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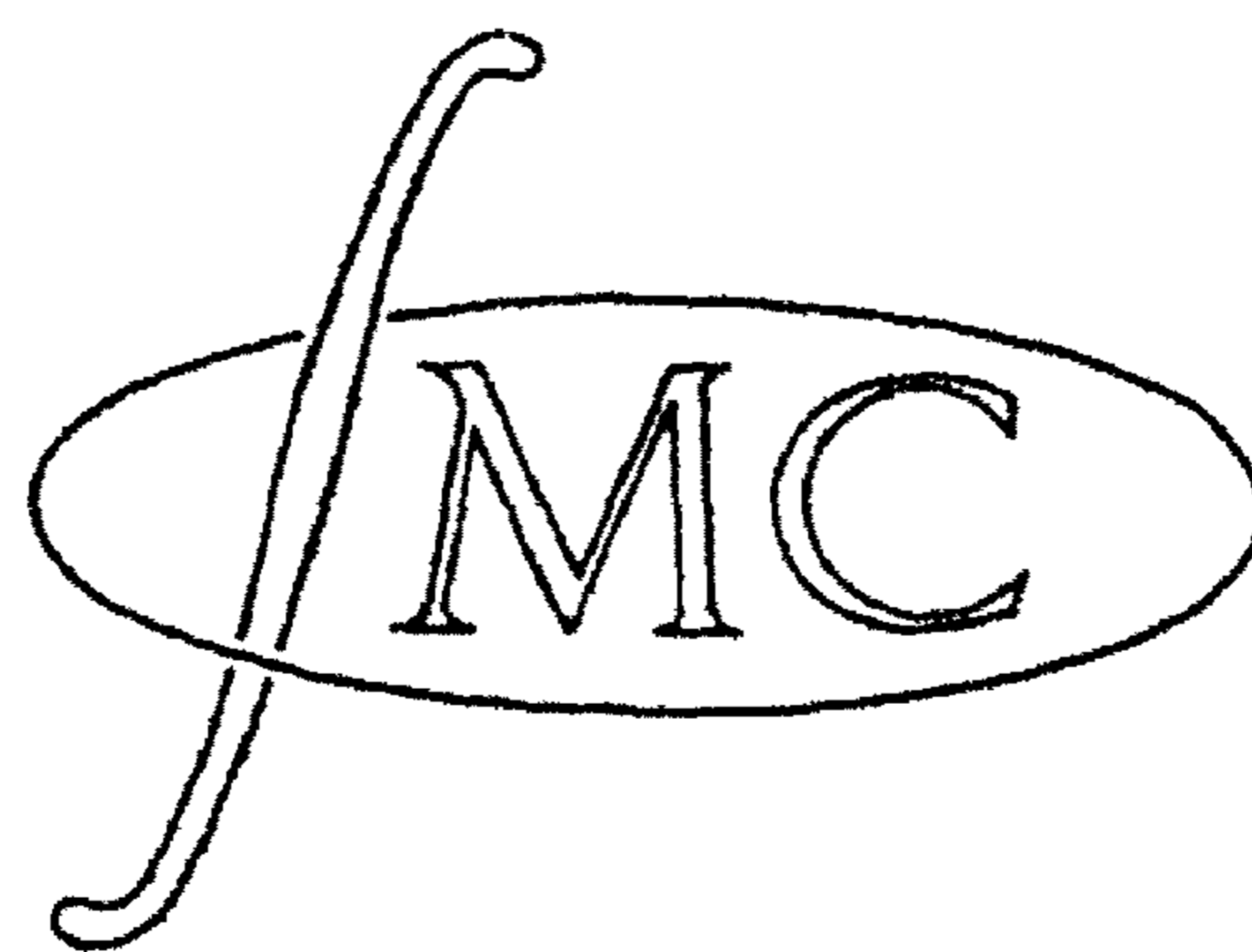
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Technical Note TN 35

Some remarks on the arithmetico-geometrical mean

by

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## Introduction.

In this note we study some properties of the arithmetico-geometrical mean  $M(a,b)$ , which is defined as follows: let  $a$  and  $b$  be two positive numbers, let  $a_n$  and  $b_n$  be defined by the following recursive relations

$$(1) \quad a_{n+1} = (a_n + b_n)/2, \quad b_{n+1} = (a_n b_n)^{1/2} \quad \text{with } a_0 = a \\ \text{and } b_0 = b,$$

$$\text{then } M(a,b) = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n.$$

In an extensive study\*) on the arithmetico-geometrical mean Gauss already derived the following two formulas which will be the starting points of our study.

$$(2) \quad \frac{\pi}{2 M(a,b)} = \int_0^{\pi/2} (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)^{-1/2} d\varphi \quad \text{and}$$

$$(3) \quad \frac{\pi}{2 M(a,b)} = 2 \int_0^{\pi/2} (\alpha^2 - \beta^2 \sin^2 \varphi)^{-1/2} d\varphi$$

$$\text{where } \alpha = a+b, \quad \beta = a-b.$$

In section 1 a series expansion for  $\{M(1+k, 1-k)\}^{-1}$  is derived valid for  $|k| < 1$ .

