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A limit case of a Volterra-Lotka system

by

H.A. Lauwerier

The periodic solution of a Volterra-Lotka system is studied for large deviation from the stationary equilibrium. An asymptotic formula for the period is derived.

KEY WORDS & PHRASES: *Volterra-Lotka system, Asymptotic approximation for the period.*

1. INTRODUCTION

This paper is a companion to the paper of GRASMAN & VELING [1] on the same subject.

We consider again a Volterra-Lotka system in the following dimensionless form

$$(1)\qquad \begin{cases} \dot{x} = \varepsilon_1 x(1-y), \\ \dot{y} = -\varepsilon_2 y(1-x), \end{cases}$$

where the variables x,y and the parameters $\varepsilon_1, \varepsilon_2$ are positive.

The solutions are periodic functions. In the x,y-plane they are represented by a one parameter family of nesting closed curves. By a proper choice of the unit of time we may take either $\varepsilon_1 = 1$ or $\varepsilon_2 = 1$. However, for reason of symmetry we keep both parameters. In the paper cited above the single independent parameter $\varepsilon = \varepsilon_1/\varepsilon_2$ is taken. The properties of the system are then studied for small values of ε. Here we study what happens when ε_1 and ε_2 are of the same order and the deviations of x and y from their equilibrium position (1,1) are large.

In the x,y-plane the curves are given by

$$(2)\qquad \varepsilon_2(x-\log x) + \varepsilon_1(y-\log y) = c.$$

The asymptotic analysis of (1) and (2) will be carried out with respect to c as a large parameter.

If in (2) the variables x and y are scaled down by $x \to cx'$ and $y \to cy'$ we obtain for $c \to \infty$ the limit form

$$(3)\qquad \varepsilon_2 x' + \varepsilon_1 y' = 1.$$

This means that in the x',y'-plane the closed curves degenerate into a triangle formed by the coordinate axes and the line (3) (see fig. 1 and fig. 2).

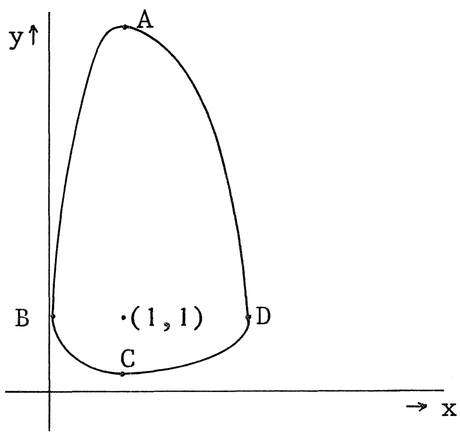

Figure 1

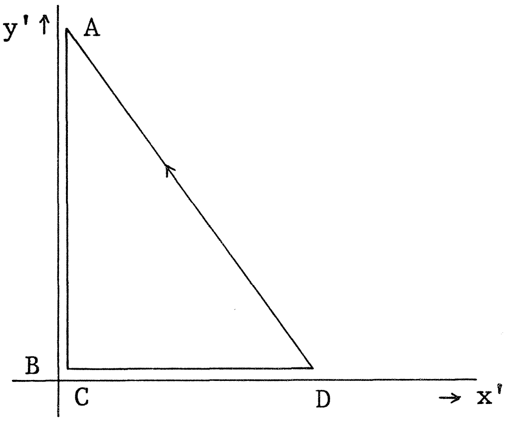

Figure 2

However, rather more detailed asymptotic analysis is needed to introduce the time variable and in particular to estimate the period for large values of c. It turns out that in fig. 1 the phase point (x,y) passes quickly from A (y maximum) to B (x minimum) but that it needs much time to make the quarter turn from B to C (y minimum). The transition from C to D (x maximum) is comparable to that from A to B. Finally the stretch from D to A is very fast. The total period T is found to be

$$T = \frac{c}{\varepsilon_1\varepsilon_2} + \frac{1}{\varepsilon_2}\log\frac{c}{\varepsilon_1} + \frac{1}{\varepsilon_1}\log\frac{c}{\varepsilon_2} + O\left(\frac{\log c}{c}\right). \tag{4}$$

