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ORTHOGONAL POLYNOMIALS IN TWO VARIABLES WHICH ARE EIGENFUNCTIONS OF TWO INDEPENDENT PARTIAL DIFFERENTIAL OPERATORS, I.

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Abstract

Let for α, β, $\gamma > -1$ and $n \geq k \geq 0$ the orthogonal polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ be defined as polynomials in u and v with "highest" term $u^{n-k}v^k$ which are obtained by orthogonalization of the sequence $1, u, v, u^2, uv, v^2, u^3, u^2v, \ldots\ldots$ with respect to the weight function $(1-u+v)^{\alpha}(1+u+v)^{\beta}(u^2-4v)^{\gamma}$ on the region bounded by the lines $1 - u + v = 0$ and $1 + u + v = 0$ and by the parabola $u^2 - 4v = 0$. Two explicit linear partial differential operators $D_1^{\alpha,\beta,\gamma}$ and $D_2^{\alpha,\beta,\gamma}$ of orders two and four, respectively, are obtained such that the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ are eigenfunctions of $D_1^{\alpha,\beta,\gamma}$ and $D_2^{\alpha,\beta,\gamma}$. It is proved that if a differential operator D has the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ as eigenfunctions then D can be expressed in one and only one way as a polynomial in $D_1^{\alpha,\beta,\gamma}$ and $D_2^{\alpha,\beta,\gamma}$. The special case $\gamma = -\frac{1}{2}$ can be reduced to Jacobi polynomials by the identity $p_{n,k}^{\alpha,\beta,-\frac{1}{2}}(x+y,xy) = \text{const.}(P_n^{(\alpha,\beta)}(x)\ P_k^{(\alpha,\beta)}(y) + P_k^{(\alpha,\beta)}(x)\ P_n^{(\alpha,\beta)}(y))$. For certain values of α, β, γ and in terms of the coordinates s, t, where $u = \cos s + \cos t$, $v = \cos s \cos t$, the operator $D_1^{\alpha,\beta,\gamma}$ is the radial part of the Laplace-Beltrami operator on certain compact Riemannian symmetric spaces of rank two.

1. Introduction

Compared with orthogonal polynomials in one variable very few things are known about orthogonal polynomials in several variables. A short survey of the subject can be found in the Bateman project [3, Chap. 12]. In the twenty years after the publication of this reference not many new results have been obtained. However, it seems to the author that there are still quite a lot of interesting problems on orthogonal polynomials in several variables, both in the general theory and in the study of the special cases. Especially those aspects of the field which are not trivial extensions of the one-variable case would be worthwile to consider.

In the one-variable case the so-called classical orthogonal polynomials are characterized by the property that these polynomials are eigenfunctions of a second order linear differential operator (cf. Bochner [2], Erdélyi [3, § 10.6]. Krall and Scheffer [5] generalized this property to the case of orthogonal polynomials in two variables as follows.
Let the class H_n of orthogonal polynomials of degree n on a two-dimensional region R with respect to a positive weight function w consist of all polynomials p(x,y) of degree n such that

$$\iint_R p(x,y)\, x^i\, y^j\, w(x,y)\, dx\, dy = 0 \qquad \text{if } i + j < n.$$

Then these orthogonal polynomials may be called classical if there exists a linear second order partial differential operator D in two variables for which each class H_n is an eigenspace, i.e. $Dp = \lambda_n p$ for all $p \in H_n$. Krall and Scheffer [5] classified all such differential operators D. Not all cases of interest can be obtained in this way. This can be seen from the example that H_n is spanned by the products $P_{n-k}^{(\alpha,\beta)}(x)\, P_k^{(\alpha,\beta)}(y)$, $k = 0,1,\ldots,n$, of two Jacobi polynomials. In this simple case and in several less trivial examples there still exists a linear second-order partial differential operator D in two variables such that each class H_n is spanned by eigenfunctions of D. But no longer two eigenfunctions belonging to the same class H_n need to have the same eigenvalue. If such an operator D exists for a particular class of orthogonal polynomials and if all eigenvalues of D have multiplicity one then there is a natural way to choose an orthogonal basis for each class H_n. This should be compared with the case of general orthogonal polynomials in two variables. Then there is usually no distinguished way to choose an orthogonal basis for H_n (cf. Erdélyi [3, § 12.1]).

The concept of "classical" orthogonal polynomials in two variables as introduced by Krall and Scheffer may be further modified. In the present paper and in one or more subsequent papers the author will consider examples of orthogonal polynomials in two variables, where for each class H_n an orthogonal basis $\{p_{n,0}, p_{n,1}, \ldots, p_{n,n}\}$ is chosen such that the following holds. There exist two algebraically independent partial differential operators D_1 of order two and D_2 of arbitrary non-zero order such that the polynomials $p_{n,k}(x,y)$ are joint eigenfunctions of D_1 and D_2.

In the example considered in the present paper the second order operator D_1 is related to compact Riemannian symmetric spaces of rank two. Apart from the two independent variables the operator $D_1 = D_1^{\alpha,\beta,\gamma}$ depends on three parameters α, β, γ. It turns out that the operator $D_2 = D_2^{\alpha,\beta,\gamma}$ has order four. The joint eigenfunctions $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ of $D_1^{\alpha,\beta,\gamma}$ and $D_2^{\alpha,\beta,\gamma}$ are polynomials in u and v with "highest" term $u^{n-k}v^k$ which are obtained by orthogonalization of the sequence
$1, u, v, u^2, uv, v^2, u^3, u^2v, \ldots$ with respect to the weight function $(1-u+v)^\alpha(1+u+v)^\beta(u^2-4v)^\gamma$ on a region bounded by the two perpendicular straight lines $1 - u + v = 0$, $1 + u + v = 0$ and by the parabola $u^2 - 4v = 0$ touching these lines. For reasons of convergence it is required that $\alpha, \beta, \gamma > -1$. These polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ form a large interesting class of orthogonal polynomials in two variables which resembles the class of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. If $\gamma = -\frac{1}{2}$ then the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ can be expressed in terms of Jacobi polynomials by the identity
$p_{n,k}^{\alpha,\beta,-\frac{1}{2}}(x+y,xy) = \text{const.}(P_n^{(\alpha,\beta)}(x)\, P_k^{(\alpha,\beta)}(y) + P_k^{(\alpha,\beta)}(x)\, P_n^{(\alpha,\beta)}(y))$.
For general values of γ the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ are not yet known in some explicit form, but the differential equations satisfied by these polynomials can be proved from the orthogonality properties.

In § 2 of this paper the operator $D_1^{\alpha,\beta,\gamma}$ is introduced and it is transformed into algebraic form. § 3 deals with the special case $\gamma = -\frac{1}{2}$.

In § 4 we prove that the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ are eigenfunctions of $D_1^{\alpha,\beta,\gamma}$. Next, in § 5 the fourth order operator $D_2^{\alpha,\beta,\gamma}$ is obtained as the product $D_2^{\alpha,\beta,\gamma} = D_+^{\alpha,\beta,\gamma} D_-^{\alpha,\beta,\gamma}$ of two second order operators. These operators $D_+^{\alpha,\beta,\gamma}$ and $D_-^{\alpha,\beta,\gamma}$ have the property that $D_-^{\alpha,\beta,\gamma} p_{n,k}^{\alpha,\beta,\gamma} = \text{const.}\ p_{n-1,k-1}^{\alpha+1.\beta+1,\gamma}$ and $D_+^{\alpha,\beta,\gamma} p_{n-1,k-1}^{\alpha+1,\beta+1,\gamma} = \text{const.}\ p_{n,k}^{\alpha,\beta,\gamma}$. Finally, in § 6 a theorem is proved stating that each differential operator which has for fixed α, β, γ all polynomials $p_{n,k}^{\alpha,\beta,\gamma}$ as eigenfunctions can be expressed in one and only one way as a polynomial in the operators $D_1^{\alpha,\beta,\gamma}$ and $D_2^{\alpha,\beta,\gamma}$.

