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A SECOND REPORT ON FUNCTIONS FROM THE STATISTICAL THEORY OF RESIDUAL CURRENTS IN TIDAL AREAS

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A second report on functions from the statistical theory of residual currents in tidal areas

by

N.M. Temme

ABSTRACT

Asymptotic approximations are given for functions occurring in a mathematical model of vorticity production by tidal currents. In an earlier report functions of the same type were considered.

KEY WORDS & PHRASES: *asymptotic expansions of Laplace type integrals, Bessel functions, tidal currents.*

1. INTRODUCTION

In an earlier report [2] we considered integrals occurring in a mathematical model of residual circulations by tidal currents. In this report we consider a simpler model, which enables us to obtain more information about some quantities than was possible in the previous case. A central point there was a first order differential equation*)

$$\frac{d\eta(\vec{k},t)}{dt} - [ik_1 \sin t - b(k)]\eta(\vec{k},t) = a(\vec{k}) \sin t,$$

where $\vec{k} = (k_1,k_2)$ is the wave number in a Fourier analysis, $k = (k_1^2+k_2^2)^{1/2}$. The above equation arose in a model for a one-directional tidal current. The simplification concerns the assumption that the current velocity vector rotates circularly. It yields the equation (with $k_1 = k\cos\theta$, $k_2 = k\sin\theta$)

$$(1.1)\qquad \frac{d\eta(\vec{k},t)}{dt} - [ik \sin(t+\theta) - b(k)]\eta(\vec{k},t) =$$

$$= a(\vec{k})[\tau_1 \sin(t+\theta) + \tau_2 \cos(t+\theta)].$$

The function b is given by

$$(1.2)\qquad b(k) = \tau_2 + \tau_3 k^2,$$

τ_1, τ_2 and τ_3 are non-negative constants and $a(\vec{k})$ is related to a stochastic forcing field, the statistics of which are prescribed.

The nontransient solution of the differential equation is written as a Fourier series

$$(1.3)\qquad \eta(\vec{k},t) = \sum_{\ell=-\infty}^{\infty} c_\ell(\vec{k}) e^{i\ell(t+\theta)}$$

with

$$(1.4)\qquad c_\ell(\vec{k}) = i(-i)^\ell a(\vec{k}) \sum_{n=-\infty}^{\infty} \frac{J_{n+\ell}(k)}{-b+in} \{\tau_1 i \frac{n}{k} J_n(k) + \tau_2 J_n'(k)\},$$

where $J_n(k)$ is the ordinary Bessel function. In order to describe the $c_\ell(\vec{k})$

*) in the above mentioned report the sign of b(k) was in error

it is convenient to introduce the functions

$$(1.5)\qquad R_m(\alpha,k) = \sum_{n=-\infty}^{\infty} \frac{J_n(k)J_{n+m}(k)}{n+\alpha}, \qquad m \in \mathbb{Z}, \quad \alpha \notin \mathbb{Z}.$$

With respect to the parameters τ_i two cases are investigated, which will be treated in the following sections. First we give some properties of the function R_m defined above.

In the previous report we used $R_0(\alpha,t)$. Explicitly we have (for a proof see [2])

$$(1.6)\qquad R_0(\alpha,k) = \frac{\pi}{\sin \pi\alpha} J_\alpha(k)J_{-\alpha}(k).$$

By using the well-known recurrence relations

$$(1.7)\qquad J_{n\pm 1}(k) = \frac{n}{k} J_n(k) \mp J'_n(k)$$

we obtain

$$(1.8)\qquad R_{\pm 1}(\alpha,k) = \frac{1}{k}\,[1 - \alpha R_0(\alpha,k)] \mp \frac{1}{2}\frac{\partial}{\partial k} R_0(\alpha,k).$$

For general m we have

$$(1.9)\qquad R_{-m}(\alpha,k) = \sum_{n=-\infty}^{\infty} \frac{J_{n-m}(k)J_n(k)}{n+\alpha} = \sum_{n=-\infty}^{\infty} \frac{J_n(k)J_{n+m}(k)}{n+\alpha+m} = R_m(\alpha+m,k).$$

Otherwise we have by using $J_{-n}(k) = (-1)^n J_n(k)$

$$(1.10)\qquad R_{-m}(\alpha,k) = \sum_{n=-\infty}^{\infty} \frac{J_{n-m}(k)J_n(k)}{n+\alpha} = \sum_{n=-\infty}^{\infty} \frac{J_{-n-m}(k)J_{-n}(k)}{-n+\alpha} =$$

$$= (-1)^{m+1} R_m(-\alpha,k),$$

and finally, by using

$$(1.11)\qquad J_{n+1}(k) + J_{n-1}(k) = \frac{2n}{k} J_n(k),$$

$$R_{m+1}(\alpha,k) = \frac{2}{k}\sum_{n=-\infty}^{\infty}\frac{(n+m)J_{n+m}(k)J_n(k)}{n+\alpha} - \sum_{n=-\infty}^{\infty}\frac{J_{n+m-1}(k)J_n(k)}{n+\alpha} =$$