However, the following more accurate expression is also obtained

$$T \approx \frac{c}{\varepsilon_1\varepsilon_2} + \frac{1}{\varepsilon_2}\left(\frac{c}{c-\varepsilon_2} + \frac{\varepsilon_1}{c}\right)\log\frac{c}{\varepsilon_1} + \frac{1}{\varepsilon_1}\left(\frac{c}{c-\varepsilon_1} + \frac{\varepsilon_2}{c}\right)\log\frac{c}{\varepsilon_2}\ . \tag{5}$$

This result may be compared with the well-known result obtained by Volterra for small disturbances of the equilibrium

$$T \approx \frac{2\pi}{\sqrt{\varepsilon_1\varepsilon_2}}\ . \tag{6}$$

2. ASYMPTOTIC ANALYSIS

From (2) the positions y_A and y_C of A and C are determined as the solutions of the transcendental equation

(7) $$y - \log y = \frac{c-\varepsilon_2}{\varepsilon_1} .$$

For $c \to \infty$ this equation can be solved in an iterative way as

$$y_A = \frac{c-\varepsilon_2}{\varepsilon_1} + \log \frac{c-\varepsilon_2}{\varepsilon_1} + \ldots,$$

or

(8) $$y_A = \frac{c}{\varepsilon_1} + \log \frac{c}{\varepsilon_1} - \frac{\varepsilon_2}{\varepsilon_1} + O\left(\frac{\log c}{c}\right) .$$

In a similar way

$$\log y_C = - \frac{c-\varepsilon_2}{\varepsilon_1} + \exp - \frac{c-\varepsilon_2}{\varepsilon_1} + \ldots,$$

or

(9) $$\log y_C = - \frac{c}{\varepsilon_1} + \frac{\varepsilon_2}{\varepsilon_1} + O(c^{-\infty}).$$

The positions x_B and x_D of B and D can easily be derived from (8) and (9) by the interchanging of ε_1 and ε_2.

We assume that the phase point (x,y) passes the points A, B, C, D at the times t_A, t_B, t_C, t_D. We write $t_B - t_A = \Delta_{AB}$ etcetera. The path ABC is determined in an iterative way as follows. As the first approximation we take (see fig. 1) $x = 0$, $\log y = -\varepsilon_2(t-t_B)$. The second approximation is next obtained from the first equation of (1)

$$\frac{d}{dt} \log x = \varepsilon_1 - \varepsilon_1 \exp -\varepsilon_2(t-t_B)$$

which gives

(10) $$\log \frac{x}{x_B} = \varepsilon_1(t-t_B) - \frac{\varepsilon_1}{\varepsilon_2} (1 - \exp -\varepsilon_2(t-t_B)).$$

This approximation appears to be sufficiently accurate. Taking $t = t_A$ and $x = 1$ we have at A

(11) $$\log \frac{1}{x_B} = -\varepsilon_1 \Delta_{AB} - \frac{\varepsilon_1}{\varepsilon_2} + \frac{\varepsilon_1}{\varepsilon_2} \exp \varepsilon_2 \Delta_{AB}.$$

If this is compared with the B-version of (9)

(12) $$\log x_B = -\frac{c}{\varepsilon_2} + \frac{\varepsilon_1}{\varepsilon_2}$$

we find in good approximation

$$\exp \varepsilon_2 \Delta_{AB} = \frac{c}{\varepsilon_1} + \varepsilon_2 \Delta_{AB}.$$

For $c \to \infty$ this can be solved as

(13) $$\Delta_{AB} = \frac{1}{\varepsilon_2} \log \frac{c}{\varepsilon_1} + \frac{\varepsilon_1}{\varepsilon_2} \frac{\log c/\varepsilon_1}{c} + \dots .$$