In section 2 a proof is given of the formula

$$(4) \quad \lim_{\varepsilon \rightarrow 0} M(1, \varepsilon) \log \frac{4}{\varepsilon} = \frac{\pi}{2}$$

In section 3 we present a relation between  $M(a,b)$  and  $M(\alpha, \beta)$  with  $\alpha = a+b$  and  $\beta = a-b$ .

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\*) see C.F. Gauss: Anziehung eines elliptischen Ringes. Herausgegeben von H. Geppert. Akad. Verlags Gesellschaft, Leipzig 1927.

All results given here are well-known, but the treatment is possibly new.

1. The series expansion of  $\{M(1+k, 1-k)\}^{-1}$

By aid of formula (2) we get for any  $\varepsilon, 0 < \varepsilon < 1$

$$\begin{aligned} \frac{1}{M(1, \varepsilon)} &= \frac{1}{2\pi} \int_0^{2\pi} (\sin^2 \varphi + \varepsilon^2 \cos^2 \varphi)^{-1/2} d\varphi = \\ &= \frac{1}{\pi} \int_0^{2\pi} e^{i\varphi} \left\{ (1+\varepsilon)e^{2i\varphi} - (1-\varepsilon) \right\}^{-1/2} \left\{ -(1-\varepsilon)e^{2i\varphi} + (1+\varepsilon) \right\}^{-1/2} d\varphi. \end{aligned}$$

Substituting  $z=e^{i\varphi}$ , we obtain the important formula

$$(5) \quad \frac{1}{M(1, \varepsilon)} = \frac{1}{\pi\sqrt{(1-\varepsilon^2)}} \oint_C \left\{ \left( z^2 - \frac{1-\varepsilon}{1+\varepsilon} \right) \left( z^2 - \frac{1+\varepsilon}{1-\varepsilon} \right) \right\}^{-1/2} dz,$$

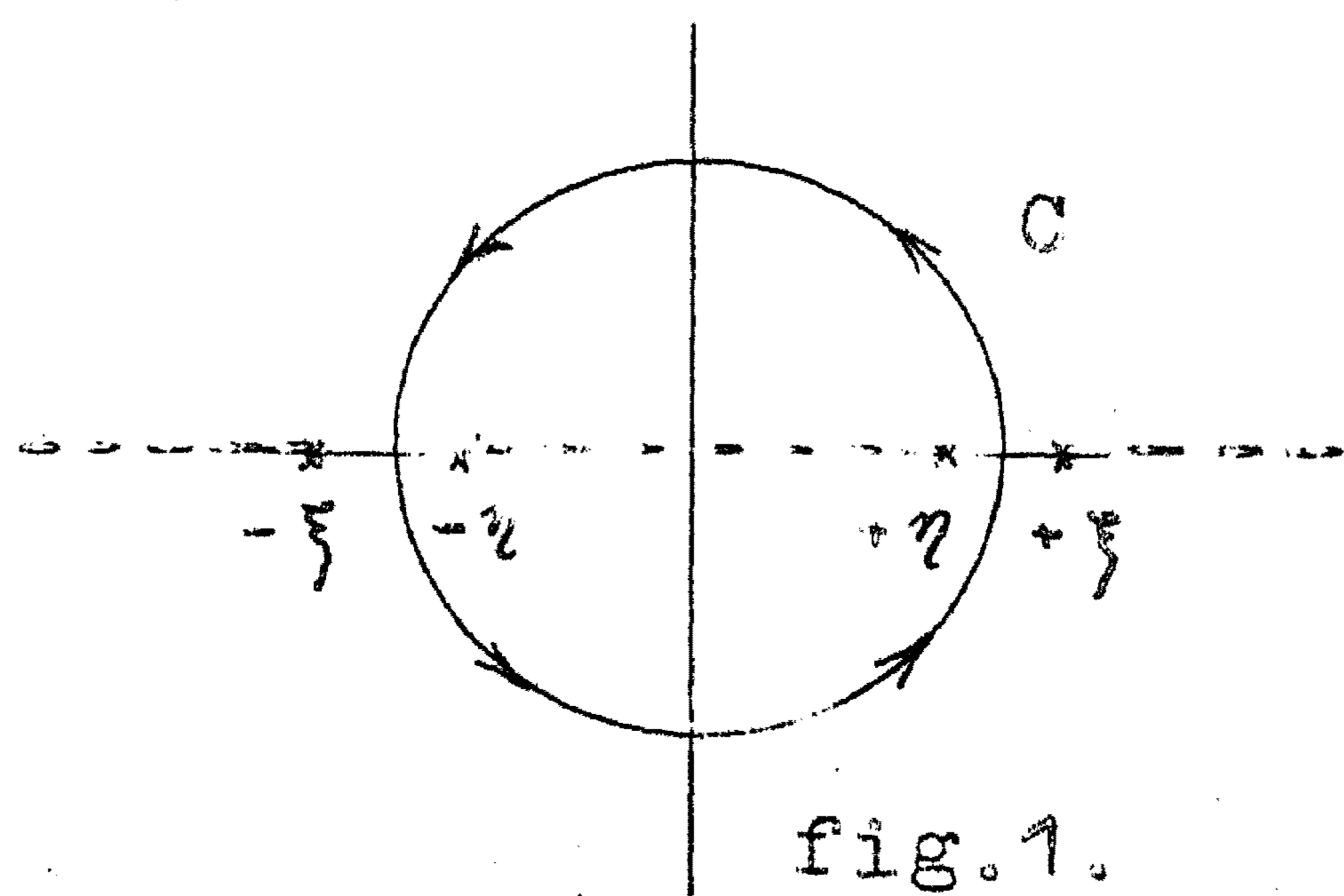
where the integration must be performed in the positive sense along the contour  $C$ .  $C$  is the unit circle around the origin of the complex plane, which has cuts

