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AMSTERDAM

AFDELING TOEGEPASTE WISKUNDE

TN 46

On the calculation of $\sum a_n^2/n$, where a_n is a Fourier coefficient.

by

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1. We consider the function f(x), a periodic function of x with period 2π, which is identically zero in nearly all of the interval $(-\pi, \pi)$. The symmetry properties of this function are such that the following cosine-expansion is possible

$$f(x) = \sum_{n=1}^{\infty} a_n \cos nx, \qquad (1.1)$$

where $$a_{2n} = 0 \qquad (n = 1, 2, \ldots). \qquad (1.2)$$

In figure 1 such a function is sketched.

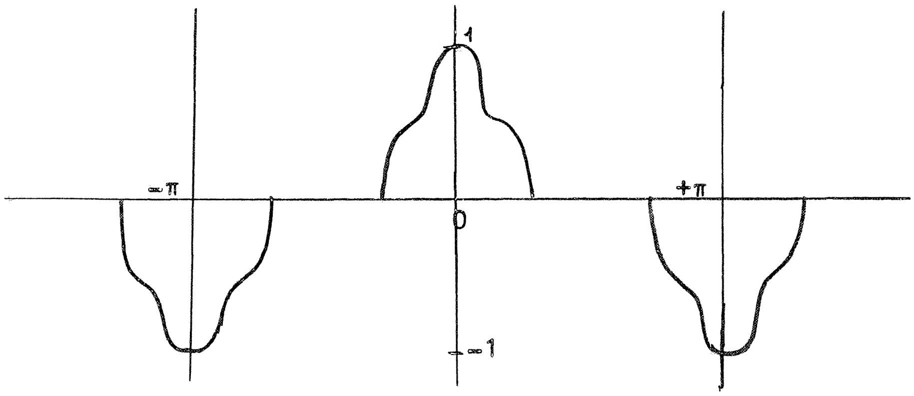


Figure 1.

It is helpful to introduce the function h(x), defined by

$$h(\pi x/\delta) = f(x). \qquad (1.3)$$

We know from alinea 1 that

$$f(x) \equiv 0 \qquad \text{for } (k-1)\pi/2 + \delta/2 \leq x \leq k\pi/2 - \delta/2 \qquad (1.4)$$

$$(k = 0, \pm 1, \ldots),$$

where δ approaches zero.

With the aid of the above we wish to compute the sum S when

$$S = \sum_{n=1}^{\infty} a_n^2/n. \qquad (1.5)$$

An obvious way to solve this problem is to invert relation (1.1) and to substitute the resulting a_n in (1.5). Since it is rather difficult to sum the series, obtained in this manner, we will use a different method.

We introduce an intermediary cosine-expansion, as follows

$$g(\pi x/\delta) = \sum_{n=1}^{\infty} a_n^2 \cos nx. \qquad (1.6)$$

On the other hand

$$g(\pi x/\delta) = (2/\pi) \int_0^{\pi} f(\xi)\ f(x-\xi)d\xi.$$

From (1.1) and (1.2) we know that

$$f(\pi/2-x) = -f(\pi/2+x),$$

so that

$$g(\pi x/\delta) = (2/\pi) \int_{-\pi/2}^{\pi/2} f(\xi)\ f(x-\xi)d\xi.$$

And applying (1.4) we have

$$g(\pi x/\delta) = (2/\pi) \int_{-\delta/2}^{\delta/2} f(\xi)\ f(x-\xi)d\xi$$

for

$$k\pi/2 - \delta/2 < x-\xi < k\pi/2 + \delta/2, \qquad (1.7)$$

$$g(\pi x/\delta) = 0 \qquad \text{in all other intervals.}$$

From (1.7) we see that

$$g[\pi(\pi-x)/\delta] = - g(\pi x/\delta).$$

Using the inversion formula for Fourier cosine-series, we find

$$a_n^2 = (2/\pi) \int_0^{\pi} g(\pi x/\delta) \cos nx\ dx. \qquad (1.8)$$

We substitute the expression (1.8) in the series (1.5) and interchange the order of integration and summation to get

$$S = (2/\pi) \int_0^{\pi} g(\pi x/\delta)\{\sum_{n=1}^{\infty} (\cos nx)/n\}dx.$$

With the aid of

$$\sum_{n=1}^{\infty} (\cos nx)/n = \mathrm{Re}\{\sum_{n=1}^{\infty} e^{inx}/n\} = -\ln|1-e^{ix}|$$

the sum becomes

$$S = -(2/\pi) \int_0^{\pi} g(\pi x/\delta) \ln|1-e^{ix}|dx$$

$$= -(2/\pi)\int_0^{\pi/2} g(\pi x/\delta)\ln|1-e^{ix}|\,dx + (2/\pi)\int_0^{\pi/2} g(\pi x/\delta)\ \ln|1+e^{-ix}|\,dx.$$

$$= (2/\pi)\int_0^{\pi/2} g(\pi x/\delta)\ \ln|\text{cotg}\ (x/2)|\,dx. \qquad (1.9)$$

The surface of integration in the (x, ξ)-plane may be found by combining (1.7) and (1.9) and is shown in figure 2. In the shaded part the product $f(\xi)\ f(x-\xi)$ is not identically equal to zero.