$$= -R_{m-1}(\alpha,k) + \frac{2(m-\alpha)}{k}R_m(\alpha,k) + \frac{2}{k}T_m(\alpha,k),$$

where

$$T_m(\alpha,k) = \sum_{n=-\infty}^{\infty} J_{n+m}(k)J_n(k).$$

It is known that $T_0(\alpha,k) = 1$. For general $m \in \mathbb{Z}$ we have

$$(1.12) \qquad T_m(\alpha,k) = \sum_{n=-\infty}^{\infty} J_n(k)\,\frac{1}{2\pi}\int_0^{2\pi} e^{ik\sin\theta - i(n+m)\theta}\,d\theta =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=-\infty}^{\infty} J_n(k)e^{-in\theta}e^{ik\sin\theta - im\theta}\,d\theta =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-im\theta}\,d\theta = \delta_{0,m}$$

where Kronecker's symbol $\delta_{\ell,m}$ is used, defined as

$$(1.13) \qquad \delta_{\ell,m} = \begin{cases} 0 & \ell \neq m \\ 1 & \ell = m \end{cases}.$$

Consequently, we have the recursion for $m \in \mathbb{Z}$

$$(1.14) \qquad R_{m+1}(\alpha,k) + R_{m-1}(\alpha,k) = \frac{2(m-\alpha)}{k}R_m(\alpha,k) + \frac{2}{k}\delta_{0,m}.$$

Remark that (1.8) leads to (1.14) with $m = 0$. This recursion may be used for the numerical evaluation of R_m when the two consecutive values are known.

It is interesting to note that (1.14) resembles the recursion relation

$$y_{m+1} + y_{m-1} = \frac{2(m-\alpha)}{x}\,y_m$$

with solutions $y_m = J_{m-\alpha}(x)$ and $y_m = Y_{m-\alpha}(x)$. This recursion is homogeneous, whereas (1.14) is inhomogeneous.

2. THE CASE $\tau_1 = 1$, $\tau_3 = 0$, τ_2 VARIABLE (≥ 0)

In this case we have from (1.2) $b = \tau_2$. We use for convenience b instead of τ_2 in the notation. We are interested in the two parts of $c_\ell(k)$ given in (1.4), i.e., we define

$$c_\ell^{(1)}(\vec{k}) = i(-i)^\ell a(\vec{k}) \sum_{n=-\infty}^{\infty} \frac{inJ_{n+\ell}(k)J_n(k)}{k(-b+in)} ,$$

(2.1)

$$c_\ell^{(2)}(\vec{k}) = -ib(-i)^\ell a(\vec{k}) \sum_{n=-\infty}^{\infty} \frac{J_{n+\ell}(k)J_n'(k)}{-b+in}$$

Especially we want to know the behaviour of $c_\ell^{(1)} c_\ell^{(1)*}$ and $c_\ell^{(2)} c_\ell^{(2)*}$ with respect to the parameter b. The density function $a(\vec{k})$ does not depend on b and will not be considered here. In fact we discuss the functions

$$G_\ell^{(1)}(k,b) = \left| \sum_{n=-\infty}^{\infty} \frac{nJ_{n+\ell}(k)J_n(k)}{k(-b+in)} \right|^2$$

(2.2)

$$G_\ell^{(2)}(k,b) = b^2 \left| \sum_{n=-\infty}^{\infty} \frac{J_{n+\ell}(k)J_n'(k)}{-b+in} \right|^2$$

and we give the asymptotic behaviour of $G_\ell^{(1)}(k,b)$ and $G_\ell^{(2)}(k,b)$ for $b \to 0$ and for $b \to \infty$ for fixed values of k and ℓ ($\ell = 0,1,2,3$).

The functions (2.1) give insight in the difference in behaviour of frictional and rotational forcing of vorticity, as the former forcing mechanism is also the major dissipative agency.

2.1. Summary of the results

For $b \to 0$ we have

$$G_\ell^{(1)}(k) = \frac{1}{k^2}\{\delta_{\ell,0} - J_0(k)J_\ell(k)\}^2 + O(b),$$

(2.3)

$$G_\ell^{(2)}(k) = J_1^2(k)J_\ell^2(k) + O(b).$$

and for $b \to \infty$ the behaviour is as follows

$$G_0^{(1)}(k) = k^2/(4b^4) + O(b^{-6})$$

(2.4)

$$G_\ell^{(1)}(k) = k^{2(\ell-1)}4^{-\ell}b^{-2\ell} + O(b^{-2\ell-2}), \qquad \ell = 1,2,3.$$

$$G_0^{(2)}(k) = k^2b^{-4} + O(b^{-6})$$

(2.5)

$$G_\ell^{(2)}(k) = k^{2(\ell-1)}4^{-\ell}b^{-2(\ell-1)} + O(b^{-2\ell}), \qquad \ell = 1,2,3.$$