from  $-\infty$  to  $-\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{1/2}$ ,

from  $-\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{1/2}$  to  $+\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{1/2}$  and

from  $+\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{1/2}$  to  $+\infty$  along the real axis, as shown in fig. 1 where

$$(6) \quad \xi = \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{1/2} \quad \text{and} \quad \eta = \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{1/2}$$



We now deform the contour  $C$  into the contour  $K$ , as shown in fig. 2.  $K$  consists of the two contracting circles with centres  $\pm\eta$  and the two lines parallel to the real axis.

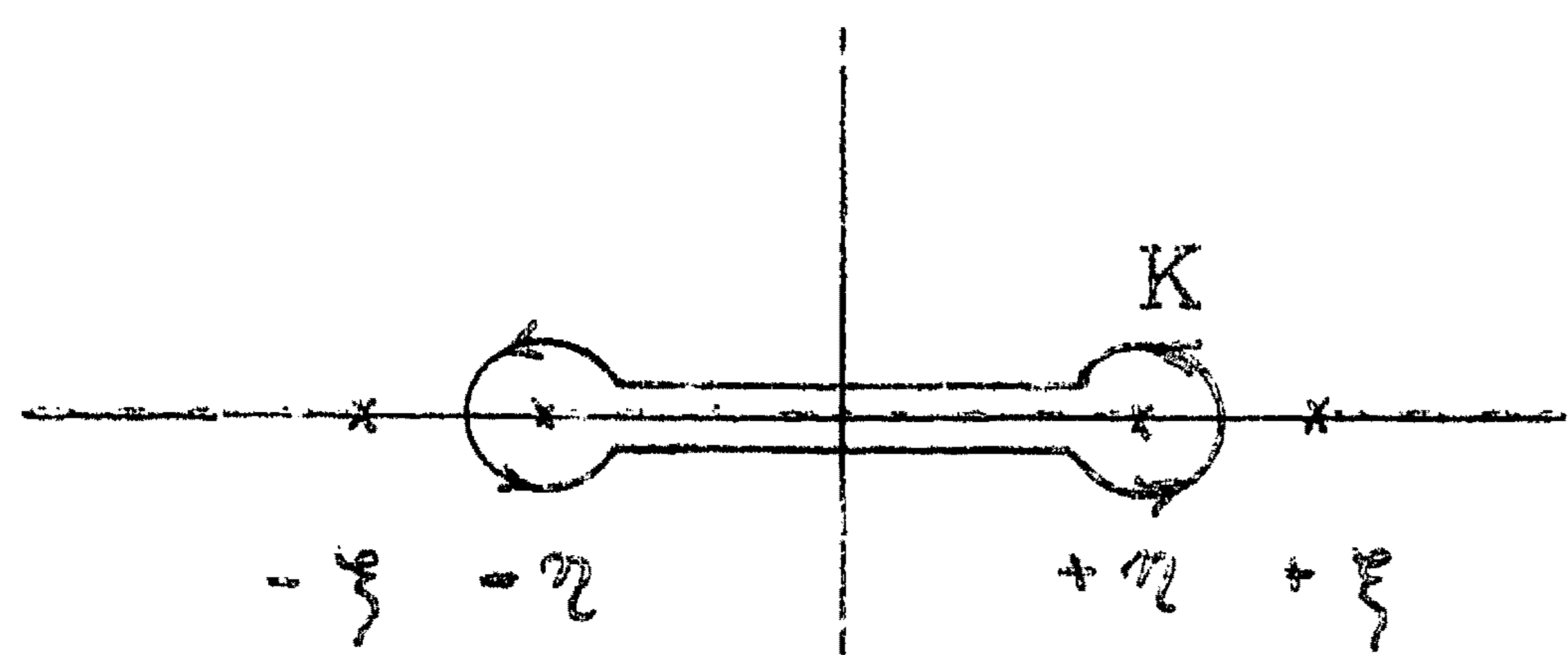


fig.2.