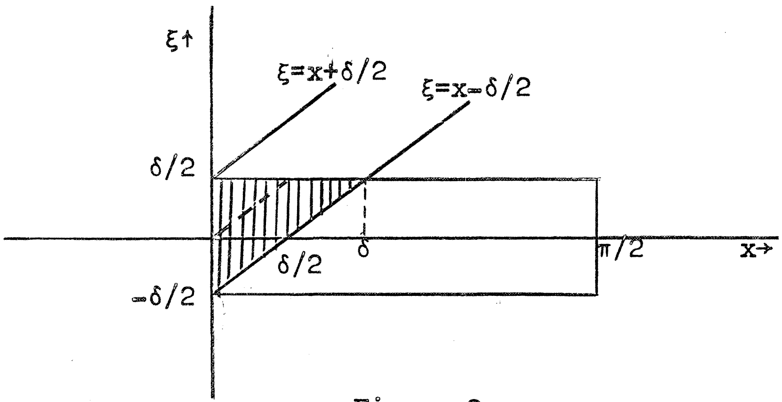


Figure 2.

Hence we find

$$S = (2/\pi)^2 \int_0^{\delta} \int_{x-\delta/2}^{\delta/2} f(\xi)\ f(x-\xi)\ \ln|\text{cotg}\ (x/2)|\ d\xi dx. \qquad (1.10)$$

Substituting (1.3) and

$$t = \pi x/\delta, \quad \tau = \pi\xi/\delta$$

in (1.10), we have

$$S = (4\delta^2/\pi^4) \int_0^{\pi} \int_{t-\pi/2}^{\pi/2} h(\tau)\ h(t-\tau)\ \ln|\text{cotg}\ (\delta t/2\pi)|\,d\tau dt.$$

For small values of δ the expansion

$$\ln|\text{cotg}\ (\delta t/2\pi)| = \ln|(2\pi/\delta t)| + O(\delta^2)$$

is valid.

Using this expansion we find an expression for S of the following form

$$S = (4\delta^2/\pi^4)\int_0^{\pi} \int_{t-\pi/2}^{\pi/2} h(\tau)\ h(t-\tau)\{\ln(\pi/\delta) + \ln 2 - \ln t + O(\delta^2)\}\,d\tau dt. \qquad (1.11)$$

The exact behaviour of S as δ approaches zero is easily found from this integral expression.

However, using the properties of h(t), we find

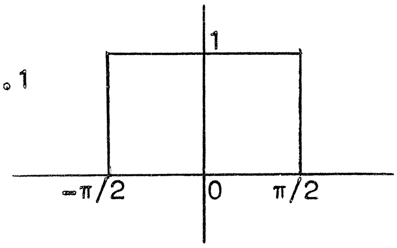
$$\int_0^{\pi}\int_{t-\pi/2}^{\pi/2} h(\tau)\, h(t-\tau)\,d\tau dt = \{\int_{-\pi/2}^{\pi/2} h(\tau)d\tau\}^2/2$$

and an interesting first approximation of S is thus obtained in the form

$$S \sim -\,(2/\pi^2)\{\int_{-\pi/2}^{\pi/2} h(\tau)d\tau\}^2\,(\delta/\pi)^2 \ln\delta. \qquad (1.12)$$

2. Some natural substitutions for h(t) are given below.

2.1



$$h(t) = 1, \quad -\pi/2 < t < \pi/2,$$
$$= 0, \quad \text{elsewhere}.$$

Using the integrals

$$\int_{-\pi/2}^{\pi/2} d\tau = \pi$$

and

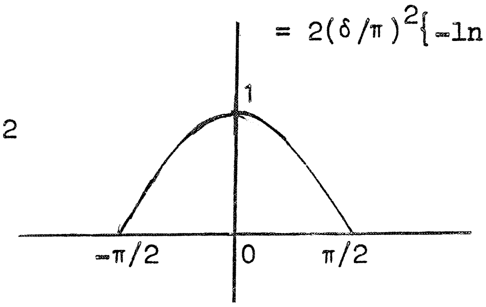
$$\int_0^{\pi}\int_{t-\pi/2}^{\pi/2} \ln t\; d\tau dt = (\pi^2/2)(\ln\pi - 3/2),$$

we find

$$S = 2(\delta/\pi)^2\{\ln(\pi/\delta) + \ln 2 - \ln\pi + 3/2 + 0(\delta^2)\}$$

$$= 2(\delta/\pi)^2\{-\ln\delta + 2{,}19 + 0(\delta^2)\}.$$

2.2



$$h(t) = \cos t, \quad -\pi/2 < t < \pi/2,$$
$$= 0 \quad , \quad \text{elsewhere}.$$