2.2. <u>A further analysis</u>

From (1.5), (1.7), (1.11) and (2.2) it follows that

$$G_\ell^{(1)}(k) = \frac{1}{k^2}|\delta_{\ell,0} - ib\, R_\ell(ib,k)|^2,$$

(2.6)

$$G_\ell^{(2)}(k) = \frac{1}{4}b^2|R_{\ell+1}(1+ib,k) - R_{\ell-1}(-1+ib,k)|^2.$$

For $b \to 0$ we have

$$ib\, R_\ell(ib,k) = J_0(k)J_\ell(k) + O(b),$$

$$b\, R_{\ell+1}(1+ib,k) = -J_1(k)J_\ell(k) + O(b),$$

$$b\, R_{\ell-1}(-1+ib,k) = J_1(k)J_\ell(k) + O(b).$$

From these results (2.3) easily follows.

Next we consider the case $b \to \infty$. This is more complicate.

We have

$$\text{(2.7)} \qquad ib\, R_\ell(-ib,k) = -\sum_{n=-\infty}^{\infty} J_n(k)J_{n+\ell}(k)\,\frac{1}{1-n/ib}\,.$$

From

$$\frac{1}{1-n/ib} = \sum_{j=0}^{N} (n/ib)^j + \frac{(n/ib)^{N+1}}{1-n/ib}$$

one obtains

$$ib\, R_\ell(-ib,k) = -\sum_{j=0}^{N} \frac{A_j(k;\ell)}{(ib)^j} + B_N$$

with

$$\text{(2.8)} \qquad A_j(k;\ell) = \sum_{n=-\infty}^{\infty} n^j J_n(k)J_{n+\ell}(k)$$

and

$$B_N = -\sum_{n=-\infty}^{\infty} J_n(k)J_{n+\ell}(k)\,\frac{(n/ib)^{N+1}}{1-n/ib}$$

Since

$$\left|\frac{1}{1-n/ib}\right| \le 1, \qquad n = 0,\pm1,\pm2,\ldots, \quad b \in \mathbb{R}$$

we have $B_N = \mathcal{O}(b^{-N-1})$, $b \to \infty$. Hence we have the asymptotic expansion

$$\text{(2.9)} \qquad ib\, R_\ell(-ib,k) \sim -\sum_{j=0}^{\infty} \frac{A_j(k;\ell)}{(ib)^j}\,, \qquad b \to \infty.$$

It remains to compute the quantities $A_j(k\ \ell)$ defined in (2.8).

From (1.12) we have

$$A_0(k;\ell) = \delta_{\ell,0}$$

$$A_1(k;\ell) = \frac{k}{2}\sum_{n=-\infty}^{\infty} J_{n+\ell}(k)[J_{n-1}(k) + J_{n+1}(k)] = \frac{k}{2}(\delta_{\ell+1,0}+\delta_{\ell-1,0}),$$

where the recursion (1.11) is used. In a similar way we obtain

$$A_2(k;\ell) = \frac{k}{2}\{\sum J_{n+\ell}(k)n\, J_{n-1}(k) + \sum J_{n+\ell}(k)n\, J_{n+1}(k)\} =$$

$$= \frac{k}{2}\{\sum J_{n+\ell}(k)(n-1)J_{n-1}(k) + \sum J_{n+\ell}(k)(n+1)J_{n+1}(k)$$

$$+ \sum J_{n+\ell}(k)J_{n-1}(k) - \sum J_{n+\ell}(k)J_{n+1}(k)\}$$

$$= \frac{k}{2}\{A_1(k;\ell+1) + A_1(k;\ell-1) + A_0(k;\ell+1) - A_0(k;\ell-1)\}$$

$$= \frac{k^2}{4}\{\delta_{\ell+2,0} + 2\delta_{\ell,0} + \delta_{\ell-2,0}\} + \frac{k}{2}\{\delta_{\ell+1,0} - \delta_{\ell-2,0}\}.$$

Generalizing this method we obtain the recursion

$$A_{j+1}(k;\ell) = \frac{k}{2}\sum J_{n+\ell}(k)n^jJ_{n+1}(k) + \sum J_{n+\ell}(k)n^jJ_{n-1}(k)$$

$$= \frac{k}{2}\{\sum J_{n+\ell}(k)\; J_{n+1}(k) \sum_{m=0}^{j}(-1)^{j-m}(n+1)^m\binom{j}{m} +$$

$$+ \sum J_{n+\ell}(k)\; J_{n-1}(k) \sum_{m=0}^{j}(n-1)^m\binom{j}{m}\}$$

$$= \frac{k}{2}\sum_{m=0}^{j}\binom{j}{m}\{A_m(k;\ell+1) + (-1)^{j-m}A_m(k;\ell-1)\}.$$