It is easily seen that for even values of n :

$$\frac{1}{2\pi i} \oint_K z^n (z^2 - \eta^2)^{-1/2} dz = \eta^n \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \dots \cdot \frac{n-1}{n}$$

Therefore we obtain from (5)

$$\frac{1}{M(1, \varepsilon)} = \frac{2}{1+\varepsilon} \sum_{\nu=0}^{\infty} \binom{-1/2}{\nu}^2 \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{2\nu},$$

or, when we define k by  $k = \frac{1-\varepsilon}{1+\varepsilon}$ , we obtain

$$(7) \quad \left\{ M(1+k, 1-k) \right\}^{-1} = \sum_{\nu=0}^{\infty} \binom{-1/2}{\nu}^2 k^{2\nu}.$$

### 3. The limit expression.

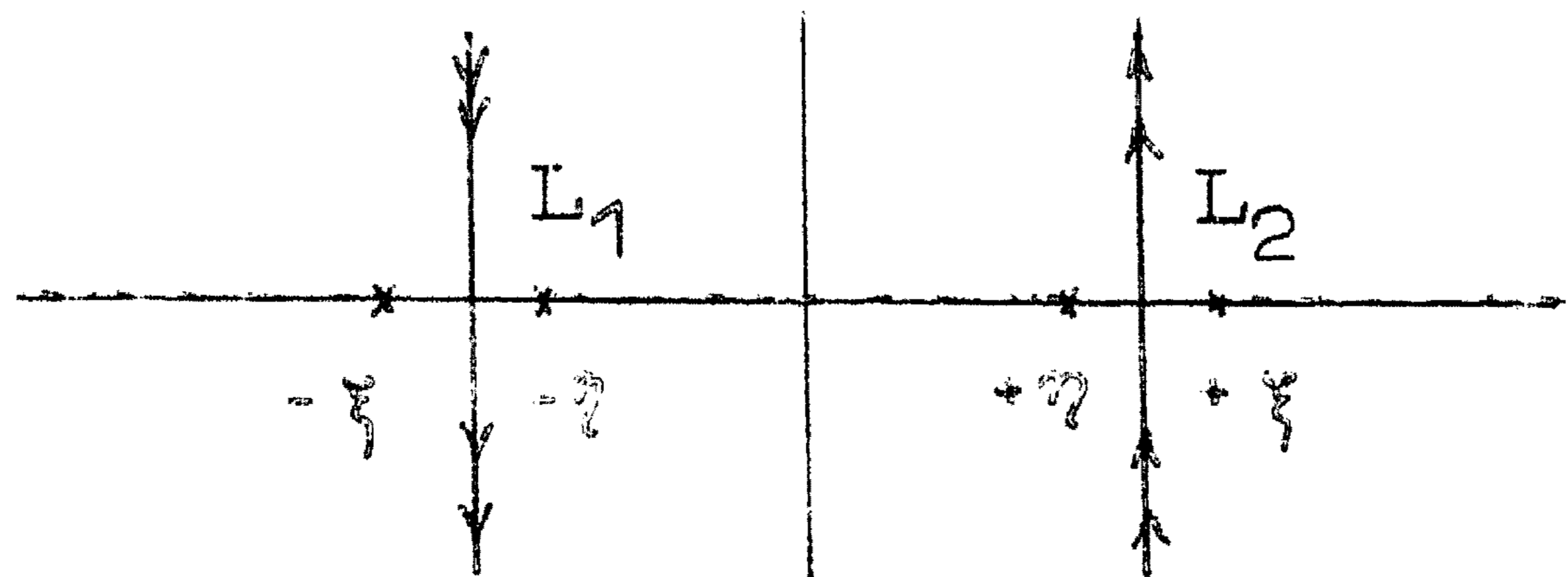
In the following we use the well known formula

$$(8) \quad \int_C f(z) (z^2 - x^2)^{-1/2} dz = \int_{-\infty}^{+\infty} f(ix \operatorname{sh} u) du,$$

Where the path of integration  $C'$  is along the imaginary axis from  $-i\infty$  to  $+i\infty$ , the complex plane has a cut along the real axis from  $-\infty$  to  $-x$  and from  $+x$  to  $+\infty$ .

We assume the convergence of the integrals.

The contour  $C$  in formula (5) will now be deformed in a contour  $L$  consisting of  $L_1$  and  $L_2$  as shown in fig.3,



where  $L_1$  and  $L_2$  intersect the real axis in the points  $-(\xi + \eta)/2$  and  $(\xi + \eta)/2$ .

After this deformation of the path of integration we may write

$$\begin{aligned} \frac{\pi \sqrt{1-\varepsilon^2}}{M(1, \varepsilon)} &= \left\{ \int_{-i\infty + \frac{\xi+\eta}{2}}^{+i\infty + \frac{\xi+\eta}{2}} - \int_{-i\infty - \frac{\xi+\eta}{2}}^{+i\infty - \frac{\xi+\eta}{2}} \right\} \left\{ (z^2 - \eta^2)(z^2 - \xi^2) \right\}^{-1/2} dz = \\ &= \int_{-i\infty}^{+i\infty} \left\{ \left( z + \frac{\xi+\eta}{2} \right)^2 - \eta^2 \right\}^{-1/2} \left\{ \left( z + \frac{\xi+\eta}{2} \right)^2 - \xi^2 \right\}^{-1/2} dz + \\ &- \int_{-i\infty}^{+i\infty} \left\{ \left( z - \frac{\xi+\eta}{2} \right)^2 - \eta^2 \right\}^{-1/2} \left\{ \left( z - \frac{\xi+\eta}{2} \right)^2 - \xi^2 \right\}^{-1/2} dz = \\ &= \int_{-i\infty}^{+i\infty} \left( z + \frac{\xi+3\eta}{2} \right)^{-1/2} \left( z + \frac{\xi-\eta}{2} \right)^{-1/2} \left( z + \frac{3\xi+\eta}{2} \right)^{-1/2} \left( z - \frac{\xi-\eta}{2} \right)^{-1/2} dz + \\ &- \int_{-i\infty}^{+i\infty} \left( z - \frac{\xi+3\eta}{2} \right)^{-1/2} \left( z - \frac{\xi-\eta}{2} \right)^{-1/2} \left( z - \frac{3\xi+\eta}{2} \right)^{-1/2} \left( z + \frac{\xi-\eta}{2} \right)^{-1/2} dz \end{aligned}$$

