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AN ELLIPTIC SINGULAR PERTURBATION PROBLEM WITH ALMOST CHARACTERISTIC BOUNDARIES

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An elliptic singular perturbation problem with almost charateristic boundaries

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1. Introduction

We investigate the Dirichlet problem for the second order, linear, elliptic differential equation

$$L_\varepsilon\phi \equiv (\varepsilon L_2+L_1)\phi = 0 \quad , \qquad 0<\varepsilon<<1 \tag{1.1}$$

in a domain $G = \{x,y : 0\leq x\leq 1, 0\leq y\leq 1\}$. When L_1 is a first order differential operator with constant coefficients the asymptotic solution of the problem exhibits a well-known boundary layer structure. The position of the boundary layers depends on the values of the coefficients of the operator L_1. In this problem the boundary values are chosen in such a way that free boundary layers do not occur in the first order approximation. The remaining boundary layers lying along the sides of the square are either parabolic or ordinary. There exist a large number of papers dealing with this subject, we mention Visik and Lyusternik [10], Knowles and Messick [7] and Eckhaus and De Jager [4]. In these studies neighborhoods of the corner points of G were excluded from approximation. In this paper it is shown that the seemingly singular behaviour of the asymptotic approximation is due to the presence of corner layers, which are visualized by applying the coordinate stretching method to both coordinates. This technique has been suggested by Eckhaus [2] and was worked out in Grasman [6]. Following this method we should obtain in straight lines the result we claim. However, there is another aspect which can be included in this study, that is the problem of almost characteristic boundaries. Let the subcharacteristics of L_1 be lines x = constant, then for the square of figure 1a and for the diamond of figure 1b we may expect different boundary layers. (The boundary values are chosen in a way that no boundary layers arise at the right-hand side of the domains.)

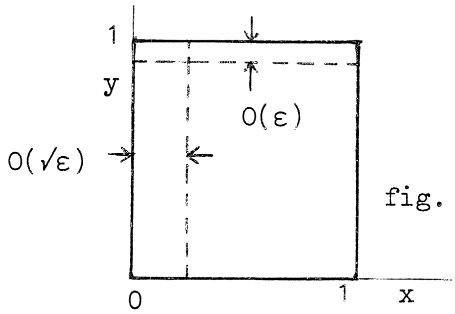


fig. 1a

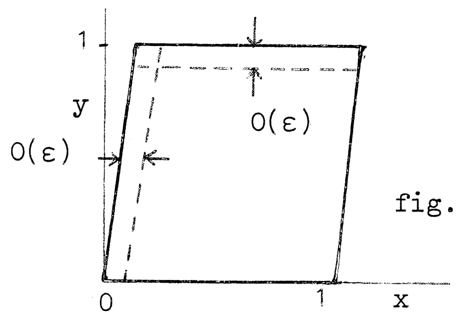


fig. 1b

It will be shown that the transition from one type of solution to the other is not so abrupt as the figures suggest.

Before formulating the problem in all details we give some definitions which can be helpful to understand the method we apply. Let us examine the behaviour of (1.1) near the corner (0,0). For that purpose we introduce a transformation of the type

$$x = \xi\varepsilon^{\alpha}, \qquad y = \eta\varepsilon^{\beta}. \tag{1.2}$$

By substituting (1.2) into (1.1) and by letting $\varepsilon \to 0$, we obtain the so-called limiting equation

$$L_0^{(\alpha,\beta)}\psi_{\alpha,\beta} = 0 \tag{1.3}$$

Before letting $\varepsilon \to 0$ we multiplied (1.1) with an appropriate power of ε so that (1.3) is a non-trivial equation with bounded coefficients. The functions $\psi_{\alpha,\beta}(\xi,\eta)$ are called formal local approximations. The integration constants of these functions are determined by matching conditions and by boundary values.

<u>Definition 1</u> A formal local approximation $\psi_{\alpha,\beta}$ is contained in $\psi_{\gamma,\delta}$, if

$$\lim_{\varepsilon\to 0} \psi_{\gamma,\delta}(\varepsilon^{\alpha-\gamma}\xi,\varepsilon^{\beta-\delta}\eta) = \psi_{\alpha,\beta}(\xi,\eta)$$

<u>Definition 2</u> A formal local approximation is called a significant approximation, if it is not contained in any other local approximation.

For a theory of singular perturbations based on more general definitions we refer to Eckhaus[3] and Grasman[6].