For the first order terms in the asymptotic expansion of the functions defined in (2.6) we need

$$A_1(k;1) = k/2$$

$$A_2(k;0) = k^2/2, \qquad A_2(k;2) = k^2/4,$$

$$A_3(k;3) = k^3/8,$$

$$A_1(k;0) = A_1(k;2) = A_1(k;3) = A_2(k;3) = 0.$$

For $G_\ell^{(1)}(k)$ we obtain

$$G_\ell^{(1)}(k) \sim \frac{1}{k^2} \left| \sum_{j=1}^{\infty} \frac{A_j(k;\ell)}{(-ib)^j} \right|^2 ,$$

and substituting the above values of $A_j(k;\ell)$ we obtain (2.4). For $G_\ell^{(2)}(k)$ we have for $b \to \infty$

$$G_\ell^{(2)}(k) \sim \frac{1}{4} b^2 \left| \frac{1}{1+ib} \sum_{j=0}^{\infty} \frac{A_j(k;\ell+1)}{(-1-ib)^j} + \frac{1}{1-ib} \sum_{j=0}^{\infty} \frac{A_j(k;\ell-1)}{(1-ib)^j} \right|^2$$

from which (2.5) easily follows.

3. THE CASE $\tau_1 = \tau_2 = 1$, τ_3 VARIABLE (≥ 0)

In this case we are interested in the integrals

$$(3.1) \qquad \int_0^\infty \int_0^{2\pi} k^\alpha < c_m(\vec{k}) c_\ell(\vec{k}) > dk d\theta$$

where < > represents an ensemble average over the stochastic function $a(\vec{k})$ contained in $c_\ell(\vec{k})$. For more details on these points we refer to Zimmerman's investigations in [3,4]. In the numerical treatment several density functions $a(\vec{k})$ where considered. Here we give the formulas for the case of a Gaussian distribution.

The integration with respect to θ in (3.1) is trivial due to the circular symmetry and due to the assumed isotropic statistics of $a(\vec{k})$. As a consequence, the quantities (3.1) are zero for $\ell \neq m$. For $\alpha = 1$, $\ell = m = 0$, (3.1) is the residual enstrophy (also considered in [2]), for $\ell = m = 1$ it is the tidal enstrophy (first harmonic), for $\ell = m = 2$ it is the tidal enstrophy (second harmonic), etc.

The integrand of (3.1) is expressed in terms of the function

$$(3.2) \qquad G_\ell(k) = \phi_\ell(k) \phi_\ell^*(k)$$

where * means the complex conjugate and

$$(3.3) \qquad \phi_\ell(k) = \frac{i^\ell}{\sqrt{2}} [e^{i\pi/4} R_{1-\ell}(1-ib,k) + e^{-i\pi/4} R_{-1-\ell}(-1-ib,k)].$$

By specifying $a(\vec{k})$ in (1.4) we write (3.1) as (with different α)

$$H_\ell^{(\alpha)}(\lambda) = \int_0^\infty k^\alpha e^{-k^2\lambda^{-2}} G_\ell(k)dk. \tag{3.4}$$

Then the enstrophy and the energy are given by

$$g_\ell(\lambda) = \lambda^{-6}H_\ell^{(5)}(\lambda),\ f_\ell(\lambda) = 2\lambda^{-4}H_\ell^{(3)}(\lambda) \tag{3.5}$$

Also the following functions are considered; they all have a physical interpretation

$$\begin{aligned} h_\ell(\lambda) &= \lambda H_\ell^{(2)}(\lambda)/H_\ell^{(3)}(\lambda), \\ j_\ell(\lambda) &= [\tfrac{1}{2}\, f_\ell(\lambda)/g_\ell(\lambda)]^{\frac{1}{2}}, \\ d_\ell(\lambda) &= h_\ell(\lambda) - j_\ell(\lambda). \end{aligned} \tag{3.6}$$

These functions are computed for $\ell = 0,1,2,3$ and $\tau_3 = 0, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}$, and $\lambda \in [10^{-2}, 10^{2}]$. The function $G_\ell(k)$ is computed for the same ℓ and τ_3 values and for $k \in [10^{-2}, 10^{2}]$.

In the following subsections we give the asymptotic behaviour of $G_\ell(k)$ for $k \to 0$ and $k \to \infty$ (§3.1) and of $H_\ell^{(\alpha)}(\lambda)$ for $\lambda \to 0$ and $\lambda \to \infty$ (§3.2) for the appropriate values of α and ℓ.

3.1 Asymptotic behaviour of $G_\ell(k)$

3.1.1 Results for $k \to 0$

From (1.6) it follows that

$$R_0(\alpha,k) = \frac{1}{\alpha}\left(1 - \frac{k^2}{2(1-\alpha^2)} + \frac{3k^4}{8(1-\alpha^2)(4-\alpha^2)} + \ldots\right).$$

Hence, by using (1.8), we have

$$R_1(\alpha,k) = \frac{k}{2\alpha(1-\alpha)} - \frac{3k^3}{8\alpha(1-\alpha^2)(2-\alpha)} + \ldots$$

and from the recursion (1.14) or directly from (1.5) we obtain

$$R_2(\alpha,k) = \frac{k^2}{4\alpha(1-\alpha)(2-\alpha)} + \ldots$$

$$R_3(\alpha,k) = \mathcal{O}(k^3), \quad R_m(\alpha,k) = \mathcal{O}(k^m), \qquad k \to 0.$$

