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D. HILHORST, H.J. HILHORST & E. MARODE

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Rigorous results on a time-dependent inhomogeneous Coulomb gas problem*)

by

D. Hilhorst, H.J. Hilhorst**) & E. Marode***)

ABSTRACT

We report results obtained by rigorous analysis of a nonlinear differential equation for the electron density n_e in a specific type of electrical discharge. The problem is essentially two-dimensional. We discuss in particular (i) the escape of electrons to infinity above a critical temperature; and (ii) the boundary layer exhibited by n_e near zero temperature.

KEY WORDS & PHRASES: *singularly perturbed nonlinear two-point boundary value problem; nonlinear parabolic equation degenerate at the origin in one space dimension; Coulomb gas; pre-breakdown discharge in an ionized gas between two electrodes*

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*) This report will be submitted for publication elsewhere.

**) Laboratorium voor Technische Natuurkunde, Postbus 5046, 2600 GA Delft, The Netherlands

***) Laboratoire de Physique des Décharges, E.S.E., Plateau du Moulon, 91190 Gif-sur-Yvette, France

In a filamentary discharge studied by Marode et al. [1,2] electrons and ions are produced with number densities n_e and n_i, respectively. The charged particles move in a background of neutrals. The discharge area is cylindrical and has its radial dimension much smaller than its longitudinal dimension. Since to a good approximation the physical situation is cylindrically symmetric, it suffices to consider a two-dimensional cross section perpendicular to the cylinder axis, in which all quantities involved are functions only of the distance r to the axis. As the ions are heavy and slow, $n_i(r,t) \equiv n_i(r)$ may be regarded as fixed on the time scale of interest. For the density $n_e(r,t)$ Marode et al. [3] use the following three equations:
(i) Coulomb's law

$$\frac{1}{r}\frac{\partial}{\partial r} rE(r,t) = 4\pi e[n_i(r) - n_e(r,t)] \tag{1}$$

where E is the electric field and -e the electron charge;
(ii) a constitutive equation for the current density j(r), consisting of a drift term and a diffusion term,

$$j(r,t) = e\mu n_e(r,t)E(r,t) + eD\frac{\partial n_e(r,t)}{\partial r} \tag{2}$$

where μ is the electron mobility and D the diffusion constant; and
(iii) the continuity equation

$$e\frac{\partial n_e(r,t)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} rj(r,t). \tag{3}$$

Both E and j are radially directed.

From Eqs. (1) - (3) a nonlinear partial differential equation for a single function can be derived. To this end we set [4]

$$u(x,t) = \int_0^{\sqrt{x}} \rho n_e(\rho,t)\,d\rho, \tag{4a}$$

$$g(x) = \int_0^{\sqrt{x}} \rho n_i(\rho)\,d\rho. \tag{4b}$$

Upon employing for the diffusion constant the Einstein relation $D = k_B T\mu/e$ (where k_B is Boltzmann's constant and T the electron temperature), putting $\varepsilon = k_B T/(2\pi e^2)$, and absorbing a factor $8\pi\mu e$ in the time scale we deduce that u satisfies

$$u_t = \varepsilon x u_{xx} + (g-u)u_x, \tag{5}$$

$$u(0,t) = 0. \tag{6}$$

By its definition $g(0) = 0$. Typically, as r increases, $n_i(r)$ rapidly falls off to zero, and hence $g(x)$ attains a limit value $g(\infty)$. The nonlinear term in Eq. (5) represents the interaction between the electrons. Without it, this equation would reduce to a linear one studied by McCauley [5] and describing the Brownian motion of a pair of opposite two-dimensional charges in each other's field. As it stands, Eq. (5) is rather reminiscent of the nonlinear equations occurring in the Thomas-Fermi theory of the atom (see, e.g., ref. [6]).

In the experimental situation that we are describing the total charge in the discharge area is positive and conserved in time. This is expressed by

$$u(\infty,t) = N_e \qquad \text{for } 0 \le t < \infty \tag{7}$$

with $0 \le N_e < g(\infty)$. One of the authors has investigated [4,7,8], by rigorous mathematical methods, the solution of Eqs. (5) and (6) for a given initial distribution $u(x,0) = u_0(x)$ and subject to condition (7) on the total charge. Here we present the main results in physical language.