The differential equation which will be studied in particular is of the form

$$\varepsilon\Delta\phi + a\varepsilon^{\gamma}\frac{\delta\phi}{\delta x} - b\frac{\delta\phi}{\delta y} - c\phi = 0 \qquad (a,b>0). \qquad (1.4)$$

The boundary values are

$$\phi(x,0) = 0, \qquad \phi(x,1) = g(x) \qquad \text{for } 0\le x\le 1, \qquad (1.5ab)$$
$$\phi(0,y) = f(y), \qquad \phi(1,y) = 0 \qquad \text{for } 0<y<1, \qquad (1.5cd)$$

where $f(y)$ and $g(x)$ are continuous functions with $f(0)\neq 0$ and $f(1)\neq g(0)$. This problem differs from the one we discussed in the beginning. Instead of moving the boundaries we rotate the subcharacteristics of L_1. It is easily seen that the effect is the same. The advantage is that in the computations less coordinate transformations are needed.

For $\varepsilon \to 0$ equation (1.4) degenerates to a first order differential equation. It can be demonstrated (see[4]) that the solution of this reduced equation only satisfies the conditions (1.5a) and (1.5d). This (trivial) solution will hold in the greater part of the domain G, only the neighborhoods of the lines $x = 0$ and $y = 1$ need to be excluded. Near these lines ϕ has a boundary layer structure. Special attention will be given to the boundary layer along the side x = 0 including the end points (0,0) and (0,1). This is done by stretching both coordinates

$$x = \xi_1 \varepsilon^{\alpha_1}, \qquad y = \eta_1 \varepsilon^{\beta_1}, \tag{1.6}$$

and

$$x = \xi_2 \varepsilon^{\alpha_2}, \qquad y = 1-\eta_2 \varepsilon^{\beta_2}. \tag{1.7}$$

Near x = 0 we may expect an ordinary boundary layer of thickness $O(\varepsilon)$ for $a>\delta>0$, $\gamma = O(\delta$ arbitrarily small but independent of $\varepsilon)$ and a parabolic boundary layer of thickness $O(\sqrt{\varepsilon})$, for $a=0$. This would suggest that a parabolic boundary layer is an unusual phenomenon in physical problems (e.g. in magnetohydrodynamics).

It is the aim of this study to demonstrate that there is a smooth transition from one type of boundary layer to the other for $a \to 0$. In such a transition interval the boundary layer will have properties of both the parabolic and the ordinary boundary layer. Although we obtain a similar formula as Comstock[1], there exist considerable differences in the interpretation of the result. For instance we do not find a boundary layer with a bubble shape. According to our idea such a type of layer occurs when the tangency of the boundary with the subcharacteristic of L_1 is of higher order, see [6].

2. The ordinary boundary layer

Substituting (1.6) and (1.7) into (1.4) and letting $\varepsilon \to 0$ we obtain various limiting equations depending on the values of $\alpha_1, \alpha_2, \beta_1, \beta_2$ and γ. Taking in account the matching principle and the boundary conditions we find a certain number of significant approximations. The value of γ determines the configuration of solutions. Such a configuration is sketched in a α,β-diagram, see for example the figures 2a and 2b.

fig. 2a

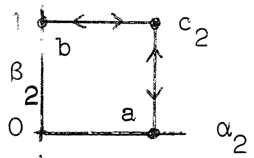


fig. 2b

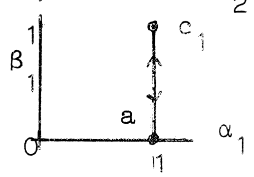
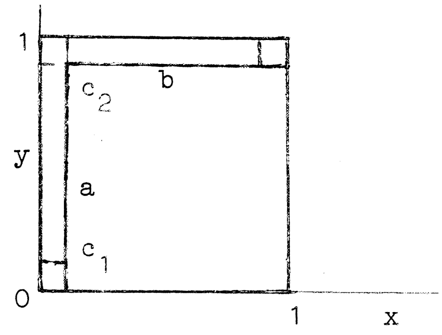


fig. 2c



For $\gamma = 0$ we distinguish the following significant cases.