The results of this paper might be applied to certain compact Riemannian symmetric spaces of rank two in order to characterize the spherical functions on these spaces as orthogonal polynomials and to obtain an explicit expression for an invariant differential operator on such a space which is independent of the Laplace-Beltrami operator.

In one or more forthcoming papers the author will consider other examples of orthogonal polynomials which are eigenfunctions of two independent partial differential operators. Some of these examples are rather elementary, because the polynomials can be written as products of Jacobi polynomials and of elementary functions. However, one particular example seems to be much deeper. In this example the operator D_1 is also related to compact symmetric spaces of rank two and the region of orthogonality is the interior of the so-called Steiner hypocycloid.

Yet another paper can be announced in which Ida Sprinkhuizen will continue the analysis of the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ introduced in the present paper. Among her results will be a Rodrigues type formula, the quadratic norm of $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ and the value of $p_{n,k}^{\alpha,\beta,\gamma}(2,1)$.

Notation. Throughout this paper the order α, β, γ may be deleted as upper index of a function or operator if no confusion is possible. For instance, we may write $p_{n,k}(u,v)$ instead of $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$.

2. The second order operator D_1

Consider for arbitrary real numbers α, β, γ the function

$$(2.1)\qquad w^{\alpha,\beta,\gamma}(s,t) = (\sin \tfrac{1}{2}s \sin \tfrac{1}{2}t)^{2\alpha+1}$$

$$\cdot\ (\cos \tfrac{1}{2}s \cos \tfrac{1}{2}t)^{2\beta+1} (\sin \tfrac{1}{2}(s+t) \sin \tfrac{1}{2}(t-s))^{2\gamma+1},$$

$$0 < s < t < \pi,$$

and the second order partial differential operator

$$(2.2)\qquad D_1^{\alpha,\beta,\gamma} = \frac{1}{w^{\alpha,\beta,\gamma}(s,t)} [\frac{\partial}{\partial s}(w^{\alpha,\beta,\gamma}(s,t)\frac{\partial}{\partial s}) + \frac{\partial}{\partial t}(w^{\alpha,\beta,\gamma}(s,t)\frac{\partial}{\partial t})].$$

Formula (2.2) can be written in explicit form as

$$(2.3)\qquad D_1^{\alpha,\beta,\gamma} = \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}$$

$$+ [(\alpha+\tfrac{1}{2}) \operatorname{cotg} \tfrac{1}{2}s - (\beta+\tfrac{1}{2}) \operatorname{tg} \tfrac{1}{2}s + (\gamma+\tfrac{1}{2}) \operatorname{cotg} \tfrac{1}{2}(s+t)$$

$$+ (\gamma+\tfrac{1}{2}) \operatorname{cotg} \tfrac{1}{2}(s-t)] \frac{\partial}{\partial s}$$

$$+ [(\alpha+\tfrac{1}{2}) \operatorname{cotg} \tfrac{1}{2}t - (\beta+\tfrac{1}{2}) \operatorname{tg} \tfrac{1}{2}t + (\gamma+\tfrac{1}{2}) \operatorname{cotg} \tfrac{1}{2}(t+s)$$

$$+ (\gamma+\tfrac{1}{2}) \operatorname{cotg} \tfrac{1}{2}(t-s)]\frac{\partial}{\partial t}\ .$$

Although the operator D_1 is defined by (2.2) only if $0 < s < t < \pi$, it follows from (2.3) that D_1 has a unique analytic continuation for all complex values of s and t except possibly on the singular lines $\sin s = 0$, $\sin t = 0$, $\sin \frac{1}{2}(s+t) = 0$, $\sin \frac{1}{2}(s-t) = 0$.

The operator D_1 has the following interpretation on certain symmetric spaces. Consider a compact Riemannian symmetric space of rank two for which the restricted root vectors have Dynkin diagram $0 \Rightarrow 0$ (cf. Araki [1, pp. 32,33]). The corresponding vector diagram is then given by figure 1.

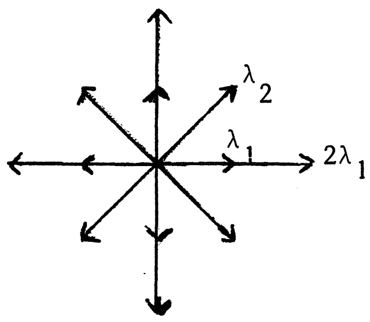

figure 1

Let the restricted roots λ_1, $2\lambda_1$ and λ_2 in figure 1 have multiplicities $2\alpha - 2\beta$, $2\beta + 1$ and $2\gamma + 1$, respectively.
Then it follows from Harish-Chandra [4, p. 270, Corollary 1] that the operator $D_1^{\alpha,\beta,\gamma}$ given by (2.2) denotes the radial part of the Laplace-Beltrami operator on such a symmetric space. The values of α, β, γ for which $D_1^{\alpha,\beta,\gamma}$ admits an interpretation on a symmetric space can be obtained from Araki [1, pp. 32, 33]. In the following the operator $D_1^{\alpha,\beta,\gamma}$ will only be considered from an analytic point of view for arbitrary real values of α, β and γ.

The singular lines of D_1 divide the (s,t)-plane into triangular regions with angles $\pi/2$, $\pi/4$, $\pi/4$ (cf. figure 2).

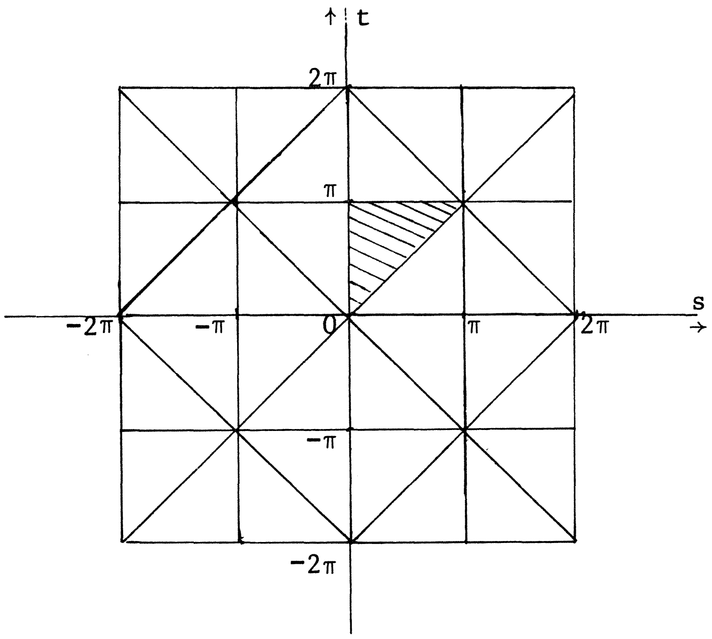

figure 2

The operator D_1 is invariant under reflections with respect to the singular lines. If a function f depending on s,t and defined for all real values of s,t is invariant under reflections with respect to these lines then, equivalently, f satisfies for all s,t the symmetry relations

$$(2.4)\qquad \begin{cases} f(s+2\pi,t) = f(s,t)\ , & f(s,t+2\pi) = f(s,t)\ , \\ f(-s,t) = f(s,t)\ , & f(s,-t) = f(s,t)\ , \\ f(t,s) = f(s,t)\ . & \end{cases}$$

Let R denote the triangular region

$$(2.5)\qquad R = \{(s,t) \mid 0 < s < t < \pi\}\ .$$

If f is continuous and satisfies (2.4) then f is completely determined by its restriction to R.