The expression for Δ_{CD} is obtained by $\varepsilon_1 \leftrightarrow \varepsilon_2$

(14) $$\Delta_{CD} = \frac{1}{\varepsilon_1} \log \frac{c}{\varepsilon_2} + \frac{\varepsilon_2}{\varepsilon_1} \frac{\log c/\varepsilon_2}{c} + \dots .$$

For the stretch BC we find by substituting $t = t_C$ and $x = 1$ in (10)

(15) $$\log \frac{1}{x_B} = \varepsilon_1 \Delta_{BC} - \frac{\varepsilon_1}{\varepsilon_2} + \frac{\varepsilon_1}{\varepsilon_2} \exp -\varepsilon_2 \Delta_{BC}.$$

If this is compared with (12) we find in good approximation

(16) $$\Delta_{BC} = \frac{c}{\varepsilon_1 \varepsilon_2} + O(c^{-\infty}).$$

For the stretch DA which is approximately of the rectilinear form

(17) $$\varepsilon_2 x + \varepsilon_1 y = c$$

we can obtain as approximate integration of the second equation of (1) by using the approximation

(18) $\dot{y} = y(c-\varepsilon_2-\varepsilon_1 y)$.

The solution is

(19) $$\log \frac{y}{c-\varepsilon_2-\varepsilon_1 y} = (c-\varepsilon_2)t + \text{constant}.$$

This approximation cannot be expected to be good at A where y is stationary. Therefore we impose the starting condition at D and determine the time needed to reach the mid-position $E(\frac{1}{2}c/\varepsilon_2, \frac{1}{2}c/\varepsilon_1)$. Then from (19) we obtain

(20) $$(c-\varepsilon_2)\Delta_{DE} = \log \frac{\frac{1}{2}c}{\varepsilon_1(\frac{1}{2}c-\varepsilon_2)} - \log \frac{1}{c-\varepsilon_1-\varepsilon_2} \approx \log \frac{c}{\varepsilon_1} + \frac{\varepsilon_2-\varepsilon_1}{c},$$

and a similar expression for Δ_{EA}.

Together we may use the approximation

(21) $$\Delta_{DA} \approx \frac{1}{c-\varepsilon_2} \log \frac{c}{\varepsilon_1} + \frac{1}{c-\varepsilon_1} \log \frac{c}{\varepsilon_2}.$$

The result (5) for the full period is merely the sum of the expressions at the right-hand sides of (13), (14), (16) and (21). The less accurate result (4) is obtained by keeping only the terms of order c, log c and a constant. We observe that the main contribution to T is due to the corner BC.

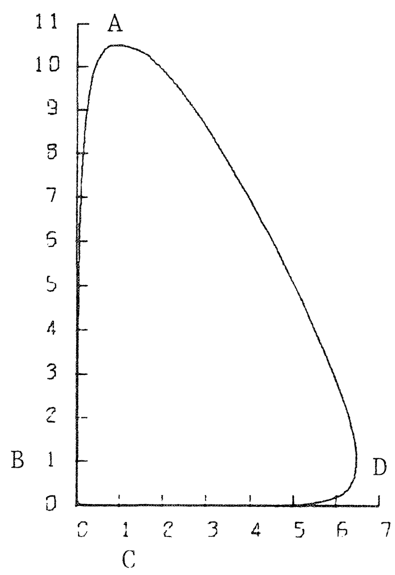

Figure 3

3. NUMERICAL ILLUSTRATION

We consider the following typical case

$$\varepsilon_1 = 0.5, \qquad \varepsilon_2 = 1.0, \qquad x_B = 0.01.$$

From (2) we obtain c = 5.115. The coordinates of the other corner points are determined by (see fig. 3)

$$\ln y_C = -8.230, \qquad x_D = 6.48, \qquad y_A = 10.59.$$

The time behaviour of x and y is sketched in figure 4.