We put now:  $\frac{\xi-\eta}{2} = \frac{1}{2} \left( \left( \frac{1+\varepsilon}{1-\varepsilon} \right)^{1/2} - \left( \frac{1-\varepsilon}{1+\varepsilon} \right)^{1/2} \right) = \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} = \gamma,$

$$\frac{\xi+3\eta}{2} = \frac{1}{2} \left( \left( \frac{1+\varepsilon}{1-\varepsilon} \right)^{1/2} + 3 \left( \frac{1-\varepsilon}{1+\varepsilon} \right)^{1/2} \right) = \frac{2-\varepsilon}{\sqrt{1-\varepsilon^2}} = \alpha,$$

$$\frac{3\xi+\eta}{2} = \frac{1}{2} \left( 3 \left( \frac{1+\varepsilon}{1-\varepsilon} \right)^{1/2} + \left( \frac{1-\varepsilon}{1+\varepsilon} \right)^{1/2} \right) = \frac{2+\varepsilon}{\sqrt{1-\varepsilon^2}} = \beta.$$

Hence we obtain:

$$\begin{aligned} \frac{\pi \sqrt{1-\varepsilon^2}}{M(1, \varepsilon)} &= \int_{-i\infty}^{+i\infty} (z^2 - \gamma^2)^{-1/2} (z+\alpha)^{-1/2} (z+\beta)^{-1/2} dz + \\ &- \int_{-i\infty}^{+i\infty} (z^2 - \gamma^2)^{-1/2} (z-\alpha)^{-1/2} (z-\beta)^{-1/2} dz \end{aligned}$$

$$= \int_{-\infty}^{+\infty} \left[ \left\{ (i\gamma \operatorname{sh} u + \alpha)(i\gamma \operatorname{sh} u + \beta) \right\}^{-1/2} - \left\{ (i\gamma \operatorname{sh} u - \alpha)(i\gamma \operatorname{sh} u - \beta) \right\}^{-1/2} \right] du,$$

where we have used formula (8).

For small values of  $\varepsilon$  we have  $\alpha = \beta + o(\varepsilon)$  and so we may write

$$\frac{\pi \sqrt{1-\varepsilon^2}}{M(1, \varepsilon)} = \int_{-\infty}^{+\infty} \left\{ (i\gamma \operatorname{sh} u + \alpha)^{-1} - (i\gamma \operatorname{sh} u - \alpha)^{-1} \right\} du + r(\varepsilon)$$

Where  $r(\varepsilon) \rightarrow 0$  for  $\varepsilon \rightarrow 0$ .

Completing the reduction we obtain finally

$$\begin{aligned} \frac{\pi \sqrt{1-\varepsilon^2}}{M(1, \varepsilon)} &= 2\alpha \int_{-\infty}^{+\infty} \frac{du}{\gamma^2 \operatorname{sh}^2 u + \alpha^2} + r(\varepsilon) = \\ &= \frac{2}{\sqrt{\alpha^2 - \gamma^2}} \ln \frac{\{\alpha + \sqrt{\alpha^2 - \gamma^2}\}^2}{\gamma^2} + r(\varepsilon) = \\ &= 2 \ln \frac{4}{\varepsilon} + r_1(\varepsilon), \text{ where } r_1(\varepsilon) \rightarrow 0 \text{ for } \varepsilon \rightarrow 0 \end{aligned}$$

We have thus obtained the desired result

$$(4) \quad \lim_{\varepsilon \rightarrow 0} M(1, \varepsilon) \ln \frac{4}{\varepsilon} = \frac{\pi}{2}.$$

### 3. The relation between $M(a, b)$ and $M(\alpha, \beta)$ .

We next study the relation between  $M(a, b)$  and  $M(\alpha, \beta)$  where  $\alpha = a+b$  and  $\beta = a-b$ .

From (3) we have

$$\frac{\pi}{2M(a, b)} = \frac{\pi}{2M\left(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2}\right)} = 2 \int_0^{\pi/2} (\alpha^2 - \beta^2 \sin^2 \varphi)^{-1/2} d\varphi.$$

The second identity is also true for  $\alpha_1 = (\alpha+\beta)/2$ , and  $\beta_1 = (\alpha\beta)^{1/2}$ , thus

$$\frac{\pi}{2M\left(\frac{\alpha_1 + \beta_1}{2}, \frac{\alpha_1 - \beta_1}{2}\right)} = 2 \int_0^{\pi/2} (\alpha_1^2 - \beta_1^2 \sin^2 \varphi)^{-1/2} d\varphi.$$