The function w defined by (2.1) is positive on the region R. The integral

$$\iint_R w^{\alpha,\beta,\gamma}(s,t)\, ds\, dt$$

is finite if and only α, β, $\gamma > -1$. From now on it will always be supposed that α, β and γ satisfy this inequality. The theorem below states that the operator D_1 is self-adjoint on R with respect to the weight function w and for an appropriate class of functions.

THEOREM 2.1. Let α, β, $\gamma > -1$. Let f and g be functions depending on s,t satisfying (2.4) which have continuous second derivatives. Then

$$(2.6)\qquad \iint_R (D_1^{\alpha,\beta,\gamma} f)\, g\, w^{\alpha,\beta,\gamma}(s,t)\, ds\, dt$$

$$= \iint_R f\, (D_1^{\alpha,\beta,\gamma} g)\, w^{\alpha,\beta,\gamma}(s,t)\, ds\, dt\ .$$

Proof. Let for positive and sufficiently small δ R_δ be a triangular region similar to and included in R such that the sides of R_δ are on distances δ from the respective sides of R. Integration by parts and application of Gauss's theorem gives

$$\iint_{R_\delta} (D_1 f)\, g\, w\, ds\, dt = \iint_{R_\delta} ((wf_s)_s + (wf_t)_t) g\, ds\, dt$$

$$= - \iint_{R_\delta} (f_s g_s + f_t g_t) w\, ds\, dt + \oint_{\partial R_\delta} \frac{\partial f}{\partial n}\, g\, w\, dl\ .$$

It follows from (2.4) that on the boundary ∂R_δ of R_δ $\frac{\partial f}{\partial n} = O(\delta)$ if $\delta \downarrow 0$. Thus

$$\int_{\partial R_\delta} \frac{\partial f}{\partial n}\, g\, w\, dl = O(\delta^{2\min(\alpha,\beta,\gamma)\ +\ 2}) \quad \text{if } \delta \downarrow 0\ .$$

Hence, by letting $\delta \downarrow 0$ it follows that

$$\iint_R (D_1 f)\, g\, w\, ds\, dt = - \iint_R (f_s g_s + f_t g_t)\, w\, ds\, dt.$$

A similar equality can be derived by reversing the roles of f and g and formula (2.6) follows. Q.e.d.

Let us transform the operator D_1 into algebraic form by the transformation

(2.7) $\quad x = \cos s\ , \quad y = \cos t\ .$

The transformed operator will also be denoted by D_1 and it equals

(2.8) $$D_1^{\alpha,\beta,\gamma} = (1-x^2)\frac{\partial^2}{\partial x^2} + (1-y^2)\frac{\partial^2}{\partial y^2}$$

$$+ [\beta - \alpha - (\alpha+\beta+2)x + (2\gamma+1)\frac{1-x^2}{x-y}]\frac{\partial}{\partial x} +$$

$$+ [\beta - \alpha - (\alpha+\beta+2)y + (2\gamma+1) \frac{1-y^2}{y-x}] \frac{\partial}{\partial y} \quad .$$

In terms of the function

$$(2.9) \qquad m^{\alpha,\beta,\gamma}(x,y) = (1-x)^{\alpha}(1+x)^{\beta}(1-y)^{\alpha}(1+y)^{\beta}(x-y)^{2\gamma+1}, \qquad -1<y<x<1 \ ,$$

we can also write

$$(2.10) \qquad D_1^{\alpha,\beta,\gamma} = \frac{1}{m^{\alpha,\beta,\gamma}(x,y)} [\frac{\partial}{\partial x}((1-x^2)m^{\alpha,\beta,\gamma}(x,y) \frac{\partial}{\partial x})$$

$$+ \frac{\partial}{\partial y} ((1-y^2)m^{\alpha,\beta,\gamma}(x,y) \frac{\partial}{\partial y})] \ .$$

The mapping $(s,t) \to (x,y)$ defined by (2.7) is a regular one-to-one mapping from each square region $\{(s,t) \mid k\pi < s < (k+1)\pi,\ l\pi < t < (l+1)\pi\}$, k,l integers, onto the square region $\{(x,y) \mid -1 < x < 1,\ -1 < y < 1\}$. In particular, the region R in the (s,t)-plane defined by (2.5) is mapped in a one-to-one and regular way onto the triangular region $\{(x,y) \mid -1 < y < x < 1\}$, which will also be denoted by R (cf. figure 3).

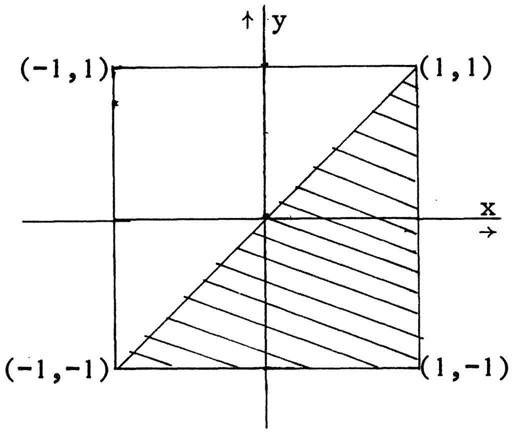

figure 3

Note that if f is a function in s and t, defined for all real values of s and t and satisfying (2.4) then there is a unique function g in x and y, defined for $x,y \in [-1,+1]$ and satisfying $g(x,y) = g(y,x)$, such that $g(\cos s, \cos t) = f(s,t)$ for all real s and t. Conversely if g is a symmetric function in x and y, defined for $x,y \in [-1,+1]$, then the function f defined by $f(s,t) = g(\cos s, \cos t)$ satisfies (2.4).

The functions $w^{\alpha,\beta,\gamma}$ and $m^{\alpha,\beta,\gamma}$ are related by the identity

$$(2.11) \qquad 2^{2\alpha+2\beta+2\gamma+3}\, w^{\alpha,\beta,\gamma}(s,t)\, ds\, dt = m^{\alpha,\beta,\gamma}(x,y)\, dx\, dy \qquad \text{on } R.$$

3. A special case: symmetrized products of Jacobi polynomials

A Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ of degree n and of order (α,β), $\alpha > -1$, $\beta > -1$, is an orthogonal polynomial of degree n on the interval $(-1, +1)$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$. The usual normalization is given by $P_n^{(\alpha,\beta)}(1) = \binom{\alpha+n}{n}$. We choose a different normalization as follows. Let $p_n^{\alpha,\beta}(x) = \text{const.}P_n^{(\alpha,\beta)}(x)$ such that $p_n^{\alpha,\beta}(x) = x^n$ + terms of lower degree. Then

$$(3.1) \qquad P_n^{(\alpha,\beta)}(x) = 2^{-n}\binom{2n+\alpha+\beta}{n}\, p_n^{\alpha,\beta}(x) .$$

There is the following pair of differential recurrence relations for Jacobi polynomials:

$$(3.2) \qquad \frac{d}{dx}\, p_n^{\alpha,\beta}(x) = n\, p_{n-1}^{\alpha+1,\beta+1}(x) ,$$

$$(3.3) \qquad (1-x)^{-\alpha}(1+x)^{-\beta}\frac{d}{dx}\left[(1-x)^{\alpha+1}(1+x)^{\beta+1}\, p_{n-1}^{\alpha+1,\beta+1}(x)\right]$$
$$= -\,(n+\alpha+\beta+1)\, p_n^{\alpha,\beta}(x) .$$

(cf. Szegö [6, formulas (4.21.7) and (4.10.1)]). By substituting the right hand side of (3.2) into the left hand side of (3.3) we obtain the second order differential equation

$$(3.4) \qquad (1-x)^{-\alpha}(1+x)^{-\beta}\frac{d}{dx}\left[(1-x)^{\alpha+1}(1+x)^{\beta+1}\frac{d}{dx}\, p_n^{\alpha,\beta}(x)\right]$$
$$= -\,n(n+\alpha+\beta+1)\, p_n^{\alpha,\beta}(x) .$$

This equation can also be written in the form

$$(3.5)\qquad [(1-x^2)\frac{d^2}{dx^2} + (\beta - \alpha - (\alpha+\beta+2)x)\frac{d}{dx}]\, p_n^{\alpha,\beta}(x)$$

$$= -\, n(n+\alpha+\beta+1)\, p_n^{\alpha,\beta}(x)$$

(cf. [6, formulas (4.2.1) and (4.2.2)]).