With (3.3) we have for $k \to 0$

$$\phi_0(k) = \frac{k}{8b(b^2+1)}[4(b+1) - 3(2+b)k^2 + \mathcal{O}(k^4)] =$$

$$= \frac{k}{2}[1+\mathcal{O}(k^2)]$$

$$\phi_1(k) = \tfrac{1}{2}(i-1)/(1-ib) + \mathcal{O}(k^2) \tag{3.7}$$

$$\phi_2(k) = \frac{k}{4}(1+i)/[(1-ib)(2-ib)] + \mathcal{O}(k^3)$$

$$\phi_3(k) = \frac{1}{8}(1-i)k^2/[(1-ib)(2-ib)(3-ib)]$$

From these results it follows that the asymptotic behaviour of $G_\ell(k)$ for $k \to 0$ is given by

$$\begin{aligned} G_0(k) &= \frac{k^2}{4} + \mathcal{O}(k^4), \\ G_1(k) &= \frac{1}{4} + \mathcal{O}(k^2), \\ G_2(k) &= \frac{k^2}{80} + \mathcal{O}(k^4), \\ G_3(k) &= \frac{k^4}{3200} + \mathcal{O}(k^6). \end{aligned} \tag{3.8}$$

Summarizing we denote these relations by

$$G_\ell(k) = \gamma_\ell k^{\beta_\ell} + \mathcal{O}(k^{\beta_\ell+2}), \qquad k \to 0 \tag{3.9}$$

where γ_ℓ and β_ℓ are readily obtained from (3.8).

3.1.2 Results for $k \to \infty$.

It is necessary to distinguish between the cases $\tau_3 > 0$ and $\tau_3 = 0$. The first case is simpler than the second one, therefore we begin with it.

3.1.2.1 $\tau_3 > 0$

Recall that $b = 1 + \tau_3 k^2$, hence $b \sim \tau_3 k^2$. The expansions in (3.7) of $\phi_\ell(k)$ are also valid for $k \to \infty$ (with other $\mathcal{O}$-terms, however). Explicitly we have for $k \to \infty$

$$
\begin{aligned}
&\phi_0(k) \sim -3/[8\tau_3^2 k],\\
&\phi_1(k) \sim -\tfrac{1}{2}\,(1+i)/[\tau_3 k^2],\\
&\phi_2(k) \sim -\tfrac{1}{4}\,(1+i)/[\tau_3^2 k^3],\\
&\phi_3(k) \sim -(1+i)/[8\tau_3^3 k^4].
\end{aligned} \tag{3.10}
$$

From (3.2) it follows that for $\tau_3 > 0$ and $k \to \infty$

$$
\begin{aligned}
&G_0(k) \sim 9/[64\tau_3^4 k^2],\\
&G_1(k) \sim 1/[2\tau_3^2 k^4],\\
&G_2(k) \sim 1/[8\tau_3^4 k^6],\\
&G_3(k) \sim 1/[32\tau_3^6 k^8].
\end{aligned} \tag{3.11}
$$

3.1.2.2. $\tau_3 = 0$

Let us first study the case $\ell = 0$. From (3.2), (3.3), (1.2), (1.6) and (1.8) it follows that

$$G_0(k) = [\mathrm{Re}\, R_1(1-i,k) - \mathrm{Im}\, R_1(1-i,k)]^2,$$

(3.12)

$$R_1(\alpha,k) = \frac{1}{k}\left[1 - \frac{\pi\alpha}{\sin\pi\alpha} J_{-\alpha}(k)J_\alpha(k)\right] - \frac{1}{2}\frac{\pi\alpha}{\sin\pi\alpha}\frac{d}{dk}[J_{-\alpha}(k)J_\alpha(k)]$$

The behaviour of $G_0(k)$ for $k \to \infty$ is found by using asymptotic expansions of the Bessel functions. In the previous subsection ($\tau_3 > 0$) the parameter α was depending on k. For instance in (3.3) we have $\alpha = \pm 1 - ib = \pm 1 - i(1+\tau_3 k^2)$. Thus if $\tau_3 > 0$ the order of the Bessel functions also depend on k. Here, in (3.12), we have $\alpha = 1-i$, a constant. This is the main difference between the cases $\tau_3 = 0$ and $\tau_3 > 0$.

From well-known results of the theory of Bessel functions we derive

$$J_{-\alpha}(x)J_\alpha(x) = \frac{1}{\pi x}[(\cos\alpha\pi + \sin 2x) + \mathcal{O}(x^{-1})]$$

(3.13)

$$\frac{d}{dx} J_{-\alpha}(x)J_\alpha(x) = \frac{2\cos 2x}{\pi x} + \mathcal{O}(x^{-2})$$

where α is a fixed parameter. From this it follows that we have using (3.12)

$$G_0(k) \sim \frac{1}{k^2}\left(1 + \frac{\cos 2k}{\sinh\pi}\right)^2, \qquad k \to \infty. \tag{3.14}$$

Comparing this result with the first of (3.11) we see that in the present case $G_0(k)$ has a damped oscillatory behaviour for large k.