Evidently,  $M(a, b) = M(a_1, b_1) = \dots = M(a_n, b_n)$  where  $a_n$  and  $b_n$  are defined according to (1), but

$$\left(\frac{\alpha_1 + \beta_1}{2} + \frac{\alpha_1 - \beta_1}{2}\right)/2 = \frac{1}{2} \left(\frac{\alpha + \beta}{2}\right) \quad \text{and}$$

$$\left(\frac{\alpha_1 + \beta_1}{2} \cdot \frac{\alpha_1 - \beta_1}{2}\right)^{1/2} = \frac{1}{2} \left(\frac{\alpha - \beta}{2}\right), \quad \text{so that}$$

$$M\left(\frac{\alpha_1 + \beta_1}{2}, \frac{\alpha_1 - \beta_1}{2}\right) = \frac{1}{2} M\left(\frac{\alpha + \beta}{2}, \frac{\alpha - \beta}{2}\right).$$

Thus it follows immediately that

$$\begin{aligned} \frac{\pi}{2M(a, b)} &= \frac{1}{2} \frac{\pi}{2M\left(\frac{\alpha_1 + \beta_1}{2}, \frac{\alpha_1 - \beta_1}{2}\right)} = \\ &= 2 \cdot \frac{1}{2} \int_0^{\pi/2} (\alpha_1^2 - \beta_1^2 \sin^2 \varphi)^{-1/2} d\varphi. \end{aligned}$$

Repeating the above argument we find

$$(9) \quad \frac{\pi}{2M(a, b)} = 2 \cdot \frac{1}{2^n} \int_0^{\pi/2} (\alpha_n^2 - \beta_n^2 \sin^2 \varphi)^{-1/2} d\varphi,$$

where  $\alpha_n$  and  $\beta_n$  are defined by  $\alpha_{n+1} = (\alpha_n + \beta_n)/2$ ,

$$\beta_{n+1} = (\alpha_n \beta_n)^{1/2}.$$

The integrand in (9) may be written as

$$(10) \quad (\beta_n^2 \cos^2 \varphi + \gamma_n^2)^{-1/2}, \quad \text{with } \gamma_n^2 = \alpha_n^2 - \beta_n^2.$$

Using Mc Laurins series we may write for this expression

$$\left\{ \beta_n \cos \varphi + \frac{\gamma_n^2}{2\beta_n \cos \varphi} - \frac{\gamma_n^4}{8\beta_n^3 \cos^3 \varphi} \left( 1 + \theta \frac{\gamma_n^2}{\beta_n^2 \cos^2 \varphi} \right)^{-3/2} \right\}^{-1}$$

with  $|\theta| < 1$

Let us now define

$$A = \beta_n \cos \varphi + \frac{\gamma_n^2}{2\beta_n \cos \varphi} \quad \text{and}$$

$$B = \frac{\gamma_n^4}{8\beta_n^3 \cos^3 \varphi} \left( 1 + \theta \frac{\gamma_n^2}{\beta_n^2 \cos^2 \varphi} \right)^{-3/2}$$

With this simplification formula (10) becomes

$$\frac{1}{A-B} = \frac{1}{A} + \frac{B}{A(A-B)}$$

and the integral (9) takes on the form

$$\frac{\pi}{2M(a,b)} = \frac{1}{2^{n-1}} \left[ \int_0^{\pi/2 - \gamma_n/\beta_n} \frac{1}{A} d\varphi + \int_0^{\pi/2 - \gamma_n/\beta_n} \frac{B}{A(A-B)} d\varphi + \int_{\pi/2 - \gamma_n/\beta_n}^{\pi/2} (\beta_n^2 \cos^2 \varphi + \gamma_n^2)^{-1/2} d\varphi \right]$$

The integrals appearing in this expression will be denoted consecutively by  $I_{1,n}$ ,  $I_{2,n}$  and  $I_{3,n}$ .

By aid of the substitution  $u = \sqrt{2} \beta_n \sin \varphi$  we obtain:



$$\begin{aligned}
 I_{1,n} &= \int_0^{\pi/2 - \gamma_n/\beta_n} \frac{2\beta_n \cos \varphi \, d\varphi}{2\beta_n^2 \cos^2 \varphi + \gamma_n^2} = \\
 &= \sqrt{2} \int_0^{\sqrt{2}\beta_n \cos \gamma_n/\beta_n} \frac{du}{\alpha_n^2 + \beta_n^2 - u^2} = \\
 &= \frac{1}{\sqrt{2} \sqrt{\alpha_n^2 + \beta_n^2}} \left\{ \ln \left[ \sqrt{\alpha_n^2 + \beta_n^2} + \sqrt{2} \beta_n \cos \frac{\gamma_n}{\beta_n} \right] - \right. \\
 &\quad \left. \ln \left| \sqrt{\alpha_n^2 + \beta_n^2} - \sqrt{2} \beta_n \cos \frac{\gamma_n}{\beta_n} \right| \right\} = \\
 (12) &= \frac{1}{\sqrt{2} \sqrt{\alpha_n^2 + \beta_n^2}} \left\{ \ln \left[ \sqrt{\alpha_n^2 + \beta_n^2} + \sqrt{2} \beta_n \cos \frac{\gamma_n}{\beta_n} \right]^2 - \right. \\
 &\quad \left. \ln \left| \alpha_n^2 + \beta_n^2 - 2\beta_n^2 \cos^2 \frac{\gamma_n}{\beta_n} \right| \right\}.
 \end{aligned}$$

