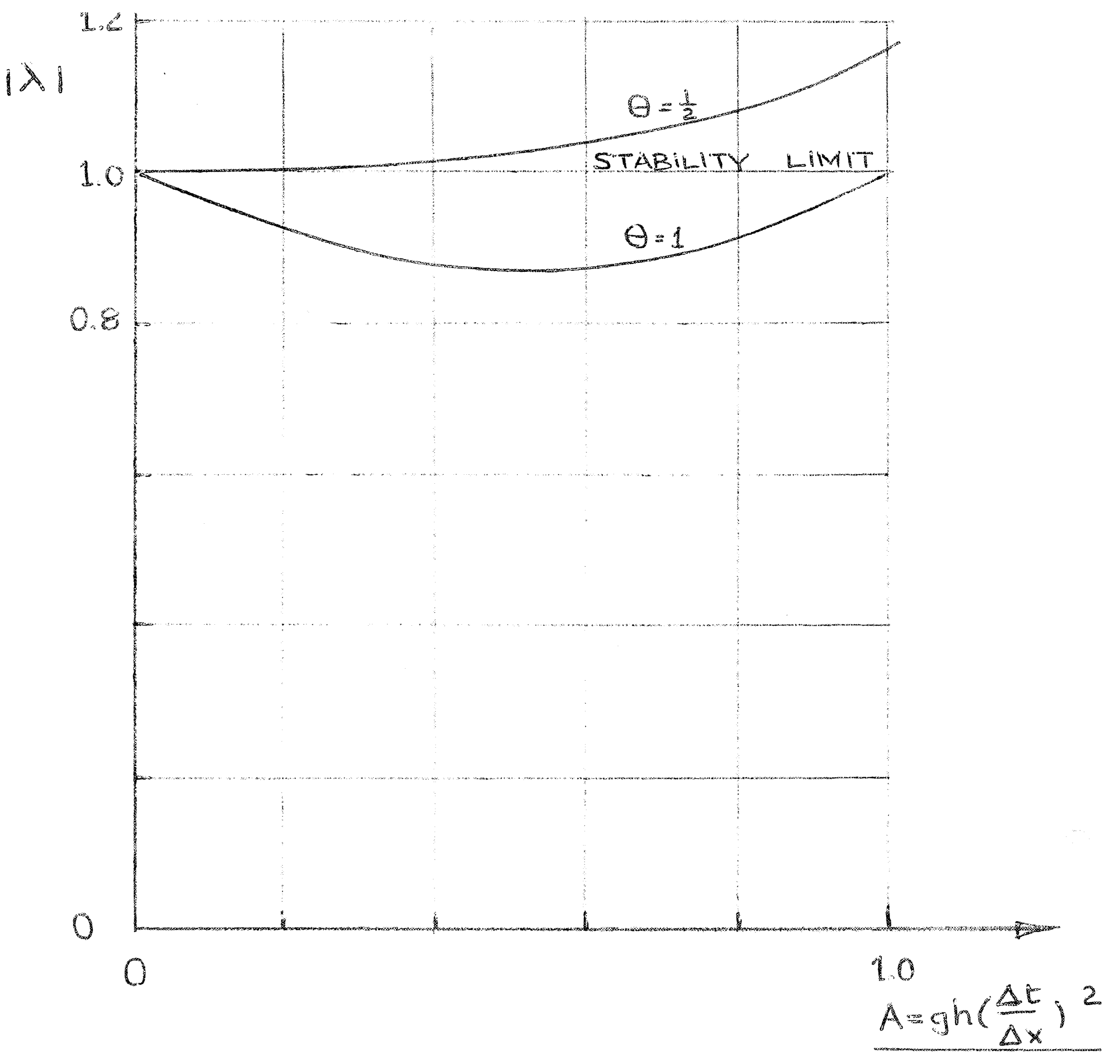
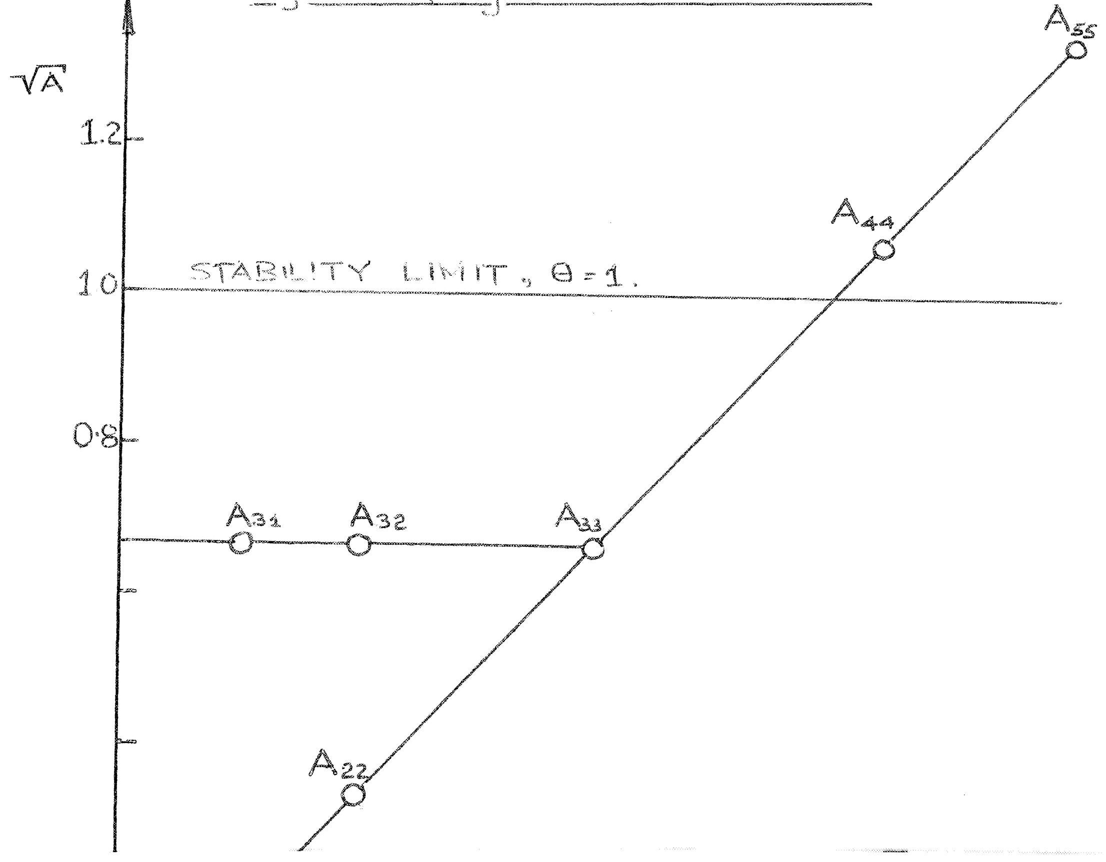