It is evident from (2.8) and (3.5) that

$$(3.6)\qquad D_1^{\alpha,\beta,-\frac{1}{2}}\, p_n^{\alpha,\beta}(x)\, p_k^{\alpha,\beta}(y)$$

$$= (-\, n(n+\alpha+\beta+1) - k(k+\alpha+\beta+1))\, p_n^{\alpha,\beta}(x)\, p_k^{\alpha,\beta}(y)\ .$$

The polynomials $p_n^{\alpha,\beta}(x)\, p_k^{\alpha,\beta}(y)$ are orthogonal polynomials in two variables on the square region $\{(x,y) \mid -1 < x < 1,\ -1 < y < 1\}$ with respect to the weight function $(1-x)^{\alpha}(1+x)^{\beta}(1-y)^{\alpha}(1+y)^{\beta}$. Let the symmetric polynomial $\tilde{p}_{n,k}^{\alpha,\beta,-\frac{1}{2}}(x,y)$ be defined by

$$(3.7)\qquad \tilde{p}_{n,k}^{\alpha,\beta,-\frac{1}{2}}(x,y) = \begin{cases} p_n^{\alpha,\beta}(x)\, p_k^{\alpha,\beta}(y) + p_k^{\alpha,\beta}(x)\, p_n^{\alpha,\beta}(y) & \text{if } n > k, \\ p_n^{\alpha,\beta}(x)\, p_n^{\alpha,\beta}(y) & \text{if } n = k\ . \end{cases}$$

Then by (3.6) the polynomial $\tilde{p}_{n,k}^{\alpha,\beta,-\frac{1}{2}}(x,y)$ is also an an eigenfunction of $D_1^{\alpha,\beta,-\frac{1}{2}}$. We have

$$(3.8)\qquad D_1^{\alpha,\beta,-\frac{1}{2}}\, \tilde{p}_{n,k}^{\alpha,\beta,-\frac{1}{2}} = (-n(n+\alpha+\beta+1) - k(k+\alpha+\beta+1))\, \tilde{p}_{n,k}^{\alpha,\beta,-\frac{1}{2}}\ .$$

If $(n,k) \neq (m,l)$ then

$$(3.9)\qquad \iint_R \tilde{p}_{n,k}^{\alpha,\beta,-\frac{1}{2}}(x,y)\, \tilde{p}_{m,l}^{\alpha,\beta,-\frac{1}{2}}(x,y)\, m^{\alpha,\beta,-\frac{1}{2}}(x,y)\, dx\, dy = 0\ ,$$

where R is the region $\{(x,y) \mid -1 < y < x < 1\}$ and where the weight function $m^{\alpha,\beta,-\frac{1}{2}}$ is given by formula (2.9).

Let N be the set of all pairs of integers (n,k) such that $n \geq k \geq 0$.

Suppose that N is lexicographically ordered. This means that $(n,k) > (m,l)$ if either $n > m$ or $n = m$ and $k > l$.

Each symmetric polynomial $p(x,y)$ is a linear combination of the symmetric polynomials $x^n y^k + x^k y^n$, $(n,k) \in N$.
Let $p(x,y) = \sum_{(m,l)\le(n,k)} c_{m,l}(x^m y^l + x^l y^m)$ for certain coefficients $c_{m,l}$, $(m,l) \in N$, such that $c_{n,k} \neq 0$. Then we say that the symmetric polynomial $p(x,y)$ has (symmetric) degree (n,k).

In particular, the symmetric polynomial $\tilde{p}^{\alpha,\beta,-\frac{1}{2}}_{n,k}(x,y)$ defined by (3.7) has symmetric degree (n,k). The term of highest degree of $\tilde{p}^{\alpha,\beta,-\frac{1}{2}}_{n,k}(x,y)$ is $x^n y^k + x^k y^n$ if $n > k$ and $x^n y^n$ if $n = k$.
The polynomials $\tilde{p}^{\alpha,\beta,-\frac{1}{2}}_{n,k}(x,y)$ can be obtained by orthogonalization of the sequence 1, $x+y$, xy, x^2+y^2, x^2y+xy^2, x^2y^2, x^3+y^3,..... with respect to the weight function $m^{\alpha,\beta,-\frac{1}{2}}$ on the region R.

Let us consider next arbitrary polynomials $q(u,v)$ in the two variables u and v. Each polynomial $q(u,v)$ is a linear combination of the polynomials $u^{n-k}v^k$, $(n,k) \in N$. Let

$$q(u,v) = \sum_{(m,l)\le(n,k)} c_{m,l}\, u^{m-l} v^l$$

for certain coefficients $c_{m,l}$, $(m,l) \in N$, such that $c_{n,k} \neq 0$. Then we say that the polynomial $q(u,v)$ has degree (n,k). Note that a polynomial of degree (n,k) has ordinary degree n.

By v.d. Waerden [7, § 33] each symmetric polynomial $p(x,y)$ can be written in one and only one way as a polynomial in the so-called elementary symmetric polynomials $x + y$ and xy.

LEMMA 3.1. Let $u = x + y$, $v = xy$. Let the symmetric polynomial $\tilde{p}(x,y)$ and the polynomial $p(u,v)$ be related by the identity $\tilde{p}(x,y) = p(u,v)$. Then the symmetric degree of $\tilde{p}(x,y)$ is equal to the degree of $p(u,v)$. If this degree is (n,k) then the coefficient of $x^n y^k + x^k y^n$ (or of $x^n y^n$ if $n = k$) for $\tilde{p}(x,y)$ is equal to the coefficient of $u^{n-k}v^k$ for $p(u,v)$.

Proof. Let $(n,k) \in N$. We shall express $u^{n-k}v^k$ as a linear combination of polynomials $x^m y^l + x^l y^m$. If $n = k$ then $v^n = x^n y^n$. If $n > k$ then it follows by the binomial formula that

$$u^{n-k}v^k = (x+y)^{n-k}(xy)^k = (x^{n-k}+y^{n-k})(xy)^k$$

$$+ \sum_{i=1}^{[\frac{1}{2}(n-k)]} c_i(x^{n-k-i}y^i + x^iy^{n-k-i})(xy)^k$$

for certain coefficients c_i . Hence

$$u^{n-k}v^k = x^ny^k + x^ky^n + \sum_{i=1}^{[\frac{1}{2}(n-k)]} c_i(x^{n-i}y^{k+i}+x^{k+i}y^{n-i})$$

$$= x^ny^k + x^ky^n + \text{ polynomial of symmetric degree less than } (n,k).$$

This proves the lemma. Q.e.d.

The transformation

(3.10) $\quad u = x + y \ , \quad v = xy$

maps the region R in the (x,y)-plane in a one-to-one and regular way onto the region $\{(u,v) \mid -v-1 < u < v+1,\ u^2-4v > 0\}$, which will also be denoted by R (cf. figure 4).