1. We take g concave and in $C^2([0,\infty))$. Then at given ε (i.e. at given temperature), there exists [4] a unique stationary solution $u_{st}(x)$ if the total number of electrons N_e is such that $N_e \le g(\infty) - \varepsilon$. In particular, when $\varepsilon \ge g(\infty)$, thermal motion prevents any electrons to be bound to the fixed ionic background. The existence of such a critical temperature is characteristic of two-dimensional Coulomb systems [9]. The main mathematical tools in treating the stationary problem are maximum principle arguments and the construction of upper and lower solutions.

2. The solution u_{st}, when it exists, has the following properties [4].

(i) It belongs to $C^2([0,\infty))$. It is strictly increasing, concave, and bounded from above by the function $\min(g(x),N_e)$. As $x \to \infty$, $u_{st}(x)$ approaches its limiting value N_e at least fast enough so that

$$n_e(r) \le n_e(r_1)\left(\frac{r^2}{x_1}\right)^{-\frac{1}{\varepsilon}[g(x_1)-N_e]}, \qquad r \to \infty, \tag{8}$$

where $r_1^2 \equiv x_1 > 0$ is arbitrary. Such power law decay is again typical of Coulomb systems in two dimensions.

(ii) As $\varepsilon \downarrow 0$, $u_{st}(x)$ converges to $\min(g(x),N_e)$ uniformly on $[0,\infty)$, and we have for the zero temperature limit of the electron density

$$\lim_{\varepsilon\downarrow 0} n_e(r) = \begin{cases} n_i(r) & r < r_0 \\ 0 & r > r_0 \end{cases} \tag{9}$$

where the critical radius r_0 is defined by the relation $g(r_0) = N_e$. At small ε there is a transition layer of width $\sim \varepsilon^{\frac{1}{2}}$, located at r_0, analogous to a Debye shielding length [3]. A uniformly valid approximate stationary solution for $\varepsilon \ll 1$ is given in [4]. It is obtained by the method of matched asymptotic expansions.

3. We consider now the time evolution problem of Eqs. (5) and (6). Suppose that the initial condition u_0 is sufficiently smooth, nondecreasing, with bounded derivative, and with $u_0(0) = 0$ and $u_0(\infty) = N_e$. Mathematically one has to find a way to deal with the degeneracy of the parabolic equation (5) in the origin. In [7] this is done via a sequence of regularized problems. The following is shown.

(i) The time evolution problem has a unique solution $u(x,t)$ such that u and u_x are bounded. In fact it satisfies $0 \le u(x,t) \le N_e$, it is nondecreasing in x for all t, and for each $t \ge 0$ we have $u(\infty,t) = N_e$.

(ii) In order to discuss the behavior of $u(x,t)$ as $t \to \infty$ we consider the function $\bar{u}_{st}$ which satisfies the steady state equation and has boundary values $\bar{u}_{st}(0) = 0$ and

$$\bar{u}_{st}(\infty) = \begin{cases} N_e & \text{if } N_e \le g(\infty) - \varepsilon & (10a) \\ g(\infty) - \varepsilon & \text{if } 0 < g(\infty) - \varepsilon < N_e & (10b) \\ 0 & \text{otherwise} & (10c) \end{cases}$$

We know from section 1 that $\bar{u}_{st}$ exists and is unique. In particular, in the case of Eq. (10c), $\bar{u}_{st}(x) \equiv 0$. Our result is that the solution $u(x,t)$ of the evolution problem converges to $\bar{u}_{st}(x)$ as $t \to \infty$, uniformly on all compact subsets of $[0,\infty)$; in the case of Eq. (10a) the convergence is actually uniform on $[0,\infty)$. The proofs are based upon the use of upper and lower solutions of the stationary problem and on a comparison theorem. Thus we have proved that all the electrons stay attached to the ions for $t \le \infty$ at temperatures such that $\varepsilon \le g(\infty) - N_e$ (case (10a)). If the temperature rises above this critical value, then some of the electrons diffuse away to infinity (case (10b)), and if it rises above a second critical value, viz. $\varepsilon = g(\infty)$, then all electrons escape to infinity (case (10c)).