For very small values of τ_3 this behaviour also occurs; also in that case, however, $G_0(k)$ ultimately behaves as in (3.10).

From the recursion in (1.14) and from (1.6) and (1.8) it follows that for $m \geq 2$

$$R_m = \mathcal{O}(k^{-1}), \qquad k \to \infty \tag{3.15}$$

and that $G_\ell(k)$ for $\ell = 1,2,\ldots$ behaves as $G_0(k)$: a damped oscillatory behaviour at infinity. More details on this point will not be given here.

3.2 The asymptotic behaviour of $H_\ell^{(\alpha)}(\lambda)$ for $\lambda \to 0$ and $\lambda \to \infty$

3.2.1 Results for $\lambda \to 0$

The asymptotic expansion of $H_\ell^{(\alpha)}(\lambda)$ for $\lambda \to 0$ is obtained as follows. Standard methods from asymptotic analysis yield using (3.4) and (3.9)

$$H_\ell^{(\alpha)}(\lambda) \sim \gamma_\ell \int_0^\infty k^\alpha e^{-k^2\lambda^{-2}} k^{\beta_\ell} dk = \frac{1}{2}\, \gamma_\ell \Gamma[(1+\alpha+\beta_\ell)/2]\lambda^{1+\alpha+\beta_\ell}.$$

From this we obtain for the functions in (3.5) and (3.6) the asymptotic behaviour for $\lambda \to 0$

$$g_\ell(\lambda) \sim \frac{1}{2}\, \gamma_\ell \Gamma(3+\tfrac{1}{2}\beta_\ell)\lambda^{\beta_\ell},$$

$$f_\ell(\lambda) \sim \gamma_\ell \Gamma(2+\tfrac{1}{2}\beta_\ell)\lambda^{\beta_\ell},$$

$$h_\ell(\lambda) \sim \Gamma(1\tfrac{1}{2}+\tfrac{1}{2}\beta_\ell)/\Gamma(2+\tfrac{1}{2}\beta_\ell),$$

$$j_\ell(\lambda) \sim (2 + \frac{1}{2}\,\beta_\ell)^{-\frac{1}{2}},$$

$$d_\ell(\lambda) = h_\ell(\lambda) - j_\ell(\lambda).$$

3.2.2 Results for $\lambda \to \infty$

To obtain the asymptotic expansion of $H_\ell^{(\alpha)}(\lambda)$ for $\lambda \to \infty$ it is convenient to write it, by means of Mellin transform technique, as follows

$$H_\ell^{(\alpha)}(\lambda) = \frac{\lambda^{\alpha+1}}{4\pi i} \int_L \lambda^{-z}\Gamma[(\alpha+1-z)/2]M[G_\ell,z]dz, \tag{3.16}$$

where

$$M[G_\ell,z] = \int_0^\infty k^{z-1}G_\ell(k)dk \tag{3.17}$$

is the Mellin transform of G_ℓ. The contour of integration L in (3.16) is a vertical, such that the singularities of the gamma function at

$$z_j = \alpha + 1 + 2j, \qquad j = 0,1,2,\ldots \tag{3.18}$$

and the singularities of $M[G_\ell, z]$ are at the right of it.

For details on this method we refer to Bleistein & Handelsman [1, Ch.4].

The poles of $M[G_\ell, z]$ can be localized by using the behaviour of G_ℓ for large k. Again, we have to distinguish between two cases: $\tau_3 = 0$, $\tau_3 > 0$.

3.2.2.1. $\tau_3 = 0$

In this event we use (3.14). It follows that $M[G_\ell, z]$ has a pole at $z = 2$. (For Re $z < 2$ the integral (3.17) converges at infinity). The values of α to be considered in (3.18) are (see (3.5) and (3.6)) are

$$\begin{aligned} \alpha &= 2, && \text{giving } z_j = 3+2j \\ \alpha &= 3, && \text{giving } z_j = 4+2j \\ \alpha &= 5, && \text{giving } z_j = 6+2j \end{aligned} \tag{3.19}$$

Hence, poles z_j are in all three cases larger than 2 (the pole of $M[G_\ell, z]$). By replacing the vertical L in (3.16) to the right we obtain by taking the residue at $z = 2$

$$H_\ell^{(\alpha)}(\lambda) = C_\ell^{(\alpha)} \lambda^{\alpha-1} + O(\lambda^{\alpha-2}), \qquad \lambda \to \infty,$$

where $C_\ell^{(\alpha)}$ is a constant, not depending on λ. For the functions in (3.5) and (3.6) we have

$$\left.\begin{aligned} & g_\ell(\lambda),\ f_\ell(\lambda) = O(\lambda^{-2}) \\ & h_\ell(\lambda),\ j_\ell(\lambda),\ d_\ell(\lambda) = O(1) \end{aligned}\right\} \qquad \lambda \to \infty \tag{3.20}$$