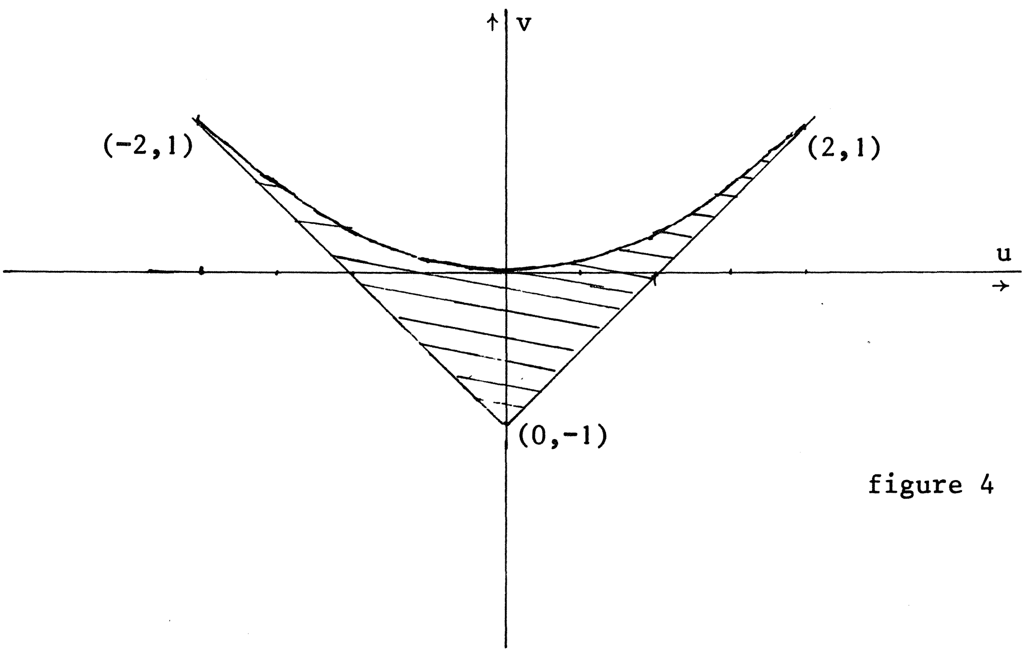

figure 4

This region is bounded by two perpendicular lines and a parabola which touches the two lines. Let the weight function $\mu^{\alpha,\beta,\gamma}$ be defined by

$$\mu^{\alpha,\beta,\gamma}(u,v)\;du\;dv = m^{\alpha,\beta,\gamma}(x,y)\;dx\;dy \quad \text{on } R. \tag{3.11}$$

Then

$$\mu^{\alpha,\beta,\gamma}(u,v) = (1-u+v)^{\alpha}(1+u+v)^{\beta}(u^2-4v)^{\gamma}. \tag{3.12}$$

Let $p_{n,k}^{\alpha,\beta,-\frac{1}{2}}(u,v) = \tilde{p}_{n,k}^{\alpha,\beta,-\frac{1}{2}}(x,y)$. Then by Lemma 3.1 the functions $p_{n,k}^{\alpha,\beta,-\frac{1}{2}}(u,v)$ are polynomials of degree (n,k) in u and v and the coefficient of $u^{n-k}v^k$ is equal to 1. If $(n,k) \neq (m,1)$ then using (3.9) and (3.12) we have the orthogonality relation

$$\iint_R p_{n,k}^{\alpha,\beta,-\frac{1}{2}}(u,v)\; p_{m,1}^{\alpha,\beta,-\frac{1}{2}}(u,v)\; \mu^{\alpha,\beta,-\frac{1}{2}}(u,v)\; du\; dv = 0\;. \tag{3.13}$$

Hence the polynomials $p_{n,k}^{\alpha,\beta,-\frac{1}{2}}(u,v)$ can be obtained by orthogonalization of the sequence $1,\; u,\; v,\; u^2,\; uv,\; v^2,\; u^3, \ldots\ldots$ on the region R with respect to the weight function $\mu^{\alpha,\beta,-\frac{1}{2}}$.

As a generalization of the polynomials $p_{n,k}^{\alpha,\beta,-\frac{1}{2}}(u,v)$ we define for $\alpha,\beta,\gamma > -1$ and $(n,k) \in N$ the polynomial $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ by the conditions

$$(3.14)\quad \begin{cases} p_{n,k}^{\alpha,\beta,\gamma}(u,v) = u^{n-k}v^k \;+\; \text{polynomial of degree less than } (n,k)\;, \\[2ex] \displaystyle\iint_R p_{n,k}^{\alpha,\beta,\gamma}(u,v)\; q(u,v)\; \mu^{\alpha,\beta,\gamma}(u,v)\; du\; dv = 0 \\[2ex] \quad \text{if } q(u,v) \text{ is a polynomial of degree less than } (n,k). \end{cases}$$

We conclude this section by deriving a fourth order differential operator which has the polynomials $\tilde{p}_{n,k}^{\alpha,\beta,-\frac{1}{2}}(x,y)$ as eigenfunctions. It follows from (3.2) and (3.3) that

(3.15) $$\frac{\partial^2}{\partial x \partial y} \tilde{p}_{n,k}^{\alpha,\beta,-\frac{1}{2}}(x,y) = n\ k\ \tilde{p}_{n-1,k-1}^{\alpha+1,\beta+1,-\frac{1}{2}}(x,y)$$

and

(3.16) $$((1-x)(1-y))^{-\alpha}((1+x)(1+y))^{-\beta} \frac{\partial^2}{\partial x \partial y} [((1-x)(1-y))^{\alpha+1}$$

$$\cdot ((1+x)(1+y))^{\beta+1} \tilde{p}_{n-1,k-1}^{\alpha+1,\beta+1,-\frac{1}{2}}(x,y)]$$

$$= (n+\alpha+\beta+1)(k+\alpha+\beta+1) \tilde{p}_{n,k}^{\alpha,\beta,-\frac{1}{2}}(x,y) .$$

Hence

(3.17) $$((1-x)(1-y))^{-\alpha}((1+x)(1+y))^{-\beta} \frac{\partial^2}{\partial x \partial y} [((1-x)(1-y))^{\alpha+1}$$

$$\cdot ((1+x)(1+y))^{\beta+1} \frac{\partial^2}{\partial x \partial y} \tilde{p}_{n,k}^{\alpha,\beta,-\frac{1}{2}}(x,y)]$$

$$= n\ k\ (n+\alpha+\beta+1)(k+\alpha+\beta+1) \tilde{p}_{n,k}^{\alpha,\beta,-\frac{1}{2}}(x,y) .$$

In § 5 the last three formulas will be generalized to the case of arbitrary γ.

4. The eigenfunctions of the second order operator D_1

If the operator $D_1^{\alpha,\beta,\gamma}$ given by (2.8) is expressed in terms of the coordinates u,v (cf. formula 3.10) then we obtain

(4.1) $$D_1^{\alpha,\beta,\gamma} = (-u^2+2v+2) \frac{\partial^2}{\partial u^2} + (-2uv+2u) \frac{\partial^2}{\partial u \partial v}$$

$$+ (u^2-2v^2-2v) \frac{\partial}{\partial v^2} + [- (\alpha+\beta+2\gamma+3)u + (2\beta-2\alpha)] \frac{\partial}{\partial u}$$

$$+ [(\beta-\alpha)u - (2\alpha+2\beta+2\gamma+5)v - (2\gamma+1)] \frac{\partial}{\partial v} .$$

In this section it will be proved that the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ defined by (3.14) are eigenfunctions of $D_1^{\alpha,\beta,\gamma}$. The proof is based on two lemmas.

LEMMA 4.1. If $D_1^{\alpha,\beta,\gamma}$ is given by (4.1) and if $(n,k) \in N$ then

$$D_1^{\alpha,\beta,\gamma}(u^{n-k}v^k) = [-n(n+\alpha+\beta+2\gamma+2) - k(k+\alpha+\beta+1)]\ u^{n-k}v^k \tag{4.2}$$

$$+ \text{ polynomial of degree less than } (n,k).$$

LEMMA 4.2. Let $\alpha,\beta,\gamma > -1$. For arbitrary polynomials $p(u,v)$ and $q(u,v)$ it holds that

$$\iint_R (D_1^{\alpha,\beta,\gamma}p(u,v))\ q(u,v)\ \mu^{\alpha,\beta,\gamma}(u,v)\ du\ dv \tag{4.3}$$

$$= \iint_R p(u,v)(D_1^{\alpha,\beta,\gamma}q(u,v))\ \mu^{\alpha,\beta,\gamma}(u,v)\ du\ dv\ .$$

Lemma 4.1 follows immediately from (4.1). Lemma 4.2 follows from Theorem 2.1 and from formulas (2.11) and (3.11).