It is easily seen that the first term of  $\frac{1}{2^{n-1}} I_{1,n}$  tends to zero for  $n \rightarrow \infty$ . As to the second term, we remark that

$$\alpha_n^2 + \beta_n^2 - 2\beta_n^2 \cos^2 \frac{\gamma_n}{\beta_n} = \alpha_n^2 - \beta_n^2 \cos 2\frac{\gamma_n}{\beta_n} = \gamma_n^2 \left\{ 1 + 2\cos(2\vartheta \frac{\gamma_n}{\beta_n}) \right\}$$

with  $|\vartheta| \leq 1$ .

$$\text{Since } \lim_{n \rightarrow \infty} \ln \left\{ 1 + 2\cos(2\vartheta \frac{\gamma_n}{\beta_n}) \right\} = 0$$

we obtain finally

$$\lim_{n \rightarrow \infty} \frac{1}{2^{n-1}} I_{1,n} = - \lim_{n \rightarrow \infty} \frac{\sqrt{2} \ln \gamma_n}{2^{n-1} \sqrt{\alpha_n^2 + \beta_n^2}}.$$

Hence we obtain by aid of formula (9)

$$\frac{\pi}{2M(a,b)} = - \lim_{n \rightarrow \infty} \frac{\frac{\sqrt{2}}{2} \ln j_n}{2^{n-1} \sqrt{\alpha_n^2 + \beta_n^2}} + \lim_{n \rightarrow \infty} \frac{1}{2^{n-1}} I_{2,n} +$$

$$+ \lim_{n \rightarrow \infty} \frac{1}{2^{n-1}} I_{3,n}.$$

It will be shown in the appendix that we have the relations

$$\lim_{n \rightarrow \infty} \frac{1}{2^{n-1}} I_{2,n} = \lim_{n \rightarrow \infty} \frac{1}{2^{n-1}} I_{3,n} = 0$$

and therefore we get:

$$(13) \quad \frac{\pi}{2M(a,b)} = \frac{-1}{M(\alpha, \beta)} \lim_{n \rightarrow \infty} \left\{ \frac{1}{2^{n-1}} \ln j_n \right\}.$$

We observe now, that  $j_n = (\alpha_n^2 - \beta_n^2)^{1/2} = (\alpha_{n-1} - \beta_{n-1})/2 =$

$$= \frac{j_{n-1}^2}{4\alpha_n}$$

and therefore  $j_n = \frac{j_0^{2^n}}{4^{2^n-1} \alpha_n \alpha_{n-1}^2 \dots \alpha_1^{2^{n-1}}}$

Thus we have the formula:

$$\frac{1}{2^{n-1}} \ln j_n = \frac{1}{2} \ln \frac{j_0}{4} - \frac{1}{2^{n-1}} \ln(\alpha_n \cdot \alpha_{n-1}^2 \dots \alpha_1^{2^{n-1}}) +$$

$$+ \frac{1}{2^{n-1}} \ln 4$$

and so we obtain the following result which relates  $M(a,b)$  with  $M(\alpha, \beta)$ .

$$(14) \quad \frac{\pi}{2} \frac{M(\alpha, \beta)}{M(a, b)} = 2 \ln \frac{4}{\sqrt{\alpha^2 - \beta^2}} + \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \ln \gamma_n$$

From this result we may obtain again the formula (4) very easily.

Let  $a=1$ ,  $b=\varepsilon$  then  $\alpha = 1+\varepsilon$ ,  $\beta = 1-\varepsilon$  and  $\alpha^2 - \beta^2 = 4\varepsilon$

Since  $|1-\alpha_n| < 2\varepsilon$  for all  $\alpha_n$ , we have

$$\lim_{\varepsilon \rightarrow 0} M(\alpha, \beta) = 1 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \ln \alpha_n = 0$$

Substituting these results into (14) we get

$$(4) \quad \lim_{\varepsilon \rightarrow 0} M(1, \varepsilon) \ln \frac{4}{\varepsilon} = \frac{\pi}{2} .$$

Appendix

It will now be shown, that  $\lim_{n \rightarrow \infty} \frac{1}{2^{n-1}} I_{2,n} =$

$$= \lim_{n \rightarrow \infty} \frac{1}{2^{n-1}} I_{3,n} = 0.$$

In order to do this we make use of the following lemma

Lemma            Let

$$(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} (1+\theta x)^{-3/2}$$

then  $\min_{0 \leq x \leq 1+\epsilon} \theta > 0$  where  $\epsilon$  is a sufficiently small positive number.

Proof: We remark that

$$(15) \quad \theta = x^{-1} \left\{ \frac{x^{4/3}}{4} \left( 1 + \frac{x}{2} - \sqrt{1+x} \right)^{-2/3} - 1 \right\},$$

Thus  $\theta$  is a continuous one-valued function of the real variable  $x$  when  $x > 0$ .

It can easily be shown that  $\theta$  is positive for  $0 < x \leq 1+\epsilon$ .  
By proving that

$$\lim_{x \rightarrow 0} \theta > 0$$

it follows that  $\theta$  assumes a positive minimum on the interval  $0 \leq x \leq 1+\epsilon$ .