with $\tau_3 = 0$

3.2.2.2. $\tau_3 > 0$

The asymptotic behaviour of $G_\ell(k)$ for $k \to \infty$ is given by (3.11). Hence, the poles of $M[G_\ell,z]$ occur at $z = 2\ell+2$ ($\ell = 0,1,2,3$). For $\ell = 0$ the situation is as in subsection 3.2.2.1 with results in (3.20). For $\ell = 1$ we have for $\alpha = 2$ (see (3.19)) a pole z_j in 3 and a pole of $M[G_1,z]$ in 4. Hence, the pole 3 gives the main contribution. For $\ell = 1$, $\alpha = 3$ the integrand of (3.16) has a double pole at $z = 4$. As a consequence, the asymptotic behaviour of $H_1^{(3)}(\lambda) = O(\ell n\lambda)$. Combining all the possible combinations of ℓ and α we obtain the following table for the asymptotic behaviour of the functions of (3.5), (3.6).

	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$
$g_\ell(\lambda)$	λ^{-2}	λ^{-4}	$\lambda^{-6}\ell n\lambda$	λ^{-6}
$f_\ell(\lambda)$	λ^{-2}	$\lambda^{-4}\ell n\lambda$	λ^{-4}	λ^{-4}
$h_\ell(\lambda)$	1	$\lambda/\ell n\lambda$	λ	λ
$j_\ell(\lambda)$	1	$(\ell n\lambda)^{\frac{1}{2}}$	$\lambda/(\ell n\lambda)^{\frac{1}{2}}$	λ
$d_\ell(\lambda)$	1	$\lambda/\ell n\lambda$	λ	λ

4. A FURTHER GENERALIZATION

Here we consider some quantities which arise in the study of a generalization of the differential equation (1.1). We use different frequencies in the forcing and damping terms of the equation and we are interested in the analogues of the G and H functions of the previous sections, especially in those components in G and H depending on the difference of the two frequencies. To be more specific, let us write the differential equation as

$$(4.1) \qquad \frac{d\eta(k,t)}{dt} = \{ik(u_1\sin\sigma_1 t + u_2\sin\sigma_2 t) - b\sigma_1\}\eta(k,t) + ka(k)\{u_1(\sin\sigma_1 t + \cos\sigma_1 t) + u_2(\sin\sigma_2 t + \cos\sigma_2 t)\},$$

where u_1, u_2, σ_1, σ_2, b are real parameters, not depending on t and k.

The nontransient solution of this equation is

$$\eta(k,t) = \sqrt{2}\ ka(k)e^{-b\sigma_1 t-ik(\cos\sigma_1 t+\gamma\cos\sigma_2 t)} \times \int^t e^{b\sigma_1\tau+ik(\cos\sigma_1\tau+\gamma\cos\sigma_2\tau)}[u_1\cos(\sigma_1\tau-\tfrac{\pi}{4}) + u_2\cos(\sigma_2\tau-\tfrac{\pi}{4})]d\tau, \tag{4.2}$$

where

$$\kappa = u_1 k/\sigma_1, \qquad \beta_1 = \sigma_1/\sigma_2, \qquad \gamma = \beta_1\beta_2 \qquad \beta_2 = u_2/u_1. \tag{4.3}$$

By expanding the exponential functions

$$e^{-i\kappa\cos\sigma_1 t},\ e^{-i\gamma\kappa\cos\sigma_2 t},\ e^{i\kappa\cos\sigma_1\tau},\ e^{i\gamma\kappa\cos\sigma_2\tau}$$

as Fourier series, as $e^{ix\cos t} = \Sigma i^m e^{imt} J_n(x)$, and by integrating the resulting series we obtain

$$\eta(k,t) = \frac{\kappa a(k)}{\sqrt{2}} e^{-i\pi/4} \sum_{r,s,n,m} i^{m+n-r-s} J_m(\kappa)\ J_n(\gamma\kappa)\ J_r(\kappa)\ J_s(\gamma\kappa) \times [u_1\{\frac{c^{i(n\sigma_2+(m+1)\sigma_1+r\sigma_1+s\sigma_2)t}}{b\sigma_1+i(m+1)\sigma_1+in\sigma_2} + + i\ \frac{e^{i(n\sigma_2+(m-1)\sigma_1+r\sigma_1+s\sigma_2)t}}{b\sigma_1+i(m-1)\sigma_1+in\sigma_2}\} + u_2\{\frac{e^{i(m\sigma_1+(n+1)\sigma_2+r\sigma_1+s\sigma_2)t}}{b\sigma_1+im\sigma_2+i(n+1)\sigma_2} + + i\ \frac{e^{i(m\sigma_1+i(n-1)\sigma_2+r\sigma_1+s\sigma_2)t}}{b\sigma_1+im\sigma_1+i(n-1)\sigma_2}\}]. \tag{4.4}$$