(iii) For the case of Eq. (10a) (with the inequality strictly satisfied) we have derived results about the rate of convergence of u to $\bar{u}_{st}$. Let the initial state have the property that $N_e - u_0(x) \le N_e(x_1/x)^\nu$ for some $x_1, \nu > 0$ satisfying $\varepsilon \le (\nu+1)^{-1}[g(x_1) - N_e]$. Then $u(x,t)$ converges to $\bar{u}_{st}(x)$ at least as fast as $t^{-1/(2p)}$ with $p = [1/\nu] + 1$, for all finite x. Furthermore, if $\nu > 1$ and $\varepsilon < \frac{1}{2}[g(\infty) - N_e]$, then u converges to $\bar{u}_{st}$ at least as fast as $t^{-\frac{1}{2}}$.

4. *Negative regions in the background charge density.* We have considered an interesting modification of the above problem obtained by also allowing negative ions to be present in the fixed background [8].
This leads to a function g which can assume minima and maxima. We studied the stationary state on a bounded domain $[O,R]$ with boundary condition $u_{st}(R) = N_e$. For non-monotone g it is nontrivial to find the zero temperature ($\varepsilon \to 0$) limit of $u_{st}(x)$ (and thus of $n_e(r)$), since the solution of the reduced differential equation (i.e. the one obtained by setting $\varepsilon = 0$) is no longer unique. To solve this problem we observe that for $\varepsilon > 0$ the solution $u_{st}(x;\varepsilon)$ minimizes the free energy functional

$$F_\varepsilon[u] = \varepsilon \int_0^R u_x \ell n u_x dx + \tfrac{1}{2} \int_0^R \frac{(g-u)^2}{x} dx, \tag{11}$$

which is readily recognized as the sum of an entropy and an electrostatic energy term.

In [8] two alternative methods were used to study the minimization of F_ε: one based on the theory of maximal monotone operators and one on duality theory. Both yield

$$\lim_{\varepsilon \downarrow 0} u_{st}(x;\varepsilon) = \inf_{0 \le u \le N_e, u' \ge 0} \tfrac{1}{2} \int_0^R \frac{(g-u)^2}{x} dx, \tag{12}$$

i.e. the limit solution of the differential equation is the physically expected minimum energy configuration. The function $u_{st}(x;0)$ is continuous [10] and can be characterized as follows: there exist intervals $[a_1,b_1]$, $[a_2,b_2],\ldots,[a_s,b_s]$, $s \ge 0$, where $u_{st}(x;0)$ takes constant values $c_1,c_2,\ldots,c_s$, respectively, and where, therefore, $n_e(r) = 0$. Outside those intervals $u_{st}(x;0) = g(x)$. The constants a_i, b_i, c_i, $i = 1,2,\ldots,s$, can be shown, finally, to be uniquely determined by the set of implicit inequalities

$$\left.\begin{aligned} &\int_x^{b_i} \frac{c_i-g(\xi)}{\xi} d\xi \ge 0 \quad \text{if} \quad c_i \ne N_e \qquad (13a) \\ &\int_{a_i}^{x} \frac{c_i-g(\xi)}{\xi} d\xi \le 0 \quad \text{if} \quad c_i \ne 0 \qquad (13b) \end{aligned}\right\} \text{for all } x \in [a_i,b_i],\ i=1,2,\ldots,s.$$

To verify this characterization of $u_{st}(x;0)$, one checks [8] that this function satisfies a variational inequality related to the minimization problem (12). In particular, if $0 < c_i < N_e$, we have the equal area construction $\int_{a_i}^{b_i}(c_i-g(\xi))\xi^{-1}d\xi = 0$. The interpretation is that the points $x = a_i$ and $x = b_i$ are at equal potential and separated by a potential barrier. Eqs. (13) may serve as the basis for a numerical algorithm to compute a_i, b_i, c_i.

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