THEOREM 4.3. Let $\alpha,\beta,\gamma > -1$ and $(n,k) \in N$. Then

$$D_1^{\alpha,\beta,\gamma}p_{n,k}^{\alpha,\beta,\gamma} = [-n(n+\alpha+\beta+2\gamma+2) - k(k+\alpha+\beta+1)]\ p_{n,k}^{\alpha,\beta,\gamma}. \tag{4.4}$$

Proof. By Lemma 4.1 the function $D_1^{\alpha,\beta,\gamma}\ p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ is a polynomial of degree (n,k). Let $(m,l) \in N$ and $(m,l) < (n,k)$. Then it follows from Lemma 4.2 and formula (3.14) that

$$\iint_R (D_1\ p_{n,k}^{\alpha,\beta,\gamma})\ p_{m,l}^{\alpha,\beta,\gamma}\ \mu^{\alpha,\beta,\gamma}\ du\ dv$$

$$= \iint_R p_{n,k}^{\alpha,\beta,\gamma}\ (D_1\ p_{m,l}^{\alpha,\beta,\gamma})\ \mu^{\alpha,\beta,\gamma}\ du\ dv = 0\ .$$

Hence $D_1\, p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ is a polynomial of degree (n,k) orthogonal to all polynomials of lower degree, so $D_1\, p_{n,k}^{\alpha,\beta,\gamma}(u,v) = \text{const.}\; p_{n,k}^{\alpha,\beta,\gamma}(u,v)$. The value of the constant follows from (4.2). Q.e.d.

5. The fourth order operator D_2

In this section we shall generalize the formulas (3.15), (3.16) and (3.17) by finding second order differential operators $D_+^{\alpha,\beta,\gamma}$ and $D_-^{\alpha,\beta,\gamma}$ such that

$$D_-^{\alpha,\beta,\gamma}\, p_{n,k}^{\alpha,\beta,\gamma} = \text{const.}\; p_{n-1,k-1}^{\alpha+1,\beta+1,\gamma} \qquad \text{and}$$

$$D_+^{\alpha,\beta,\gamma}\, p_{n-1,k-1}^{\alpha+1,\beta+1,\gamma} = \text{const.}\; p_{n,k}^{\alpha,\beta,\gamma}\ .$$

Then $D_+^{\alpha,\beta,\gamma}\, D_-^{\alpha,\beta,\gamma}$ is a fourth order operator which has the polynomials $p_{n,k}^{\alpha,\beta,\gamma}$ as eigenfunctions.

Let us define

$$\text{(5.1)} \qquad D_-^{\alpha,\beta,\gamma} = D_-^{\gamma} = \frac{1}{2(x-y)^{2\gamma+1}}\left[\frac{\partial}{\partial x}\Big((x-y)^{2\gamma+1}\frac{\partial}{\partial y}\Big) + \frac{\partial}{\partial y}\Big((x-y)^{2\gamma+1}\frac{\partial}{\partial x}\Big)\right]$$

and

$$\text{(5.2)} \qquad D_+^{\alpha,\beta,\gamma} = ((1-x)(1-y))^{-\alpha}((1+x)(1+y))^{-\beta}\, D_-^{\gamma}$$
$$.\ ((1-x)(1-y))^{\alpha+1}((1+x)(1+y))^{\beta+1}\ .$$

These operators are generalizations of the differential operators in (3.15) and (3.16), respectively.

LEMMA 5.1. Let $\alpha,\beta,\gamma > -1$. For arbitrary symmetric polynomials $p(x,y)$ and $q(x,y)$ it holds that

$$\text{(5.3)} \qquad \iint_R (D_-^{\gamma}\, p(x,y))\, q(x,y)\, m^{\alpha+1,\beta+1,\gamma}(x,y)\, dx\, dy =$$

$$= \iint_R p(x,y)\, (D_+^{\alpha,\beta,\gamma}\, q(x,y))\; m^{\alpha,\beta,\gamma}(x,y)\, dx\, dy\ .$$

Proof. Let $Q(x,y) = ((1-x)(1-y))^{\alpha+1}((1+x)(1+y))^{\beta+1}\ q(x,y)$.
Then we have to prove that

$$\iint_R \left[\frac{\partial}{\partial x}\left((x-y)^{2\gamma+1}\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial y}\left((x-y)^{2\gamma+1}\frac{\partial p}{\partial x}\right)\right] Q\, dx\, dy$$

$$= \iint_R p\left[\frac{\partial}{\partial x}\left((x-y)^{2\gamma+1}\frac{\partial Q}{\partial y}\right) + \frac{\partial}{\partial y}\left((x-y)^{2\gamma+1}\frac{\partial Q}{\partial x}\right)\right] dx\, dy\ .$$

Denote the left hand side of this equality by I_1 and the right hand side by I_2. Let

$$I_3 = -\iint_R \left(\frac{\partial p}{\partial x}\frac{\partial Q}{\partial y} + \frac{\partial p}{\partial y}\frac{\partial Q}{\partial x}\right)(x-y)^{2\gamma+1}\, dx\, dy\ .$$

Let ∂R denote the boundary of R. Integration by parts and application of Gauss's theorem gives

$$I_1 = I_3 + \oint_{\partial R} (x-y)^{2\gamma+1}\, Q\left(\frac{\partial p}{\partial y}\, dy - \frac{\partial p}{\partial x}\, dx\right)$$

and

$$I_2 = I_3 + \oint_{\partial R} (x-y)^{2\gamma+1}\, p\left(\frac{\partial Q}{\partial y}\, dy - \frac{\partial Q}{\partial x}\, dx\right)\ .$$

On the sides $y = -1$ and $x = 1$ the contour integrals vanish since Q and $\frac{\partial Q}{\partial x}$ vanish for $y = -1$ and Q and $\frac{\partial Q}{\partial y}$ vanish for $x = 1$. The contour integrals evidently vanish on the side $x = y$ if $\gamma > -\frac{1}{2}$. Otherwise we have to use that the normal derivatives of p and Q on the line $x = y$ vanish. Thus it is proved that $I_1 = I_3 = I_2$. Q.e.d.

If the operators D_-^{γ} and $D_+^{\alpha,\beta,\gamma}$ are expressed in terms of u and v then we obtain the formulas

(5.4) $$D_-^{\gamma} = \frac{\partial^2}{\partial u^2} + u\frac{\partial^2}{\partial u\partial v} + v\frac{\partial^2}{\partial v^2} + (\gamma + \frac{3}{2})\frac{\partial}{\partial v} \quad ,$$

(5.5) $$D_+^{\alpha,\beta,\gamma} = (1-u+v)^{-\alpha}(1+u+v)^{-\beta}D_-^{\gamma}(1-u+v)^{\alpha+1}(1+u+v)^{\beta+1}$$

$$= (-u^2+v^2+2v+1)\frac{\partial^2}{\partial u^2} + (-u^3+uv^2+2uv+u)\frac{\partial^2}{\partial u\partial v}$$

$$+ (-u^2v+v^3+2v^2+v)\frac{\partial^2}{\partial v^2} + [(\alpha-\beta)(u^2-2v-2)$$

$$+ (\alpha+\beta+2)(uv-u)]\frac{\partial}{\partial u} + [(\alpha+\beta+\gamma+\frac{7}{2})(-u^2+2v)$$

$$+ (\alpha-\beta)(uv-u) + (2\alpha+2\beta+\gamma+\frac{11}{2})v^2 + (\gamma+\frac{3}{2})]\frac{\partial}{\partial v}$$

$$+ [(\alpha-\beta)(\alpha+\beta+\gamma+\frac{5}{2})u + (\alpha+\beta+2)(\alpha+\beta+\gamma+\frac{5}{2})v$$

$$+ (\alpha-\beta)^2 + (\gamma+\frac{1}{2})(\alpha+\beta+2)] \quad .$$

LEMMA 5.2. Let the operators D_-^{γ} and $D_+^{\alpha,\beta,\gamma}$ be given by (5.4) and (5.5). Let $(n,k) \in N$. If $k > 0$ then

(5.6) $D_-^{\gamma}(u^{n-k}v^k) = k(n+\gamma+\frac{1}{2})u^{n-k}v^{k-1}$ + polynomial of degree less than $(n-1,k-1)$,

(5.7) $D_+^{\alpha,\beta,\gamma}(u^{n-k}v^{k-1}) = (k+\alpha+\beta+1)(n+\alpha+\beta+\gamma+\frac{3}{2})u^{n-k}v^k$ + polynomial of degree less than (n,k).