The r, s, n, m - values run from $-\infty$ to $+\infty$. It is clear that the series (4.4) can be written as

$$\sum_{p,q,r,s} a_{p,q,r,s}\ e^{i(p+r)\sigma_1 t+i(q+s)\sigma_2 t} \tag{4.5}$$

This series can be split up into

$$\sum_j \alpha_j^{(1)} e^{ij\sigma_1 t} + \sum_j \alpha_j^{(2)} e^{ij\sigma_2 t} + \sum_j \alpha_j^{(3)} e^{ij(\sigma_1+\sigma_2)t}$$
$$+ \sum_j \alpha_j^{(4)} e^{ij(\sigma_2-\sigma_1)t} + \ldots \text{ (ad infinitum).}$$

Here we are interested in the coëfficient $\alpha_1^{(4)}$. It arises in (4.5) if $p + r = -1$, $q + s = 1$. Hence

$$\alpha_1^{(4)} = \sum_{p,q} a_{p,q,-J-p,1-q}.$$

When applying this to (4.4), we infer that the Fourier coëfficient of $e^{i(\sigma_2-\sigma_1)t}$ is given by

$$\alpha_1^{(4)} = ia(k)\ \phi_{\sigma_2-\sigma_1}(\kappa)$$

with $\phi_{\sigma_2-\sigma_1}$ given by the double sum

$$\phi_{\sigma_2-\sigma_1}(\kappa) = \frac{\kappa e^{i\pi/4}}{\sqrt{2}} \sum_{p,q} \frac{J_{p-1}(\kappa)\ J_{q+1}(\gamma\kappa)}{b+ip+iq/\beta_1} \{A_{p,q}+\beta_2\ B_{p,q}\}, \tag{4.6}$$

with

$$A_{p,q} = J_{p+1}(\kappa)\ J_q(\gamma\kappa) + iJ_{p-1}(\kappa)\ J_q(\gamma\kappa),$$

$$B_{p,q} = J_p(\kappa)\ J_{q+1}(\gamma\kappa) + iJ_p(\kappa)\ J_{q-1}(\gamma\kappa),$$

and β_1, β_2 defined in (4.3).

Then we define the function (the analogue of (3.2))

$$G_{\sigma_2-\sigma_1}(\kappa) = \phi_{\sigma_2-\sigma_1}(\kappa)\ \phi^*_{\sigma_2-\sigma_1}(\kappa)$$

and furthermore

$$\psi_{\sigma_2-\sigma_1}(\kappa) = \text{phase}[\phi_{\sigma_2-\sigma_1}(\kappa)]; \tag{4.5}$$

the relation between ϕ, G and ψ is

$$\phi_{\sigma_2-\sigma_1}(\kappa) = G^{\frac{1}{2}}_{\sigma_2-\sigma_1}(\kappa)\ e^{i\psi_{\sigma_2-\sigma_1}(\kappa)} ,$$

where we take $\psi_{\sigma_2-\sigma_1}(\kappa) \in (-\pi,\pi]$. The analogues of (3.4) and (3.5) are also considered. We define

$$H^{\alpha}_{\sigma_2-\sigma_1}(\lambda) = \int^{\infty} \kappa^{\alpha}\ e^{-\kappa^2\lambda^{-2}}\ G_{\sigma_2-\sigma_1}(\kappa)\ d\kappa$$

$$g_{\sigma_2-\sigma_1}(\lambda) = \lambda^{-6}\ H^{(5)}_{\sigma_2-\sigma_1}(\lambda)$$

$$f_{\sigma_2-\sigma_1}(\lambda) = 2\lambda^{-4}\ H^{(3)}_{\sigma_2-\sigma_1}(\lambda).$$

The function $G_{\sigma_2-\sigma_1}(\kappa)$ is computed for

$$\beta_1 = .966, \quad \beta_2 = .1,.4,.7,1.0, \quad b = 1, \quad k \in [10^{-2},10^{2}],$$

the function $\psi_{\sigma_2-\sigma_1}(\kappa)$ for

$$\beta_1 = .966, \quad \beta_2 = .4, \quad b = 10^{-2},10^{-1},1,10, \quad k \in [10^{-2},10^{2}]$$

and $g_{\sigma_2-\sigma_1}(\lambda)$, $f_{\sigma_2-\sigma_1}(\lambda)$ for

$$\beta_1 = .966, \quad \beta_2 = .4, \quad b = 1, \qquad \lambda \in [10^{-2},10^{2}].$$

5. SOME REMARKS ON THE COMPUTATIONS

For the computations of $G_{\ell}(k)$ we used the representations (3.2) with (3.2) and (1.5). The series in (1.5) converges very fast, since for large n we have

$$J_n(k)J_{n+m}(k) \sim \frac{(k/2)^{2n+m}}{n!\ (n+m)!} , \qquad m \geq 0;$$

for m < 0 we used (1.10). The λ-integrals $H_{\ell}^{(\alpha)}(\lambda)$ in (3.4) were computed by using a trapezoidal rule (after a suitable transformation). For details see the previous report [2]. The function of Section 4 are computed as their analogues of Section 3.

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