(a) *The ordinary boundary layer along the line* $x = 0$

For $\alpha_k = 1$, $\beta_k = 0$ ($k = 1$ or 2) we have the limiting equation

$$\frac{\partial^2 U}{\partial \xi_k^2} + a\frac{\partial U}{\partial \xi_k} = 0 \tag{2.1}$$

The significant approximation satisfying boundary condition (1.5c) has the form

$$U(\xi_k, y) = f(y)e^{-a\xi_k} . \tag{2.2}$$

(b) *The ordinary boundary layer along the line* $y = 1$

$$\frac{\partial^2 V}{\partial \eta_2^2} + b\frac{\partial V}{\partial \eta_2} = 0 \qquad (\alpha_2=0, \beta_2=1) \tag{2.3}$$

Clearly this problem has the following solution

$$V(x, \eta_2) = g(x)e^{-b\eta_2} . \tag{2.4}$$

(c) *The corner layers near* (0,0) *and* (0,1)

For $\alpha_k = \beta_k = 1$ the limiting equations are

$$\frac{\partial^2 W_k}{\partial \xi_k^2} + \frac{\partial^2 W_k}{\partial \eta_k^2} + a\frac{\partial W_k}{\partial \xi_k} + (-1)^k \frac{\partial W_k}{\partial \eta_k} = 0 \quad , \quad k = 1,2 \tag{2.5}$$

(k=1 corresponds with the corner layer near (0,0),and k=2 with the corner layer near (0,1), see figures 2.abc). Tha boundary values are, respectively

$$W_1(\xi_1, 0) = 0 \qquad W_1(0, \eta_1) = f(0), \tag{2.6ab}$$

$$W_2(\xi_2, 0) = g(0) \qquad W_2(0, \eta_2) = f(1). \tag{2.6cd}$$

Moreover, the functions W_k are required to satisfy the matching conditions

$$W_1(\xi_1,\eta_1) = f(0)e^{-a\xi_1} \qquad \text{for } \eta_1 \to \infty, \qquad (2.7a)$$

$$W_2(\xi_2,\eta_2) = f(1)e^{-a\xi_2} \qquad \text{for } \eta_2 \to \infty, \qquad (2.7b)$$

$$W_2(\xi_2,\eta_2) = g(0)e^{-b\eta_2} \qquad \text{for } \xi_2 \to \infty, \qquad (2.7c)$$

For the solutions we write

$$W_1(\xi_1,\eta_1) = \frac{4f(0)}{\pi i}\, e^{-\frac{1}{2}a\xi_1} \int_{-\infty}^{+\infty} \exp\{-\tfrac{1}{2}\xi_1\sqrt{a^2+b^2+4\lambda^2} +$$

$$+ \tfrac{1}{2}\eta_1(b+2i\lambda)\}\, \frac{\lambda}{4\lambda^2+b^2}\, d\lambda \qquad (2.8a)$$

and

$$W_2(\xi_2,\eta_2) = g^* e^{-b\eta_2} + f(1)e^{-a\xi_2} + \qquad (2.8b)$$

$$\frac{4g^*}{\pi i}\, e^{-\frac{1}{2}a\xi_2} \int_{-\infty}^{+\infty} \exp\{-\tfrac{1}{2}\xi_2\sqrt{a^2+b^2+4\lambda^2} - \tfrac{1}{2}\eta_1(b-2i\lambda)\}\, \frac{\lambda}{4\lambda^2+b^2}\, dt +$$

$$\frac{4f(1)}{\pi i}\, e^{-\frac{1}{2}b_2} \int_{-\infty}^{+\infty} \exp\{-\tfrac{1}{2}\eta_2\sqrt{a^2+b^2+4\lambda^2} - \tfrac{1}{2}\xi_2(a-2i\lambda)\}\, \frac{\lambda}{4\lambda^2+a^2}\, dt ,$$

$$g^* = g(0) . \qquad (2.8c)$$

Since for $\lambda = \frac{1}{2}ib$ the integrand of (2.8a) has a pole with a residu independent of η_1, it is easily seen that condition (2.7a) is satisfied by taking a contour as given in figure 3 and by letting $R,\eta_1 \to \infty$. The properties (2.7bc) follow straight forward from (2.8b).

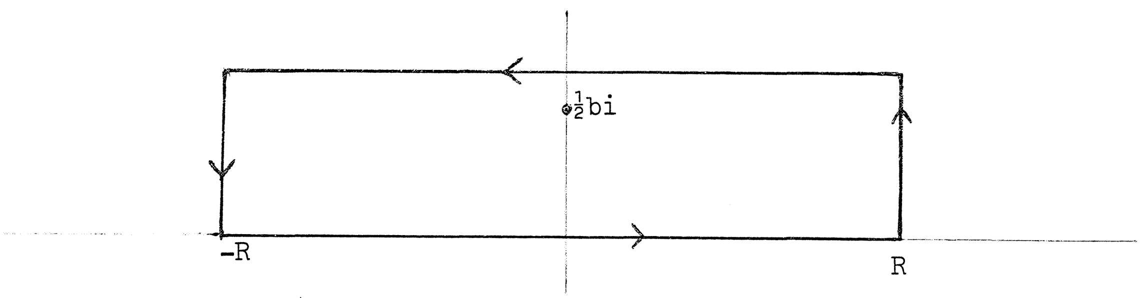


fig. 3

3. The transition boundary layer

For $0<\gamma<\frac{1}{2}$ we observe some changes compared with the situation of the preceding section. The local solutions in the corners follow from (2.8ab) by taking a = 0. The ordinary boundary layer increases and will have a thickness of order $O(\varepsilon^{1-\gamma})$. Moreover, two new significant approximations are considered