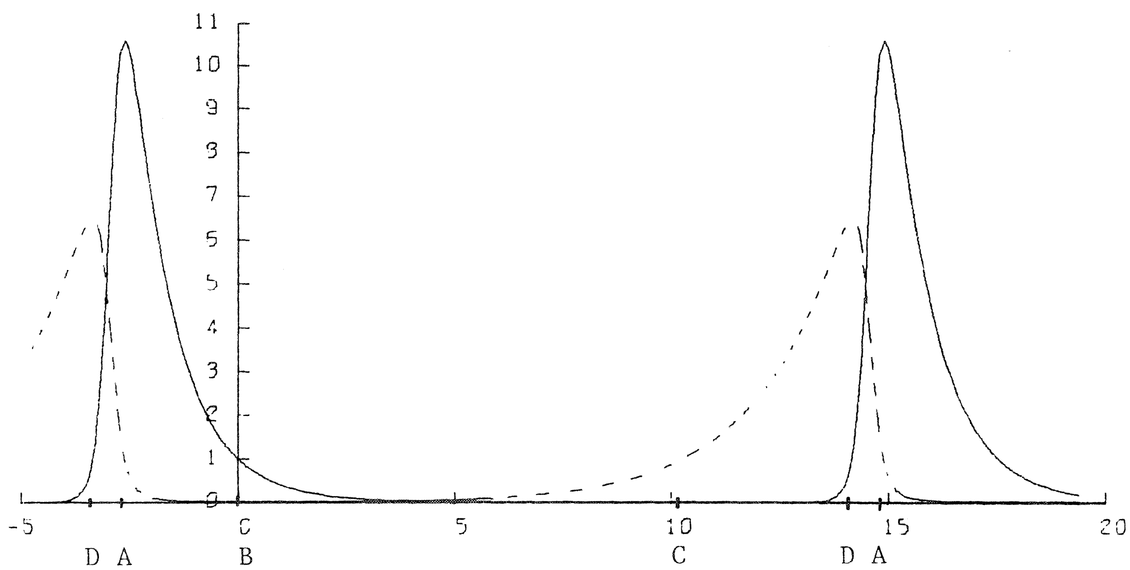

Figure 4

The correct solution of (11) is $\Delta_{AB} = 2.55$. Our approximate formula (13) gives 2.56. The correct solution of $\Delta_{CD} = 3.91$ and the approximation (14) gives 3.90.

Together we find

$$T = 2.56\ (AB) + 10.23\ (BC) + 3.90\ (CA) + 0.92\ (DA) = 17.61.$$

If this result is compared to the corresponding figures in the paper by GRASMAN & VELING it should be born in mind that the time variable contains the scaling factor ε_1. Thus our value $\varepsilon_1 T = 8.80$ corresponds with $\varepsilon_1 T_G = 8.64$ according to formula (11) in the above cited paper. The true numerical value $\varepsilon_1 T_N$ obtained from a computer program is 8.76. More numerical data are collected below. The author wishes to thank G.J.M. LAAN for his assistance in carrying out the computations.

	x_B	c	$\varepsilon_1 T$	$\varepsilon_1 T_G$	$\varepsilon_1 T_N$
a. $\varepsilon_1 = 0.1$, $\varepsilon_2 = 1$	0.01	4.72	7.13	7.10	7.11
	0.05	3.15	5.21	5.17	5.17
	0.1	2.50	4.37	4.30	4.31
b. $\varepsilon_1 = 0.5$, $\varepsilon_2 = 1$	0.01	5.12	8.80	8.64	8.76
	0.05	3.55	6.88	6.70	6.87
	0.1	2.90	6.05	5.86	6.09
c. $\varepsilon_1 = \varepsilon_2 = 1$	0.01	5.62	10.43		10.41
	0.05	4.05	8.45		8.56
	0.1	3.40	7.59		7.82

REFERENCES

[1] GRASMAN, J. & E. VELING, *An asymptotic formula for the period of a Volterra-Lotka system*, Math. Biosc. 18 (1973), 185-189.