Fig 1 (fig 2 is almost identical)



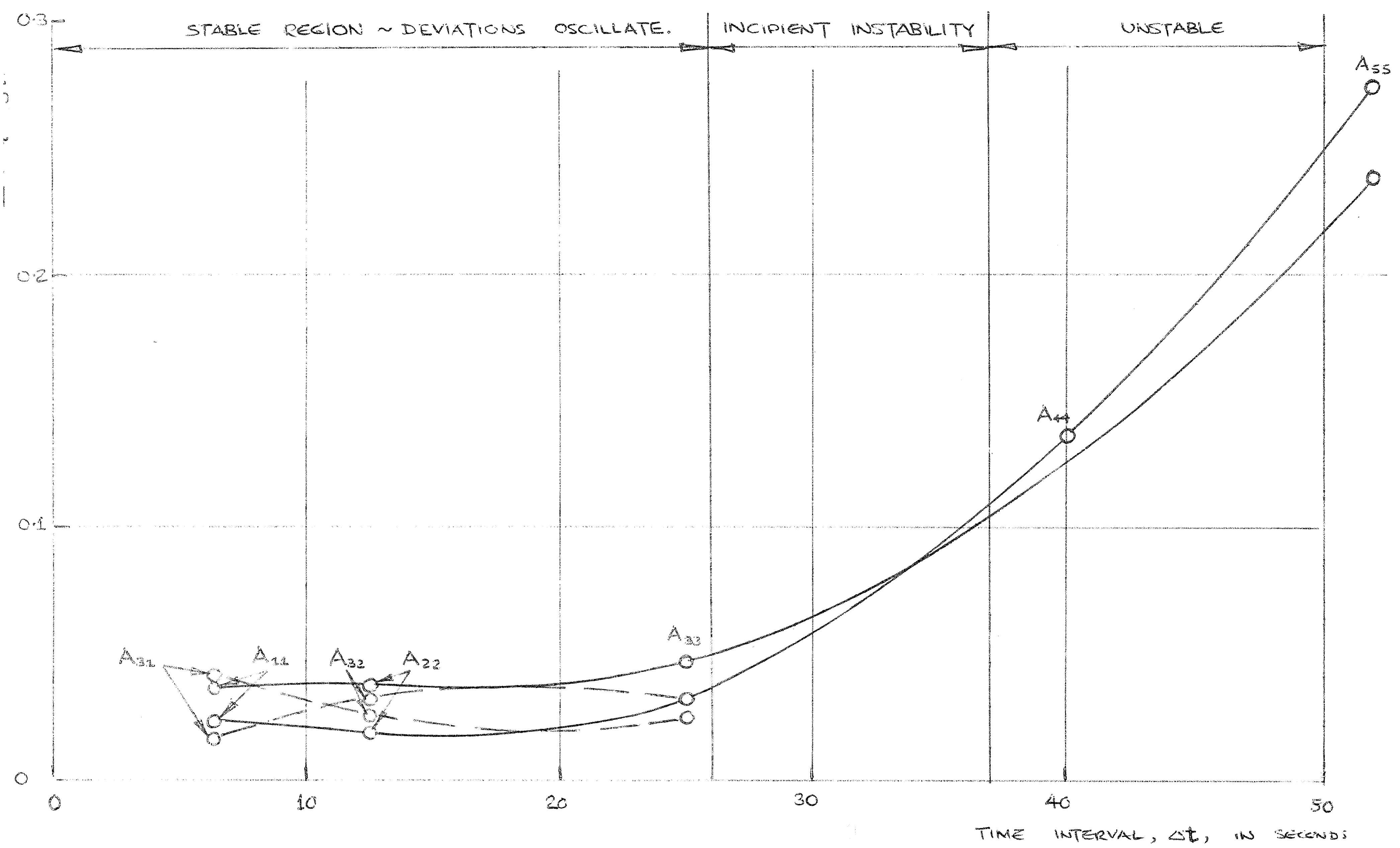


Fig. 6. (fig 4 and fig. 5 omitted from this note).
figs 7 and 8 also omitted.