fig. 4a

fig. 4b

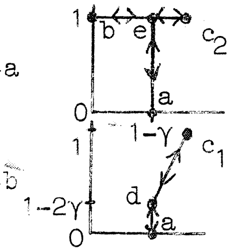
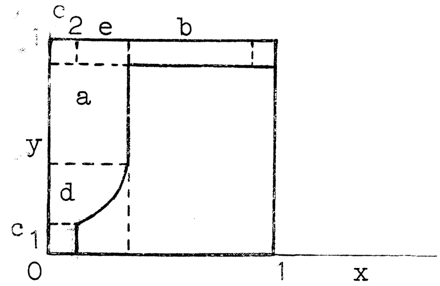


fig. 4c



(d) *The intermediate layer*

The function $X(\xi_1,\eta_1)$ has to satisfy the limiting equation

$$\frac{\partial^2 X}{\partial \xi^2} + a\frac{\partial X}{\partial \xi_1} - b\frac{\partial X}{\partial \eta_1} = 0 \qquad (\alpha_1=1-\gamma,\beta_1=1-2\gamma), \tag{3.1}$$

and the boundary conditions

$$X(0,\eta_1) = f(0) \qquad , \qquad X(\xi_1,0) = 0. \tag{3.2ab}$$

The matching conditions are

$$\lim_{\varepsilon\to 0} W_1(\varepsilon^{\alpha-1}\xi_1,\varepsilon^{2(\alpha-1)}\eta_1;a=0) = \lim_{\varepsilon\to 0} X(\varepsilon^{\alpha-1+\gamma}\xi_1,\varepsilon^{2(\alpha-1+\gamma)}\eta_1),$$

$$1-\gamma<\alpha<1 \ , \tag{3.3a}$$

and

$$\lim_{\eta_1\to\infty} X(\xi_1,\eta_1) = f(0)e^{-a\xi_1} \ . \tag{3.3b}$$

It appears that the solution of this problem already has some properties of the well-known parabolic boundary layer solution in the case of charateristic boundaries (3.4)

$$X(\xi_1,\eta_1) = \frac{f(0)}{2}\left\{e^{-a\xi_1}\mathrm{erfc}\left(\frac{\xi_1}{2}\sqrt{\frac{b}{\eta_1}} - \frac{a}{2}\sqrt{\frac{\eta_1}{b}}\right) + \mathrm{erfc}\left(\frac{\xi_1}{2}\sqrt{\frac{b}{\eta_1}} + \frac{a}{a}\sqrt{\frac{\eta_1}{b}}\right)\right\}$$

We used the name "intermediate layer" to express that the layer forms a linkage between the corner layer and the ordinary boundary layer. For increasing γ this layer grows in thickness with $O(\varepsilon^{1-\gamma})$ and in length with $O(\varepsilon^{1-2\gamma})$.

The solution (3.4) satisfies condition (3.3a) which can be understood as follows. From (3.4) we deduce that

$$X(\xi_1,\eta_1)\simeq f(0)\ \mathrm{erfc}\left(\frac{\xi_1}{2}\sqrt{\frac{b}{\eta_1}}\right) \qquad \text{for } \eta_1/\xi_1^2 = O(1),\ 0<\xi_1<<1 .$$

For the proof that on the other hand

$$W_1(\xi_1,\eta_1,a=0)\simeq f(0)\ \mathrm{erfc}\left(\frac{\xi_1}{2}\sqrt{\frac{b}{\eta_1}}\right) \text{ for } \eta_1/\xi_1^2 = O(1),\ \xi_1>>1$$

we refer to Temme [8]. He shows that for $\xi=\rho\sin\nu, \eta = \rho\cos\nu$ the expression (2.8a) (with a=0) behaves as

$$\frac{f(0)}{\pi i}\ e^{\frac{1}{2}\rho\ \cos\nu}\int_{-\infty}^{+\infty}\exp\{-\tfrac{1}{2}\rho\ \sin\nu\sqrt{b^2+4\lambda^2}+\rho\cos\lambda.$$

$$.\ i\lambda\}\ \frac{4\lambda}{4\lambda^2+b^2}\,d\lambda \simeq$$

$$f(0)\ \mathrm{erfc}\left(\sqrt{\rho b}\,\sin\frac{\nu}{2}\right) \quad , \qquad \rho>>1\ ,\quad 0\le\nu\le\pi/2.$$