The Taylor series for  $(1+x)^{1/2}$  is

$$1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + O(x^4),$$

We thus obtain, substituting this into (15),

$$\begin{aligned} \theta &= x^{-1} \left\{ \frac{x^4}{4} - \frac{x^2}{3} - \frac{x^3}{16} + O(x^4) \right\}^{-2/3} - 1 \\ &= x^{-1} \left\{ \left( 1 - \frac{x}{2} + O(x^2) \right)^{-2/3} - 1 \right\} = \\ &= x^{-1} \left\{ \frac{x}{3} + O(x^2) \right\} = \frac{1}{3} + O(x) \end{aligned}$$

which proves the lemma, moreover it follows that

$$(16) \quad \min_{0 \leq x \leq 1+\varepsilon} \theta \leq \frac{1}{3}.$$

We proceed now to estimate  $I_{2,n}$ .

If,  $0 \leq \varphi \leq \pi/2 - \beta_n/\gamma_n$  then

$$\frac{\gamma_n^2}{\beta_n^2 \cos^2 \varphi} \leq \frac{\gamma_n^2}{\beta_n^2} \left\{ \frac{\gamma_n}{\beta_n} - \frac{\gamma_n^3}{6\beta_n^3} + \dots \right\}^{-2} = 1 + O(\gamma_n^2),$$

Thus for sufficiently large  $n$  we may state that

$$\frac{\gamma_n^2}{\beta_n^2 \cos^2 \varphi} \leq 1 + \varepsilon \quad \text{where } \varepsilon \text{ is a small positive number.}$$

From the lemma, proved above, follows the existence of a positive minimum  $\mu$  for  $\theta$  in formula (11), when  $\varphi$  varies from 0 to  $\frac{\pi}{2} - \beta_n/\gamma_n$ ; moreover we know from (16) that  $\mu < 1/2$ .

Taking all this into consideration, we can estimate  $I_{2,n}$  as follows

$$I_{2,n} = \int_0^{\pi/2 - \beta_n/\gamma_n} \frac{\gamma_n^4}{8\beta_n^3 \cos^3 \varphi \left( 1 + \theta \frac{\gamma_n^2}{\beta_n^2 \cos^2 \varphi} \right)^{3/2} \left( \beta_n \cos \varphi + \frac{\gamma_n^2}{2\beta_n \cos \varphi} \right)} \frac{1}{(\beta_n^2 \cos^2 \varphi + \gamma_n^2)^{1/2}} d\varphi <$$

$$\begin{aligned}
 &< \frac{\gamma_n^4}{8} \int_0^{\pi/2} \frac{\beta_n \cos \varphi \, d\varphi}{(\beta_n^2 \cos^2 \varphi + \mu \gamma_n^2)^3} = \\
 &= - \frac{\gamma_n^4}{8} \int_0^{\beta_n} \frac{du}{(u^2 - (\beta_n^2 + \mu \gamma_n^2))^3}
 \end{aligned}$$

A trivial calculation <sup>\*)</sup> shows that the last integral is equal to

$$\frac{\gamma_n^4}{8} \left[ \frac{\beta_n}{4\delta_n^2 (\delta_n^2 - \beta_n^2)^2} + \frac{3\beta_n}{8\delta_n^4 (\delta_n^2 - \beta_n^2)} + \frac{3}{16\delta_n^5} \ln \frac{(\delta_n + \beta_n)^2}{\delta_n^2 - \beta_n^2} \right],$$

where we have substituted  $\delta_n^2 = \beta_n^2 + \mu \gamma_n^2$ .

For sufficiently large  $n$  the following estimate holds:

$$\ln \frac{1}{\delta_n^2 - \beta_n^2} < \frac{1}{\delta_n^2 - \beta_n^2} < \frac{1}{(\delta_n^2 - \beta_n^2)^2}.$$

Therefore we may write

$$I_{2,n} < K \frac{\gamma_n^4}{(\delta_n^2 - \beta_n^2)^2} = \frac{K}{\mu^2} \quad \text{where } K \text{ is some positive constant.}$$

From this it follows immediately that:

$$\lim_{n \rightarrow \infty} \frac{1}{2^{n-1}} I_{2,n} = 0.$$

$$\text{Finally we estimate } I_{3,n} = \int_{\pi/2 - \gamma_n/\beta_n}^{\pi/2} \frac{\gamma_n}{\beta_n} (\beta_n^2 \cos^2 \varphi + \gamma_n^2)^{-1/2} d\varphi.$$

The length of the integration interval is  $\gamma_n/\beta_n$ , the maximum value of the integrand  $1/\gamma_n$ , and hence

\*) see e.g. H.B.Dwight: Tables of Integrals and other Mathematical Data.

. The MACMILLAN COMPANY. New York, 1955.  
p. 30.

$$I_{3,n} \leq \frac{\gamma_n}{\beta_n} \cdot \frac{1}{\gamma_n} = \frac{1}{\beta_n},$$

and thus

$$\lim_{n \rightarrow \infty} \frac{1}{2^{n-1}} I_{3,n} = 0.$$