With the aid of

$$\int_{-\pi/2}^{\pi/2} \cos\tau\; d\tau = 2$$

and of

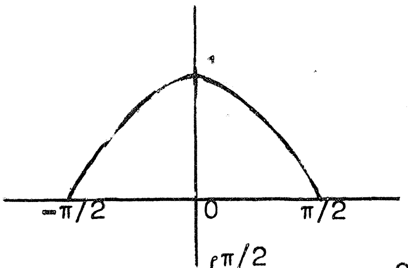
$$\int_0^{\pi}\int_{t-\pi/2}^{\pi/2} \cos\tau\,\cos(t-\tau)\,\ln t\,d\tau dt = 1 + 2\ln\pi - (\pi/2)\,\mathrm{Si}(\pi) - \mathrm{Cin}(\pi),$$

formula (1.11) gives

$$S = (8\delta^2/\pi^4)\{\ln(\pi/\delta) + \ln 2 - \tfrac{1}{2} - \ln\pi + (\pi/4)\,\mathrm{Si}(\pi) + (\tfrac{1}{2})\,\mathrm{Cin}(\pi) + O(\delta^2)\}$$

$$= (8\delta^2/\pi^4)\{-\ln\delta + 2{,}47 + O(\delta^2)\}.$$

2.3



$$h(t) = (2/\pi)^2\{(\pi/2)^2 - t^2\}, \quad -\pi/2 < t < \pi/2,$$
$$= 0, \quad \text{elsewhere}.$$

We use
$$\int_{-\pi/2}^{\pi/2} \{(\pi/2)^2 - \tau^2\}d\tau = (4/3)(\pi/2)^3$$

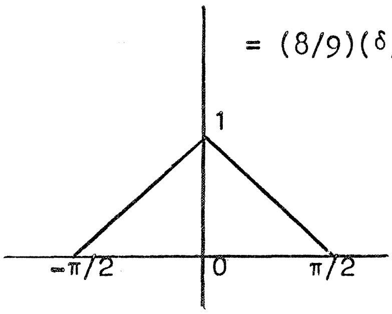
and
$$\int_0^{\pi}\int_{t-\pi/2}^{\pi/2} \{(\pi/2)^2 - \tau^2\}\{(\pi/2)^2 - (t-\tau)^2\}\ln t\,d\tau dt = (\pi^6/72)\{\ln\pi - 1{,}75\},$$

to get the result

$$S = (8/9)(\delta/\pi)^2\{\ln(\pi/\delta) + \ln 2 - \ln\pi + 1{,}75 + O(\delta^2)\}$$

$$= (8/9)(\delta/\pi)^2\{-\ln\delta + 2{,}44 + O(\delta^2)\}.$$

2.4



$$h(t) = (2/\pi)(\pi/2 - |t|), \quad -\pi/2 < t < \pi/2,$$
$$= 0, \quad \text{elsewhere}.$$

The integrals become

$$\int_{-\pi/2}^{\pi/2} (\pi/2 - |\tau|)d\tau = (\pi/2)^2$$

and
$$\int_0^{\pi}\int_{t-\pi/2}^{\pi/2} (\pi/2-|\tau|)(\pi/2-|t-\tau|)\ln t\,d\tau dt = (\tfrac{1}{2})(\pi/2)^4\{\ln\pi + 0{,}33\ln 2 - 2{,}08\},$$

so that

$$S = (\tfrac{1}{2})\ (\delta/\pi)^2\{\ln(\pi/\delta) + \ln 2 - \ln \pi - 0{,}33 \ln 2 + 2{,}08 + O(\delta^2)\}$$

$$= (\tfrac{1}{2})\ (\delta/\pi)^2\{-\ln \delta + 2{,}54 + O(\delta^2)\}.$$

N.B.

This research was instigated by some calculations in the dissertation of J. VERWEEL, "Magnetic properties of some ferroxplana single crystals" (Amsterdam, December 1966).

The distribution of magnetic charges for the special case of 180° Bloch walls of thickness δ at a relative distance a, is a function which behaves in a similar manner as the function f(x) in section 1, while the energy of the magnetic field can be represented in the form (1.5). By means of the formulae deduced in this note, the energy belonging to different charge distributions may be easily calculated.