Furthermore

(5.8) $$D_-^{\gamma}\, u^n = n(n-1)\, u^{n-2} \quad .$$

LEMMA 5.3. Let $\alpha,\beta,\gamma > -1$. For arbitrary polynomials $p(u,v)$ and $q(u,v)$ it holds that

$$(5.9)\qquad \iint_R (D_-^\gamma\, p(u,v))\, q(u,v)\, \mu^{\alpha+1,\beta+1,\gamma}(u,v)\, du\, dv$$

$$= \iint_R p(u,v)\,(D_+^{\alpha,\beta,\gamma} q(u,v))\mu^{\alpha,\beta,\gamma}(u,v)\, du\, dv\ .$$

Lemma 5.2 follows immediately from (5.4) and (5.5). Lemma 5.3 follows from Lemma 5.1 by using (3.11).

THEOREM 5.4. Let $\alpha,\beta,\gamma > -1$ and $(n,k) \in N$. Then

$$(5.10)\qquad D_-^\gamma\, p_{n,k}^{\alpha,\beta,\gamma} = \begin{cases} k(n+\gamma+\frac{1}{2})\, p_{n-1,k-1}^{\alpha+1,\beta+1,\gamma} & \text{for } k > 0 \\ 0 & \text{for } k = 0\ , \end{cases}$$

$$(5.11)\qquad D_+^{\alpha,\beta,\gamma}\, p_{n-1,k-1}^{\alpha+1,\beta+1,\gamma} = (k+\alpha+\beta+1)(n+\alpha+\beta+\gamma+\tfrac{3}{2})\, p_{n,k}^{\alpha,\beta,\gamma} \quad \text{for } k > 0\ .$$

Proof. By Lemma 5.3 we have

$$(5.12)\qquad \iint_R (D_-^\gamma\, p_{n,k}^{\alpha,\beta,\gamma})\, p_{m,l}^{\alpha+1,\beta+1,\gamma}\, \mu^{\alpha+1,\beta+1,\gamma}\, du\, dv$$

$$= \iint_R p_{n,k}^{\alpha,\beta,\gamma}\, (D_+^{\alpha,\beta,\gamma}\, p_{m,l}^{\alpha+1,\beta+1,\gamma})\, \mu^{\alpha,\beta,\gamma}\, du\, dv\ .$$

It follows from Lemma 5.2 that $D_-^\gamma\, p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ is a polynomial which has degree $(n-1,k-1)$ if $k > 0$, degree not larger than $(n-2,n-2)$ if $k = 0$, $n \geq 2$, and which is the zero polynomial if $k = 0$, $n = 0$ or 1. Similarly $D_+^{\alpha,\beta,\gamma}\, p_{m,l}^{\alpha+1,\beta+1,\gamma}(u,v)$ is a polynomial of degree not larger than $(m+1,l+1)$. First we prove (5.10) for $k > 0$. Let $(m,l) < (n-1,k-1)$ in (5.12). Then the right hand side of (5.12) is equal to zero by the orthogonality property of the polynomials with respect to the weight function $\mu^{\alpha,\beta,\gamma}$.

Thus the left hand side of (5.12) vanishes. By using the orthogonality property of the polynomials with respect to the weight function $\mu^{\alpha+1,\beta+1,\gamma}$ it follows that $D_-^{\gamma}\, p_{n,k}^{\alpha,\beta,\gamma} = \text{const.}\; p_{n-1,k-1}^{\alpha+1,\beta+1,\gamma}$ and the constant is given by (5.6). If $k=0$, $n \geq 2$ then (5.10) follows by choosing $(m,l) \leq (n-2,n-2)$ in (5.12). Formula (5.10) is evident if $(n,k) = (0,0)$ or $(1,0)$. Formula (5.11) can be derived in a similar way as (5.10) by putting $(n,k) < (m+1,l+1)$ in (5.12). Q.e.d.

Let the fourth order partial differential operator $D_2^{\alpha,\beta,\gamma}$ be defined by

$$(5.13) \qquad D_2^{\alpha,\beta,\gamma} = D_+^{\alpha,\beta,\gamma}\, D_-^{\gamma}$$

COROLLARY 5.5. Let $\alpha,\beta,\gamma > -1$ and $(n,k) \in N$. Then

$$(5.14) \qquad D_2^{\alpha,\beta,\gamma}\, p_{n,k}^{\alpha,\beta,\gamma} = k(k+\alpha+\beta+1)(n+\gamma+\tfrac{1}{2})(n+\alpha+\beta+\gamma+\tfrac{3}{2})\; p_{n,k}^{\alpha,\beta,\gamma} .$$

We conclude this section by expressing the operator D_2 in terms of s and t. It follows that

$$(5.15) \qquad D_2^{\alpha,\beta,\gamma} = \frac{1}{4w_1(s,t)w_2(s,t)} [\frac{\partial}{\partial s} (w_2(s,t)\frac{\partial}{\partial t} + \frac{\partial}{\partial t}(w_2(s,t) \frac{\partial}{\partial s}]$$

$$\cdot \frac{w_1(s,t)}{w_2(s,t)} [\frac{\partial}{\partial s} (w_2(s,t) \frac{\partial}{\partial t}) + \frac{\partial}{\partial t} (w_2(s,t) \frac{\partial}{\partial s}] ,$$

where

$$w_1(s,t) = (\sin \tfrac{1}{2}s \sin \tfrac{1}{2}t)^{2\alpha+1} (\cos \tfrac{1}{2}s \cos \tfrac{1}{2}t)^{2\beta+1} ,$$

$$w_2(s,t) = (\sin \tfrac{1}{2}(t+s) \sin \tfrac{1}{2}(t-s))^{2\gamma+1} .$$

Observe that

$$D_2 = \frac{\partial^4}{\partial s^2 \partial t^2} + \text{ terms of order less than four.}$$

6. The algebra of differential operators generated by D_1 and D_2

In the previous sections we have obtained two differential operators D_1 and D_2 for which the polynomials $p_{n,k}^{\alpha,\beta,\gamma}$ are eigenfunctions. It is a natural problem to find the general form of the differential operators which admit the polynomials $p_{n,k}^{\alpha,\beta,\gamma}$ as eigenfunctions. This question will be considered in the present section.

From now on it is supposed that the parameters α,β,γ are fixed and larger than -1. We shall write $p_{n,k}$ instead of $p_{n,k}^{\alpha,\beta,\gamma}$.

LEMMA 6.1. Let the differential operator

$$D = \sum_{(m,l)\le(n,k)} a_{m,l}(u,v)\left(\frac{\partial}{\partial u}\right)^{m-l}\left(\frac{\partial}{\partial v}\right)^{l} \tag{6.1}$$

admit the polynomials $p_{m,l}(u,v)$ as eigenfunctions. Then D is completely determined by the eigenvalues of $p_{m,l}$, $(m,l) \le (n,k)$, and $a_{m,l}(u,v)$ is a polynomial in u and v of degree not larger than (m,l).