Thus for $\nu\sqrt{\rho} = O(1)$, $\rho>>1$ we have

$$W_1 \simeq f(0)\ \mathrm{erfc}\left(\frac{\rho\nu}{2}\sqrt{\frac{b}{\rho}}\right).$$

(e) *The ordinary boundary layer along* $y=1$ *near* (0,1)

The significant approximation has to satisfy equation (2.3)($\alpha_2=1-\gamma,\beta_2=1$) and match three other significant approximations:

$$Y(\xi_2,\eta_2) = g(0)\ e^{-b\eta_2} \qquad \text{for} \quad \xi_2\to\infty ,$$

$$W_2(\xi_2,\eta_2) = Y(0,\eta_2) \qquad \text{for} \quad \xi_2\to\infty ,$$

$$Y(\xi_2,\eta_2) = f(1)\ e^{-a\xi_2} \qquad \text{for} \quad \eta_2\to\infty .$$

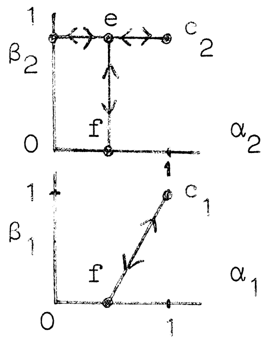
If we substitute the values a=0 and $g^*=g(0)-f(1)$ in (2.8b), then it is readily verified that the function

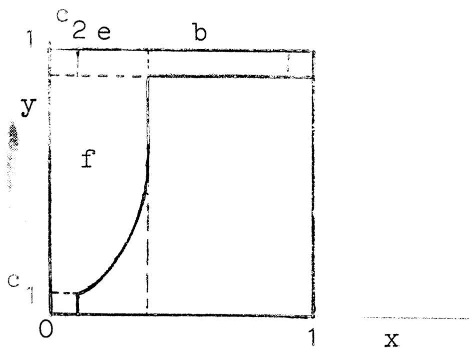
$$Y(\xi_2,\eta_2) = (g(0)-f(1)e^{-a\xi_2})e^{-b\eta_2} + f(1)e^{-a\xi_2} \qquad (3.5)$$

fullfills all conditions including the boundary condition $Y(\xi_2,0)=g(0)$.

4. The parabolic boundary layer

For $\gamma=\frac{1}{2}$ the ordinary boundary layer vanishes, and the intermediate layer transforms into:





(f) *The parabolic boundary layer*

$$\frac{\partial^2 Z}{\partial \xi_k^2} + a\frac{\partial Z}{\partial \xi_k} - b\frac{\partial Z}{\partial y} - cZ = 0 \qquad (\alpha_k=\tfrac{1}{2}, \beta_k=0, \quad k=1 \text{ or } 2). \qquad (4.1)$$

The solution that satisfies the boundary conditions is written as

$$Z(\xi_k,y) = \frac{\xi_k}{2}\sqrt{\frac{b}{\pi}}\, e^{-\frac{a}{2}\xi_k} \int_0^y f(p)\, \exp\left\{\frac{-\xi_k^2 b}{4(y-p)} - (\tfrac{1}{4}a^2+c)(y-p)\right\}\frac{dp}{(y-p)^{3/2}}, \qquad (4.2)$$

so that the boundary layer solution (3.5) transforms into

$$Y(\xi_2,\eta_2) = \{g(0) - Z(\xi_2,1)\}e^{-b\eta_2} + Z(\xi_2,1) .$$

From (4.2) we may conclude that Z remains regular for $a \to 0$. Therefore, $\gamma > \frac{1}{2}$ has not to be considered as a separate case. This means that the parabolic boundary layer solution is represented by (4.2) in alle cases where the boudary coincides with the subcharacteristic of L_1 with an accuracy of $O(\sqrt{\varepsilon})$.

Remark. For the construction of a uniform asymptotic approximation the reader is referred to [6] where for the Dirichlet problem of (1.1) in a stricktly convex domain G it is proved that by an appropriate composition of significant approximations the exact solution is approximated uniformly in $\bar{G}$ within a certain accuracy.
For this problem a similar proof can be given.

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