Proof. Let $D\, p_{m,l} = \lambda_{m,l}\, p_{m,l}$. Then

$$a_{0,0}(u,v) = \lambda_{0,0}\ ,$$

$$(m-l)!\,l!\ a_{m,l}(u,v) = \lambda_{m,l}\, p_{m,l}(u,v)$$

$$- \sum_{(\mu,\lambda)<(m,l)} a_{\mu,\lambda}(u,v)\left(\frac{\partial}{\partial u}\right)^{\mu-\lambda}\left(\frac{\partial}{\partial v}\right)^{\lambda} p_{m,l}(u,v)\ .$$

This recurrence relation completely determines the coefficients $a_{m,l}(u,v)$. It follows by complete induction with respect to $(m,l) \in N$ that $a_{m,l}(u,v)$ is a polynomial in u and v of degree not exceeding (m,l). Q.e.d.

COROLLARY 6.2. Let D and D' be two differential operators which admit the polynomials $p_{n,k}(u,v)$ as eigenfunctions. Then $DD' = D'D$.

LEMMA 6.3. Let the differential operator D be given by (6.1), where the coefficients $a_{m,l}(u,v)$ are polynomials in u and v. Let D be transformed in terms of s and t, $u = \cos s + \cos t$, $v = \cos s \cos t$, and write

$$D = \sum_{i+j\leq n} b_{ij}(s,t)\left(\frac{\partial}{\partial s}\right)^i\left(\frac{\partial}{\partial t}\right)^j , \qquad (s,t) \in R.$$

Then the functions b_{ij} have unique extensions to one-valued analytic functions, regular for all complex s and t except possibly on the lines $\sin s = 0$, $\sin t = 0$, $\sin \frac{1}{2}(s+t) = 0$ and $\sin \frac{1}{2}(s-t) = 0$. The operator D is invariant under the reflections
$(s,t) \to (-s,t)$, $(s,t) \to (s,-t)$ and $(s,t) \to (t,s)$.

Proof. The Lemma follows by

$$a_{m,l}(u,v)\left(\frac{\partial}{\partial u}\right)^{m-l}\left(\frac{\partial}{\partial v}\right)^l = a_{m,l}(\cos s + \cos t, \cos s \cos t)$$

$$\cdot \left[\frac{\cos s}{\sin s(\cos t - \cos s)}\frac{\partial}{\partial s} + \frac{\cos t}{\sin t(\cos s - \cos t)}\frac{\partial}{\partial t}\right]^{m-l}$$

$$\cdot \left[\frac{1}{\sin s(\cos s - \cos t)}\frac{\partial}{\partial s} + \frac{1}{\sin t(\cos t - \cos s)}\frac{\partial}{\partial t}\right]^l .$$

Q.e.d.

LEMMA 6.4. Let D be a differential operator of order m which admits the polynomials $p_{n,k}(u,v)$ as eigenfunctions. Then in terms of s and t the operator D equals

$$(6.2) \qquad D = \sum_{l=0}^{m} c_l\left(\frac{\partial}{\partial s}\right)^{m-l}\left(\frac{\partial}{\partial t}\right)^l + \text{terms of lower order with constant}$$

coefficients c_l.

Proof. By Corollary 6.2 D commutes with D_1 and D_2. We have

$$D_1 = \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2} + \text{terms of lower order} ,$$

$$D_2 = \frac{\partial^4}{\partial s^2 \partial t^2} + \text{terms of lower order} .$$

Let D be given by (6.2), where $c_l = c_l(s,t)$ may depend on s,t. The vanishing of the terms of order m+1 in the operator $DD_1 - D_1D$ implies that

$$\frac{\partial}{\partial s} c_l(s,t) + \frac{\partial}{\partial t} c_{l-1}(s,t) = 0 , \quad 1 \le l \le m ,$$

$$\frac{\partial}{\partial s} c_0(s,t) = 0 ,$$

$$\frac{\partial}{\partial t} c_m(s,t) = 0 .$$

The vanishing of the terms of order m+3 in the operator $DD_2 - D_2D$ implies that

$$\frac{\partial}{\partial t} c_l(s,t) + \frac{\partial}{\partial s} c_{l-1}(s,t) = 0 , \quad 1 \le l \le m ,$$

$$\frac{\partial}{\partial t} c_0(s,t) = 0 ,$$

$$\frac{\partial}{\partial s} c_m(s,t) = 0 .$$

It follows that $\frac{\partial}{\partial s} c_l(s,t) = 0 = \frac{\partial}{\partial t} c_l(s,t), \quad 0 \le l \le m$.
Hence the coefficients c_l are constants. Q.e.d.

Clearly, each differential operator D which is a polynomial in the commuting differential operators D_1 and D_2 admits the polynomials $p_{n,k}$ as eigenfunctions. We can now prove the converse statement.

THEOREM 6.5. Let D be a differential operator which admits the polynomials $p_{n,k}$ as eigenfunctions. Then D can be expressed in one and only one way as a polynomial in the operators D_1 and D_2.

Proof. Suppose that there exist differential operators admitting the polynomials $p_{n,k}$ as eigenfunctions which can not be expressed as polynomials in D_1 and D_2. Let D be such an operator of minimal order m. The operator D can be written in the form (6.2). Since D satisfies the symmetries mentioned in Lemma 6.3 the m^{th} order part of D must be a symmetric polynomial $(\frac{\partial}{\partial s})^2$ and $(\frac{\partial}{\partial t})^2$. Then there exists a polynomial Q in two variables such that the operator $Q((\frac{\partial}{\partial s})^2 + (\frac{\partial}{\partial t})^2, (\frac{\partial}{\partial s})^2(\frac{\partial}{\partial t})^2)$ is equal to the m^{th} order part of D. Hence $D - Q(D_1,D_2)$ is a differential operator of order less than m which admits the polynomials $p_{n,k}(u,v)$ as eigenfunctions, so $D - Q(D_1,D_2)$ is a polynomial in D_1 and D_2 and, therefore, D is a polynomial in D_1 and D_2. This is a contradiction. It follows that each operator D admitting the polynomials $p_{n,k}$ as eigenfunctions is equal to a certain polynomial $Q(D_1,D_2)$.

Next we have to prove that the polynomial Q is uniquely determined by D. Let $Q(D_1,D_2) = 0$ and suppose that Q is not the zero polynomial. Then

$$Q(u,v) = \sum_{m=0}^{n} \sum_{l=0}^{[\frac{1}{2}m]} c_{m,l}\, u^{m-2l} v^{l} ,$$

where not all the coefficients $c_{n,l}$, $l = 0,1,\ldots,[\frac{1}{2}n]$ are zero.
It follows that

$$Q(D_1,D_2) = \sum_{m=0}^{n} \sum_{l=0}^{[\frac{1}{2}m]} c_{m,l} \left(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2} + \text{first order operator}\right)^{m-2l}$$

$$\cdot \left(\frac{\partial^2}{\partial s^2}\frac{\partial^2}{\partial t^2} + \text{third order operator}\right)^{l}$$

$$= \sum_{l=0}^{[\frac{1}{2}n]} c_{n,l} \left(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}\right)^{n-2l} \left(\frac{\partial^2}{\partial s^2}\frac{\partial^2}{\partial t^2}\right)^{l}$$

$$+ \text{operator of order less than } 2n .$$

Since $Q(D_1,D_2) = 0$ the n^{th} order part of $Q(D_1,D_2)$ vanishes, hence $c_{n,l} = 0$ for $l = 0,1,\ldots,[\frac{1}{2}n]$. This is a contradiction. Hence Q is the zero polynomial if $Q(D_1,D_2)$ is the zero operator. Q.e.d.

References

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